

Lecture 1

Harmonic Maps

1° Geroch Conj (T^3, g) $\mathbb{Q}$, $R_g \geq 0$?

scalar cur.



R. Schoen - S.-T. Yau 1978.

positive mass conj.

2° Frenkel's Conj: $(M^{2n}, \omega, J) \cong \mathbb{C}P^n$

Y. Siu - S.-T. Yau 1980

3° Homeomorphic Sphere thm $(M^{n \geq 4}, g)$ $1 < \sec \leq 4$

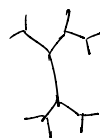
$\pi_1(M) = \{0\}$

$\frac{\max \sec(p)}{p} < \frac{7}{m} < 4$

M. Micalef - D. Moore 1988 $\Rightarrow M \cong S^n$

4° super-Rigidity of p-adic group

M. Gromov - R. Schoen 1989



Smooth $\rightarrow$ non-smooth

5° Higher Teichmüller Theory

6° Finite Time Extinction of Ricci flow $\Rightarrow$ Harmonic maps, prove

T. Colding - W. Minicozzi 03. 07

7° Existence of minimal surfaces in Higher-Codin
D. Chang - X.Z. Yesterday

$$u : (M^n, g) \rightarrow (N^m, h) \hookrightarrow \mathbb{R}^L$$

• Def: Energy density $\rightarrow$ Harmonic energy

$$e(u)(x) := \frac{1}{2} |\nabla u|^2(x)$$

$$E|_{\Omega}(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2(x) d\text{vol}_g(x)$$

$\Omega \subset M$

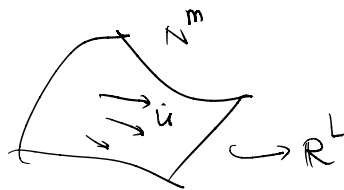
First Variation

$$u_t : \underset{t \in \mathbb{R}}{(-\varepsilon, \varepsilon)} \times \overset{\text{space } \Omega}{\Omega} \longrightarrow (N^m, h) \hookrightarrow \mathbb{R}^L$$

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} E(u_t) &= \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \int_{\Omega} \langle \nabla u_t, \nabla u_t \rangle d\text{Vol}_g \\ &= \int_{\Omega} \langle \nabla u, \nabla \dot{u} \rangle d\text{vol}_g \end{aligned}$$

$$\frac{d}{dt} \Big|_{t=0} u_t = \dot{u} \in T(u^*(TN))$$

$\nwarrow$ tangent vector field



$$\frac{d}{dt} \Big|_{t=0} E(u_t) = - \int_{\Omega} \langle \nabla u, \dot{u} \rangle d\text{vol}_g$$

$$(\dot{u} \in C_c(\cdot)) \Rightarrow \Delta u(x) \perp \dot{u}(x) \quad \forall \text{ a.e. } x \in \Omega$$

$$\Rightarrow \Delta u(x) \perp T_{u(x)} N \quad \forall x \in \Omega$$

⬆ bundle's section

Claim: $\Delta u(x) = -A^N(u(x)) (\nabla u(x), \nabla u(x))$ $T^\perp N = \text{span}\{v_1, \dots, v_{L-m}\}$

A^N : 2nd fundamental form of $N^m \hookrightarrow \mathbb{R}^L$

$$A^N(v, w) := -(\nabla_v^{\mathbb{R}^n} w)^\perp = \sum_{j=1}^{L-m} \langle \nabla_v^{\mathbb{R}^n} v_j, w \rangle v_j$$

$$= -A^N(\nabla_{x_\alpha} u, \nabla_{x_\beta} u) g^{\alpha\beta}$$

$TN \times TN \rightarrow T^\perp N$

PP: " $\Leftarrow$ " ✓ tangent vec. $\downarrow$ normal vec.

" $\Rightarrow$ " • $\nabla u(x) \perp v_j(u(x))$

$$0 = \langle \nabla u(x), v_j(u(x)) \rangle$$

$$\Rightarrow 0 = \text{div}_M \langle \nabla u(x), v_j(u(x)) \rangle$$

$$= \langle \Delta u(x), v_j \rangle + \langle \nabla_{x_\alpha} u, \nabla_{v_j} v_j(u(x)) \rangle g^{\alpha\beta}$$

$$\Rightarrow \Delta u(x) = -A(u(x)) (\nabla u, \nabla u) \cdot v_j$$

Def $u \in C^2(M, N)$ is a harmonic map.

iff $\frac{d}{dt} \Big|_{t=0} E(u_t) = 0 \quad \forall \text{ variation } u_t$

(HME)

$$\Leftrightarrow \Delta u \mp A(u(x)) (\nabla u(x), \nabla u(x)) = 0,$$

Intrinsic point of view

$$du \in \Gamma(T^*M \otimes u^*TN)$$

$$du(v) := u_*(v) \in T_{u(x)} N, \quad v \in T_x M$$

$$\gamma'' = D_{\gamma'} \gamma'$$

$$\frac{d^2}{dt^2} \Big|_{\gamma} \gamma(t) = -A^N(\gamma')(\gamma', \gamma')$$

Def: $u \in W^{1,2}(M, N) = \{ v \in W^{1,2}(M, \mathbb{R}^L), v(x) \in N, \text{ for a.e. } x \}$
 is weakly harmonic if $0 = \int \underbrace{\langle \nabla u(x), \nabla u(x) \rangle}_{\downarrow \substack{\forall \eta \in W^{1,2}(M, \mathbb{R}^L) \cap L^\infty}} - \underbrace{A(u(x))(\nabla u, \nabla u)}_{\downarrow} \eta \, dx$

Rank: $W^{1,p}(M^n, N^m) = \begin{cases} p > n & \text{Banach manifold} \\ p \leq n & \times \end{cases} \quad \because W^{1,p} \hookrightarrow C^\alpha$

Def $u \in W^{1,2}(M, N)$ is "energy minimizing"

iff $\forall \Omega \subset\subset M, \forall v \in W^{1,2}(M, N)$

$$u - v \in W^{1,2}_0(\Omega, \mathbb{R}^L)$$

$$\Rightarrow E(u) \leq E(v)$$

Def $u \in W^{1,2}$ is "stationary HM"

iff $\cdot u$ is weakly HM

$\cdot \forall \Omega \subset\subset M, \forall \phi_t \in C_c^\infty(\Omega, \Omega)$

$$\left. \frac{d}{dt} \right|_{t=0} E(u \circ \phi_t) = 0$$

Rmk: $\dot{u} = \left. \frac{d}{dt} \right|_{t=0} (u \circ \phi_t) = du_x \left(\left. \frac{\partial \phi_s}{\partial s} \right|_{s=0} = \bar{X} \right) \in L^2 \text{ vector field.}$

Bochner formula (M^n, g, χ^α)

$$\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + \text{Ric}^M(\nabla u, \nabla u) - \sum_{\alpha, \beta} R^N(u_\alpha, u_\beta, \chi_\alpha, \chi_\beta)$$

$$\nwarrow |\nabla du|^2 \quad du \in T(\tau^* M \otimes u^* TN)$$

2nd Variational formula

$$u_t: (-\varepsilon, \varepsilon) \times M \rightarrow N \hookrightarrow \mathbb{R}^L \in C^2$$

$$H.M.E. \quad (\Delta u)^\perp = 0$$

$$\dot{u} = \frac{\partial}{\partial t} \Big|_{t=0} u \in T(u^* TN)$$

$$\ddot{u} = \frac{\partial^2}{\partial t^2} \Big|_{t=0} u \in P(u^* TN)$$

$$\frac{d^2}{dt^2} \Big|_{t=0} E(u_t) = \int_M |\nabla \dot{u}|^2 + \underbrace{\int_M \langle \nabla u, \nabla \ddot{u} \rangle}_0 - \sum_\alpha \int_M R^N(\dot{u}, u_\alpha, \dot{u}, u_\alpha)$$

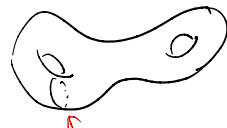
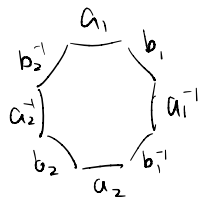
Application

Preissman's thm (Do Carmo) (N^n, h) closed Riem mfd

$\text{Sec } h < 0 \Rightarrow$ Any abelian subgrp of $\pi_1(M)$ has rank ≤ 1 .

e.x. π^2 ?

$$(\Sigma_{g \geq 2}^2, g_{hyp})$$

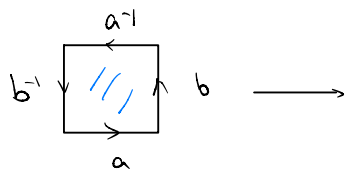


$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

$$\Leftrightarrow 1 \notin \pi_1(N) \supset \underbrace{\mathbb{Z}}_a \oplus \underbrace{\mathbb{Z}}_b \Rightarrow " \Rightarrow "$$

Claim: $\exists u_0: \pi^2 \rightarrow N$ s.t. $u_{0,x}: \pi_1(\pi^2) \rightarrow \pi_1(N)$ inj

abelian
 $a \cdot b \cdot a^{-1} \cdot b^{-1} = 0$
 $\downarrow$
 trivial



$\Rightarrow$ H.M.

$$u_\infty: \pi^2 \rightarrow (N^m, g)$$

Thm (Eells-Sampson)

$$u_0: (M^n, g) \rightarrow (N^n, h) \quad \sec_h \leq 0$$

$$\Rightarrow \begin{cases} \partial_t u = \Delta u + A(u)(\nabla u, \nabla u) \\ u(0, x) = u_0(x) \end{cases}$$

$$\lim_{t \rightarrow +\infty} u(t, x) \xrightarrow{C^k} u_0 \quad \text{H.M.}$$

Claim: u_∞ has rank ≤ 1

≥ 0

$$\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + \underbrace{\text{Ric}(\cdot)}_{\geq 0, \text{ flat}} - \sum_{\alpha, \beta} R(u_\alpha, u_\beta, u_\alpha, u_\beta)$$



① $|\nabla u|^2 = \text{const.}$

② $\sec < 0 \Rightarrow u_\alpha \parallel u_\beta$

重新给出 Yam first paper's proof.

Bochner-Formula $(M, g, \chi^\alpha) \rightarrow (N, h) \hookrightarrow \mathbb{R}^L$

$$\left(\frac{1}{2} |\nabla u|^2 \right)_{\beta\beta} = \frac{1}{2} (u_\alpha^i u_\alpha^i)_{\beta\beta} = (u_\alpha^i u_{\beta\alpha}^i)_{\beta} = \underbrace{u_{\alpha\beta}^i \cdot u_{\alpha\beta}^i}_{\text{(Hess } u^i)_{\alpha\beta}} + u_\alpha^i u_{\beta\alpha\beta}^i$$

$$= u_{\alpha\beta}^i \cdot u_{\alpha\beta}^i + u_\alpha^i u_{\beta\beta\alpha}^i$$

$$+ u_\alpha^i \cdot \text{Ric}_{\alpha\beta}^M u_\beta^i$$

$$= |\text{Hess } u^i|^2 + \langle u_\alpha, (\Delta u)_\alpha \rangle$$

$$+ \text{Ric}^M(\nabla u, \nabla u)$$

$$\langle u_\alpha, (\Delta u)_\alpha \rangle = \langle u_\alpha, (A(u)(\nabla u, \nabla u))_\alpha \rangle$$

$$u_\alpha \perp A = - \langle u_{\alpha\alpha}, A(u)(\nabla u, \nabla u) \rangle$$

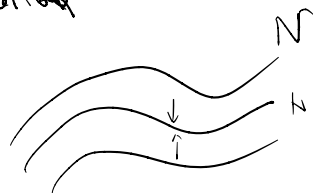
$$= - \left(\sum_{\alpha} A(u)(u_\alpha, u_\alpha) \right)^2$$

$$\text{Claim: } |\text{Hess}_M u^i|^2 = |\nabla du|^2 + \sum_{\alpha, \beta} A(u)(u_\alpha, u_\beta) \cdot A(u)(u_\alpha, u_\beta)$$

$$\bullet \pi: N_N \rightarrow N$$

$$\bullet p = d\pi_{p \in N}: \mathbb{R}^L \rightarrow T_p N, \text{ orthogonal projection}$$

$$\bullet dP_p = AN$$



$$P(u_\alpha) = u_\alpha$$

$$u_{\alpha\beta} = (P(u_\alpha))_\beta = d P(u_\alpha, u_\beta) + P(u_{\alpha\beta})$$

$$= A(u_\alpha, u_\beta) + D du$$

$$\therefore \frac{1}{2} \Delta |du|^2 = |D du|^2 + Ric^M(du, du)$$

$$+ \underbrace{\sum_{\alpha, \beta} A_{u_\alpha u_\beta} A_{u_\alpha u_\beta} - (A_{u_\alpha u_\beta})^2}_{\text{Gauss - Codazzi}}$$

$$- R^N_{\alpha\beta\gamma\delta} u_\alpha u_\beta u_\gamma u_\delta$$

$$0 = \tilde{R}^M_{ijkl} = R^M_{ijkl} - h_{ik} h_{jl} + h_{il} h_{jk}$$

Convexity of Energy.

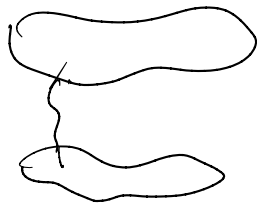
$$u \in C^2 \quad H, M. \quad \text{Sec} \leq 0 \quad \pi_1(N) = \{0\}.$$

Let U be an arbitrary $C^2(M, N)$

$$\left. \begin{array}{l} (N, h) \\ \cdot \text{Sec} \leq 0 \\ \cdot \pi_1(N) = \{0\} \end{array} \right\} \Rightarrow \exists \text{ homotopy: } F: [0, 1] \times M \rightarrow N$$

$$\cdot F(0, x) = u(x)$$

$$\cdot F(1, x) = U(x)$$



$$\cdot t \mapsto F(t, x)$$

geodesic from
 $u(x)$ to $v(x)$

Rmk : $M = \Omega$, $u|_{\partial\Omega} = v|_{\partial\Omega}$

$\cdot$ Write $u_t(x) = F(t, x)$ $\ddot{u}_t = \nabla_{\dot{u}_t} \dot{u}_t = 0$

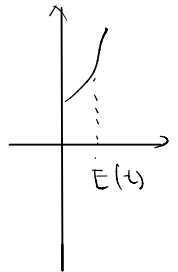
$$\frac{d^2}{dt^2} \Big|_{t=0} E(u_t) = \int_{\Omega/M} |\nabla \dot{u}|^2 + \int_M \langle \nabla u_t, \nabla \ddot{u}_t \rangle$$

≥ 0

$$- \sum_{\alpha} \int_M R^N(\dot{u}_t, u_{\alpha}, \dot{u}_t, u_{\beta})$$

≥ 0

$$\Rightarrow \begin{cases} \frac{d^2}{dt^2} \Big|_{t=0} E(u_t) \geq 0 \\ \frac{d}{dt} \Big|_{t=0} E(u_t) = 0 \end{cases} \quad \hat{=} \quad \begin{aligned} E(u_1) &\geq E(u_0) \\ E(v) &\geq E(u) \end{aligned}$$



Prop $\text{Sec}_u \leq 0$

$$\Rightarrow E(u) \leq E(v)$$

$\cdot$ if u, v both harmonic

$$\Rightarrow E(u) = E(v)$$

$$\Rightarrow |\nabla \dot{u}| \equiv 0 \quad \dots \rightarrow \dot{u}_t \equiv 0 \quad \Rightarrow u(x) = v(x)$$

① Dirichlet

$$u|_{\partial\Omega} = v|_{\partial\Omega}$$

$$\Rightarrow \dot{u}_t|_{\partial\Omega} \equiv 0$$

$$0 = \int_{\Omega} |\nabla u|^2 \geq \frac{1}{C} \int_{\Omega} |u|^2$$

(N, h) , $\sec \leq 0$

$$\Rightarrow \frac{1}{2} \Delta |\nabla u|^2 \geq |\nabla^2 u|^2 - C |\nabla u|^2$$

$\Rightarrow$ Prop

$$u \in C^2(B_1^n, N)$$

$$\sec_N \leq 0$$

$$\Rightarrow \sup_{x \in B_{1/2}} |\nabla u|^2(x) \leq C_1 \int_{B_1} |\nabla u|^2$$

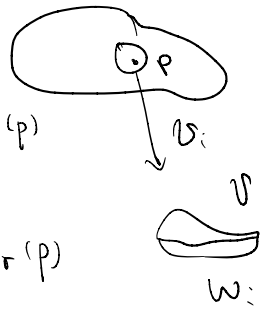
$$u \in B_r(x_0), \quad |\nabla u|^2(x) \sim \frac{1}{r^2}$$

$$\int_{B_r(x_0)} |\nabla u|^2 d\text{vol}_g \sim r^{n-2}$$

$$\therefore \Rightarrow \sup_{B_{r/2}(x_0)} r^2 |\nabla u|^2(x) \leq C \frac{1}{r^{n-2}} \int_{B_r(x_0)} |\nabla u|^2$$

$$U_i \longrightarrow W_i : B_{r_0}(p) \subset M \longrightarrow (N, h)$$

harmonic

$$\text{Let } \bar{U}_i : M \longrightarrow N \quad \begin{cases} W_i & x \in B_r(p) \\ \bar{U}_i & x \notin B_r(p) \end{cases}$$


$$W \searrow E(V_i) \geq E(\bar{U}_i) \searrow W$$

$$E(V_i) - E(\bar{U}_i) \searrow 0$$

By convexity

$$\begin{aligned} E(V_i) - E(\bar{U}_i) &= E(U_i) - E(W_i) \\ &\geq \int |\nabla U_i|^2 \geq \frac{1}{c} \int |U_i|^2 \\ &\geq \int d_{\mathbb{R}^L}^2 (U_i(x), W_i(x)) \end{aligned}$$

$$*. \quad W_i : B_{r_0/2}(p) \longrightarrow N$$

$$\Downarrow C^\infty$$

$$W_\infty : B_{r_0/2}(p) \longrightarrow N \quad C^\infty \text{ harmonic.}$$

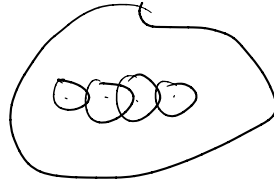
$$\Rightarrow \|W_i - V_i\|_{L^2} \rightarrow 0$$

$$\Rightarrow U_i \rightarrow W_i \text{ in } W^{1,2} \Rightarrow \Rightarrow \text{in } L^2$$

$$\Rightarrow U_\infty = W_\infty \text{ in } B_{r_0/2}(p)$$

$$2) \quad U_\infty \overset{\text{homotopic}}{\sim} U_0$$

$$\Leftarrow U_\infty \sim U_i$$



covered by balls.

$$V_i: B_{r_j}(p_j) \rightarrow N$$

is

$$W_i \Rightarrow U_\infty \text{ in } B_{r_j/2}(p_j)$$

$$\Rightarrow \begin{cases} U_\infty, C^\infty \text{ H.M.} \\ U_\infty \sim V_i \sim U_0 \end{cases}$$

$$\Rightarrow E(U_\infty) \geq W$$

$$\Rightarrow E(U_\infty) = W.$$

□

Lecture 2.

• Existence of Minimal Surfaces in Higher Codim

- Plateau Problem C. Morrey 1950s

- Incompressible minimal surface R. Schoen - S.-T. Yau

$$(M^n, g), \pi_1(M) \supset \pi_1(\Sigma_{g \geq 1}^2) \xrightarrow{1978} \exists \text{ minimal } \Sigma_g \hookrightarrow M \quad \text{incompressible}$$

- Sachs - Uhlenback Theory 1981

$$(M^n, g) \text{ non-aspherical } \pi_k(M) \neq 0, k \geq 2$$

$$\exists \text{ minimal } S^2 \hookrightarrow M$$

- D. Cheng - X. Zhou, 2026

$$\bullet n=4,5 \Rightarrow \exists \text{ min } \Sigma_g^2 \hookrightarrow (M^n, g)$$

$$\bullet H_k(M, \mathbb{Z}) \neq \{0\} \quad k=3,4,5 \quad \nearrow$$

• Poincaré's thm (M^n, g) $\sec g \leq 0$,

$\Rightarrow$ Free Abelian subgroup of $\pi_1(M)$ is cyclic

$$\simeq \mathbb{Z} / \text{rank} \leq 1$$

Regularity

Thm (Helein, 1990s).

Any weakly harmonic map $\mapsto \int_{\Sigma} \langle \nabla u, \nabla v \rangle - A(u)(\nabla u, \nabla v) \rangle \, d\mu_{\Sigma}$

$u: \Omega^2 \rightarrow (N^m, h)$ is C^∞
 $\uparrow$
 convex domain

• Stationary Harmonic Map $u: \Omega^{\geq 3} \rightarrow (N^m, h)$

1° u is weakly harmonic map (critical pt for ambient deformation)

2° $\forall \phi_s: (-\epsilon, \epsilon) \times \Omega \rightarrow \Omega \rightsquigarrow \chi(x) = \frac{\partial}{\partial s} \Big|_{s=0} \phi_s(x) \in \chi_c(\Omega)$

$\frac{d}{ds} \Big|_{s=0} E(u \circ \phi_s) = 0$ (unit. for domain vari.)

• Monotonicity for E for Stationary H.M. $u \circ \phi_s$

1° $0 = \frac{d}{ds} \Big|_{s=0} E(u \circ \phi_s, g)$

$= \frac{d}{ds} \Big|_{s=0} E(u, g_s)$

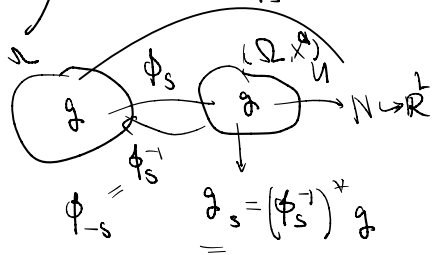
$\frac{\partial}{\partial s} = -X = \frac{d}{ds} \Big|_{s=0} \frac{1}{2} \int_{\Omega} g^{\alpha\beta} u_\alpha u_\beta \sqrt{\det g_s} dx^1 \wedge \dots \wedge dx^n$

• $g_s^{\alpha\beta} = g^{\alpha\beta} + s(L_{-X} g)^{\alpha\beta} + o(s)$

Taylor exp.

• $(g_{\alpha\beta})_s = g_{\alpha\beta} + s(L_{-X} g)_{\alpha\beta} + o(s)$

$(L_{-X} g)_{\alpha\beta} = \frac{\partial}{\partial s} \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right\rangle_g$



$$\hat{\phi}_s : (-\epsilon, \epsilon) \times \Omega \rightarrow \Omega$$

s x^α

$$= \langle \nabla_{\partial_s} \partial_\alpha, \partial_\beta \rangle + \langle \partial_\alpha, \nabla_{\partial_s} \partial_\beta \rangle$$

$$= \langle \nabla_{\partial_\alpha} \partial_s, \partial_\beta \rangle + \langle \partial_\alpha, \nabla_{\partial_\beta} \partial_s \rangle$$

$$= \langle \nabla_{\partial_\alpha} x, \partial_\beta \rangle + \langle \partial_\alpha, \nabla_{\partial_\beta} x \rangle$$

$$g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha \quad = X_{,\gamma}^\beta + X_{,\beta}^\alpha$$

$$(L - x \cdot g)_{\alpha\beta} = -X_{,\beta}^\alpha - X_{,\alpha}^\beta$$

$$\Rightarrow \frac{\partial}{\partial s} \Big|_{s=0} (g_s)^{\alpha\beta} = X^{\alpha\beta}_{,\beta} + X^{\beta\alpha}_{,\alpha}$$

$$\begin{aligned} \frac{\partial}{\partial s} \Big|_{s=0} \sqrt{\det g_{\alpha\beta}} &= \frac{1}{2} \sqrt{\det g} \cdot \text{Tr}_g \left(\frac{\partial}{\partial s} \Big|_{s=0} g_s \right) \\ &= -\frac{1}{2} \sqrt{\det g_0} \cdot g^{\alpha\beta} (X_{\alpha\beta} + X_{\beta\alpha}) \end{aligned}$$

$$\Rightarrow = \left(\frac{1}{2} \right) \int_\Omega (X^{\alpha\beta}_{,\beta} + X^{\beta\alpha}_{,\alpha}) u_\alpha \cdot u_\beta - \frac{1}{2} g^{\alpha\beta} (X_{\alpha\beta} + X_{\beta\alpha}) |\nabla u|^2 \, d\text{vol}_g(x)$$

$$0 = \int_\Omega (X^{\alpha\beta}_{,\beta} + X^{\beta\alpha}_{,\alpha}) \left[u_\alpha \cdot u_\beta - \frac{1}{2} |\nabla u|^2 g_{\alpha\beta} \right] \, d\text{vol}_g$$

Stress-Energy Tensor

$$S(u)_{\alpha\beta}$$

Yang-Mills
bundle $\rightarrow_{SU(2)} \mathbb{R} \quad \text{cpl. rank} = 2$
 (M^4, g)

$SU(2)$ -valued, 1-form

$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

$$A_t = (\text{id} + a_{ij} t)$$

$$\ln \det(A_t) = t \cdot \text{tr}(a_{ij}) + o(t)$$

$$G_t = G_0 (\text{id} + t G_0^{-1} G_t)$$

$$\frac{\partial}{\partial t} \Big|_{t=0} \det(G_t) = \det(G_0) \cdot \text{tr}_{G_0} \left(\frac{\partial}{\partial t} G_t \right)$$

$$A = d + w$$

$$F_A = dw + w \wedge w.$$

$$YM(w) = \int (F_A)^2 \text{dvol}_g \quad \frac{1}{t^{n-4}}$$

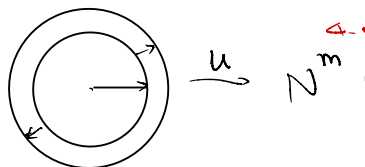
2-dim Harmonic map



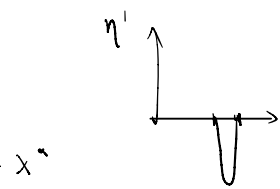
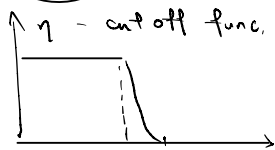
4-dim Yang-Mills

$$\cdot 2^\circ \quad \Omega \subseteq \mathbb{R}^n, \quad o \in \Omega$$

Choose $\vec{X}(x) = \vec{x} \cdot \eta(|x|)$
vector field



$$\xrightarrow{u} N^m$$



$$\begin{aligned} \vec{X}^{\alpha\beta} &= \frac{\partial}{\partial x^\beta} (\eta(r) x^\alpha) \\ &= \eta(r) \delta_\beta^\alpha \\ &\quad + \eta'(r) \frac{\partial r}{\partial x^\beta} x^\alpha \\ &= \eta(r) \delta_\beta^\alpha + \eta'(r) \frac{x^\beta}{r} x^\alpha \end{aligned}$$

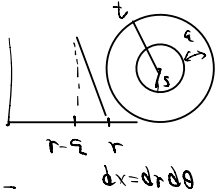
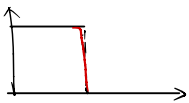
$$\Rightarrow 0 = \int \left(\eta(r) \delta_{\alpha\beta} + \eta'(r) \frac{x^\beta}{r} x^\alpha \right) \left[u_\alpha u_\beta - \frac{1}{2} |\nabla u|^2 \delta_{\alpha\beta} \right] \text{dvol}_x$$

$$\frac{\partial}{\partial r} = \frac{x_\alpha}{r} \frac{\partial}{\partial x_\alpha} B_1^n(o)$$

$$= \int_{B_1^n(o)} \eta |\nabla u|^2 - \frac{n}{2} \eta |\nabla u|^2 + \eta'(r) r |\nabla_{\partial r} u|^2 - \frac{1}{2} \eta' |\nabla u|^2 r$$

$$\Rightarrow \frac{1}{2} (n-2) \int_{\Omega} \eta |\nabla u|^2 = \int_{\Omega} r \eta'(r) \left(|\nabla_{\partial r} u|^2 - \frac{1}{2} |\nabla u|^2 \right)$$

$$\eta(x) \rightarrow \chi_{[0,r]}$$



$$\Rightarrow \frac{1}{2} (n-2) \int_{B_r(o)} |\nabla u|^2 = -r \int_{\partial B_r(o)} \left[|\nabla_{\partial r} u|^2 - \frac{1}{2} |\nabla u|^2 \right]$$

$$E(r) = \frac{1}{2} \int_{B_r(o)} |\nabla u|^2, \quad E'(r) = \frac{1}{2} \int_{\partial B_r(o)} |\nabla u|^2$$

$$\Rightarrow (n-2) E(r) = -r \int_{\partial B_r(0)} |\partial_r u|^2 + r \cdot E'(r)$$

$$E'(r) - \frac{n-2}{r} E(r) = \int_{\partial B_r(0)} |\partial_r u|^2$$

$$\forall 0 < s < t < 1$$

$$\Rightarrow \frac{1}{t^{n-2}} \int_{B_t^n} |\nabla u|^2 - \frac{1}{s^{n-2}} \int_{B_s^n(0)} |\nabla u|^2 = 2 \int_{B_t^n \setminus B_s^n} \frac{|\partial_r u|^2}{r^{n-2}} dx$$

A priori Estimates

Thm (Σ -Regularity for C^2 -H.M.)

$$u: B_1^n(0) \rightarrow (N^m, h) \hookrightarrow \mathbb{R}^L, \quad C^2\text{-H.M.}$$

$$\exists \varepsilon = \varepsilon(n, m, N) > 0, \text{ s.t.}$$

$$\text{if } \int_{B_1^n} |\nabla u|^2 < \varepsilon$$

$$\Rightarrow \sup_{x \in B_{\frac{1}{2}}(0)} |\nabla u|^2(x) \leq C \int_{B_1} |\nabla u|^2(x)$$

Rmk: ① if $\sec_N \leq 0$. ✓

$$\text{② } r^2 \sup_{x \in B_{\frac{r}{2}}(x_0)} |\nabla u|^2(x) \leq C \frac{1}{r^{n-2}} \int_{B_r(x_0)} |\nabla u|^2(x).$$

• Bochner formula.

$$\frac{1}{2} \Delta |\nabla u|^2 = |\nabla^2 u|^2 + \text{Ric}(\nabla u, \nabla u) - R^N(u_\alpha, u_\beta, u_\alpha, u_\beta)$$

$$\bullet R^N \leq 0. \Rightarrow \frac{1}{2} \Delta |\nabla u|^2 \geq -C |\nabla u|^2$$

$$\bullet \Rightarrow \Delta \left(\frac{1}{2} |\nabla u|^2 \right) \geq -C_1 |\nabla u|^2 - C_2 |\nabla u|^2 |\nabla u|^2$$

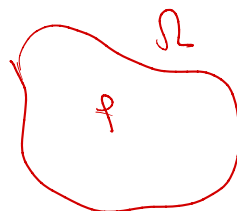
Rmk : Density

$$(H)_u(x) = \lim_{r \downarrow 0} \frac{1}{r^{n-2}} \int_{B_r(x)} |\nabla u|^2$$

Lemma $x \mapsto (H)_u(x)$ is upper semi-continuous.

$$x_i \rightarrow x$$

$$\overline{\lim} (H)_u(x) \in (H)_u(x)$$

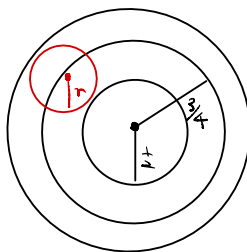


infinitely often
 $\text{dist}(x, \partial \Omega) |f| \leq C.$

"Contradiction Argument"

$$\bullet \forall x \in B_{3/4}, \forall r \in (0, \frac{1}{4})$$

$$\begin{aligned} \frac{1}{r^{n-2}} \int_{B_r^n(x)} |\nabla u|^2 &\leq \frac{1}{(\frac{1}{4})^{n-2}} \int_{B_{\frac{1}{4}}^n(x)} |\nabla u|^2 \\ &\leq \frac{1}{(\frac{1}{4})^{n-2}} \int_{B_1^n} |\nabla u|^2 \\ &\leq C \cdot \varepsilon. \end{aligned}$$



$$(\star) \quad \text{dist}^2(x, \partial B_{3/4}) |\nabla u|^2(x)$$

$$\text{good} \quad \begin{cases} x \in B_{1/2}(0) & \checkmark \\ x \in \partial B_{3/4}(0) & = 0 \end{cases}$$



$$\exists x_i \in B_{3/4}^n(0) \text{ s.t.}$$

$$d(x_i, \partial B_{3/4}^n)^2 |\nabla u|^2(x_i) = \sup_{B_{3/4}} (\sim)$$

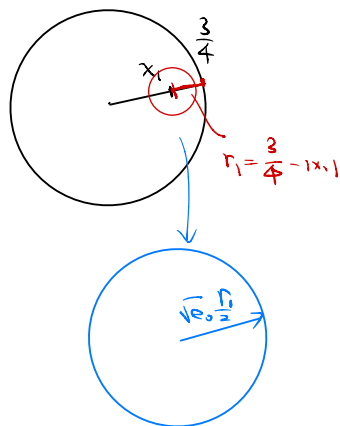
$$\geq d(x, \partial B_{3/4}^n)^2 |\nabla u|^2(x), \quad \forall x \in B_{3/4}^n$$

$$e_0 := |\nabla u|^2(x_i) \quad (\rightarrow +\infty)$$

Rescale $B_{r_1/2}^n(x_1) \longrightarrow B_{\frac{1}{\sqrt{e_0}} \frac{r_1}{2}}^n(0)$

$x \longmapsto \sqrt{e_0} (x - x_1)$

$v = u_{x_1, \sqrt{e_0}}(y) = u\left(x_1 + \frac{y}{\sqrt{e_0}}\right)$



(*) claim: $\sup_{B_{\frac{1}{\sqrt{e_0}} \frac{r_1}{2}}^n(0)} |\nabla v|^2 \leq 4$

Pf $\forall x \in B_{\frac{r_1}{2}}(x_1), \quad d(x, \partial B_{3/4})^2 |\nabla u|^2(x) \leq r_1^2 e_0$

$\sup_{x \in B_{\frac{r_1}{2}}(x_1)} d(x, \partial B_{3/4})^2 \geq \frac{r_1^2}{4}$

$\Rightarrow |\nabla u|^2(x) \leq 4e_0 \xrightarrow{\text{Rescale}}$

~

□

$\Rightarrow \therefore B_{\frac{1}{\sqrt{e_0}} \frac{r_1}{2}}^n(0)$

$\Delta \frac{1}{2} |\nabla v|^2 \geq -C_1 |\nabla v|^2$

① $\sqrt{e_0} \frac{r_1}{2} \geq 1$

D-N-M $\rightarrow I = |\nabla v|^2(0) \leq C \int |\nabla v|^2$

$= C \frac{1}{(\sqrt{e_0})^{n-2}} \int_{B_{\frac{1}{\sqrt{e_0}}}(x_1)} |\nabla u|^2 \leq C \cdot C' \cdot \varepsilon$

" $\Rightarrow \Leftarrow$ " Contradiction

$$\textcircled{2} \quad r_0 = \sqrt{e_0} \cdot \frac{r_1}{2} \leq 1$$

$$d(x, \partial B_{\frac{r_1}{4}})^2 u_1^2 \leq e_0 r_1^2 \leq 4, \quad \text{Actual.} \quad \checkmark$$

Rmk: ε -Regularity holds for stationary H.M.

Compactness

Cor . u_i : C^2 -H.M. from $\Omega \rightarrow \{N^m, h\}$

$$u_i \xrightarrow{W^{1,2}} u \quad \leftarrow \text{may be not smooth.}$$

The singular set $\mathcal{S}(u)$ has locally finite $\mathcal{H}^{n-2}$ -measure!

u is C^∞ on $\Omega \setminus \mathcal{S}(u)$

Pf . Claim $\mathcal{S}(u) = \{x \in \Omega : \exists r_0 > 0 \text{ s.t. } \forall r \in (0, r_0)$

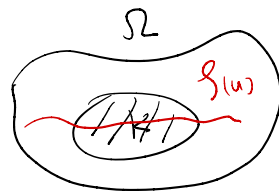
$$\downarrow \text{by Contradiction} \quad \lim_{i \rightarrow +\infty} \frac{1}{r^{n-2}} \int_{B_r^n(x)} |\nabla u_i|^2 \geq \varepsilon_0 \}$$

of ε -regular

• $\mathcal{S}(u)$ is relatively closed in Ω

Pf $\sim$ upper semi-cont of $\bigoplus_u^{n-2} \langle \cdot, \cdot \rangle$

• Estimate $H^{n-2}(\mathcal{S}(u) \cap K)$



pf: $\forall x \in \mathcal{S}(u) \cap K, \exists \delta > 0,$

s.t. $\forall r \in (0, \delta), \lim$

• cover $\mathcal{S}(u) \cap K$ by $B_{r_j}^n(x_j)$ s.t.

$\{B_{r_j}^n(x_j)\}$ pairwise disjoint.

$$\forall x_j \text{ for } j \gg 1, \quad \frac{1}{\left(\frac{r_j}{2}\right)^{n-2}} \int_{B_{r_j/2}^n} |\nabla u|^2 \geq \varepsilon.$$

$$\sum_j \left(r_j^{n-2}\right) \leq \frac{C}{\varepsilon} \sum_j \int_{B_{r_j/2}^n(x_j)} |\nabla u|^2$$

$$\downarrow$$

$$H^{n-2}(\mathcal{S}(u) \cap K)$$

$$\leq \frac{C}{\varepsilon} \int_{\Omega} |\nabla u|^2$$

• Regularity for energy Minimizing Maps

• Prop: $n \geq 3, \phi_0(x) = \frac{x}{|x|}, B^n \rightarrow \mathcal{S}^{n-1} = \partial B^n$
is energy min.

Thm $n \geq 3, u \in W^{1,2}(B^n, N)$
energy min. map

$\mathcal{S}(u)$ is closed

$$\begin{cases} n=3 & \mathcal{S}(u) \text{ discrete} \\ n>3 & \text{Hausdorff dim}(\mathcal{S}(u)) \leq n-3 \end{cases}$$

Sketch

0°) Monotonicity formula.



$$\frac{1}{t^{n-2}} \int_{B_t^n(p)} |\nabla u|^2 - \frac{1}{s^{n-2}} \int_{B_s^n(p)} |\nabla u|^2 = \int_{B_t^n \setminus B_s^n} \frac{|\partial_r u|^2}{|x-p|^{n-2}}$$

$$\partial_r = \nabla \frac{x-p}{|x-p|}$$

$$\mathcal{H}_u^{n-2}(p) = \lim_{t \downarrow 0} \frac{1}{t^{n-2}} \int_{B_t^n(p)} |\nabla u|^2 \quad \text{upper semi-continuous.}$$

$$\text{if } \exists r > 0, \quad \frac{1}{r^{n-2}} \int_{B_r^n(p)} |\nabla u|^2 = \mathcal{H}_u^{n-2}(p)$$

$$\Rightarrow \partial_r u = 0 \quad \text{for a.e. } x \in B_r^n(p)$$

$$u(y) = u\left(r, \frac{y-p}{|y-p|}\right)$$

1°) ε -regularity

$$\text{Thm If } \int_{B_1(0)} |\nabla u|^2 < \varepsilon_0$$

$$\Rightarrow u \in C^\infty(B_{1/2}(0)), \quad \sup_{B_{1/2}(0)} |\nabla u|^2(x) \leq C \int_{B_1} |\nabla u|^2$$

Gor. $f(u) = \{x \in \Omega, \quad \bigoplus_u^{n-2}(x) \geq \varepsilon_0\} = \{x: \bigoplus_u^{n-2}(x) > 0\}$

Rmk: $x \in \Omega \setminus f(u)$ regular pt.

$$\frac{1}{r^{n-2}} \int_{B_r^n(x)} \underbrace{|\nabla u|^2(y)}_{\text{bdd}} \leq \frac{1}{r^{n-2}} C \cdot r^n$$

$$\sim C r^2 \downarrow 0$$

Rmk: $I(p, r) = \frac{\frac{1}{r^{n-2}} \int_{B_p(r)} |\nabla u|^2}{\frac{1}{r^{n-1}} \int_{\partial B_p(r)} d(u(x), Q)^2 d\text{vol}}$

Almgren's

Frequency

$$= r \frac{\int_{B_p(r)} |\nabla u|^2}{\int_{\partial B_p(r)} d(u(x), Q)^2}$$

to do ε -regularity?

Step 2 Blow up Analysis / - E minimizing

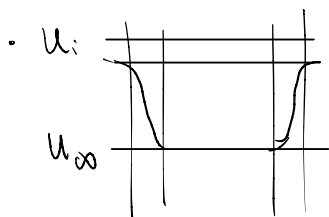
Lem: $u_i \in W^{1,2}(\Omega, N)$ E-min. maps

Stationary harmonic map is wrong.

$$u_i \xrightarrow{W^{1,2}} u_\infty \in W^{1,2}(\Omega, N)$$

$$\Rightarrow \begin{cases} \textcircled{1} u_\infty \text{ is "E-min"} \\ \textcircled{2} u_i \xrightarrow{W^{1,2}} u_\infty \end{cases}$$

$$\Rightarrow \liminf E(u_i) > E(u_\infty)$$



Def $x_0 \in \text{Sing}(u) \xrightarrow{W^{1,2}} u_{x_0, r_i}(x) = u(x_0 + r_i x) \rightarrow \phi \rightarrow \text{tangent map}$

Prop: ϕ satisfies

(1) ϕ is E -minimizing

$$(2) \quad 0 < \langle \psi |_{u(p)}^{n-2} = \langle \psi |_{\phi}^{n-2} = \frac{1}{r^{n-2}} \int_{B_r^n(o)} |\nabla \phi|^2$$

homogeneous degree = 0

(3) (upper semi-continuous)

$$\underset{\phi}{\text{argmax}} \, l_0^{n-2} = \max_{x \in \mathbb{R}^n} \underset{\phi}{\text{argmax}} \, l_1^{n-2}(x)$$

$$(4) \quad S(\phi) := \left\{ y \in \mathbb{R}^n : \begin{pmatrix} H \end{pmatrix}_{\phi}^{n-2}(y) = \begin{pmatrix} H \end{pmatrix}_{\phi}^{n-2}(0) \right\} \quad \mathbb{R}^n$$

is a linear subspace of $\mathbb{R}^n$

$$\phi(x+y) = \phi(y) + y \in S(\phi).$$

$\mathbb{R}^n$
 $\left\{ \begin{matrix} \textcircled{1} \\ \phi \end{matrix} \right\}^{n-2} (0)$
 $\text{span} \rightarrow$
 $\textcircled{1}^{n-2} (0)$
 ϕ
 $\vec{y}$
 $\vec{x}$
 $\vec{0}$
 不是 $\vec{y}$ 的 max. 而是 $\vec{0}$ 的方向

Pf of (4)

$$\left. \begin{array}{l} \text{by (2)} \\ \text{(3)} \end{array} \right\} \begin{array}{l} \phi(x) = \phi(\lambda x) \quad \forall \lambda > 0. \quad (1) \\ \phi(y + \lambda x) = \phi(y + x) \quad (2) \end{array}$$

$$\phi(x) \stackrel{(1)}{=} \phi(\lambda x) = \phi(y + (\lambda x - y))$$

$$\stackrel{(2)}{=} \phi(y + \frac{1}{\lambda^2}(\lambda x + y))$$

$$= \phi(y + \frac{x}{\lambda} - \frac{y}{\lambda^2})$$

$$\stackrel{(1)}{=} \phi(x + \underbrace{(\lambda - \frac{1}{\lambda})}_{\downarrow 1} y)$$

$$y, z \in S(\phi) \Rightarrow y + z \in S(\phi).$$

$$\phi(x + y + z) = \phi(x).$$

Remark : $S(\phi) \subset \text{Sng}(\phi) \Rightarrow \mathcal{H}^{n-2}(S(\phi)) = 0$
 $\Leftrightarrow \dim(S(\phi)) \leq n-3$

Claim : u is stationary H.M.

$$\Rightarrow \mathcal{H}^{n-2}(\text{Sng}(u)) = 0$$

$$\sum r_j^2 \leq C \int_{U_\delta(\text{Sng}(u))} |\nabla u|^2 \xrightarrow{\epsilon L^2} 0.$$

$$H^n \rightarrow 0 \text{ as } \delta \rightarrow 0$$

Def (Stratification) $\forall 0 \leq j \leq n-3$

$$\mathcal{S}_j(u) = \{ p \in \mathcal{S}_g(u) : \text{all tangent map } \phi \text{ of } u \text{ at } p, \dim(S(\phi)) \leq j \}$$

further, $\mathcal{S}_0(u) \subset \mathcal{S}_1(u) \subset \dots \subset \mathcal{S}_{n-3}(u)$.

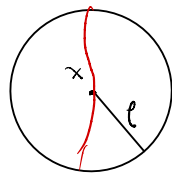
Step 3 Thm: $\forall 0 \leq j \leq n-3$.

$$\dim_{\mathcal{H}}(\mathcal{S}_j(u)) \leq j$$

Proof ① Claim: $x \in \mathcal{S}_j(u)$ $\forall \delta > 0, \exists \varepsilon = \varepsilon(x, u, \delta) > 0$,

s.t. $\forall \rho \in (0, \varepsilon]$

$$\eta_{x, \rho} \left\{ y \in B_\rho^n(x) : \left(\mathcal{H}_u^{n-2}(y) \right) \geq \left(\mathcal{H}_u^{n-2}(x) \right) - \varepsilon \right\}$$



$$\eta_{x, \rho} : y \mapsto \frac{y-x}{\rho}$$

$\subset \delta$ -nbhd of some
 j -dim plane $L_{x, \rho}$.

② Fix $\delta > 0, \forall i \in \mathbb{N}$

$$\mathcal{S}_{j,i} := \{ x \in \mathcal{S}_j(u) : \text{Claim holds for } \varepsilon = \frac{1}{i} \}$$

$$\mathcal{S}_j = \bigcup_{i \in \mathbb{N}} \mathcal{S}_{j,i}$$

$$\left(\mathcal{H}_u^n(x) + \frac{1}{i} \right) \geq \left(\mathcal{H}_u^n(y) \right) \geq \left(\mathcal{H}_u^n(x) \right) - \frac{1}{i}$$

$$j \in \mathbb{N}, \quad \mathcal{S}_{j,i,q} = \{x \in \mathcal{S}_{j,i} \mid \bigotimes_{u=1}^{n-2} (x) \in (\frac{q-1}{i}, \frac{q}{i}]\}$$

$$\mathcal{S}_j = \bigcup_{j,i \in \mathbb{N}} \mathcal{S}_{j,i,q}$$

$$\Rightarrow \forall x, y \in \mathcal{S}_{j,i,q},$$

$$\eta \left(\bigcup_{y \in \mathcal{S}_{j,i,q}} \left(\mathcal{S}_{j,i,q} \cap B_\epsilon(y) \right) \right) < \delta\text{-nbhd of } L_{y,p}$$

③ A set satisfy δ -approx property

$\mu^{\delta+\beta(\delta)}$ finite measure

$\beta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.



这样的 ϕ 不是一维的。

the tangent space 维数 - 2

更强。

Lecture 3

- $u: (B_1^n(x_0), g) \longrightarrow (N^m, h) \hookrightarrow \mathbb{R}^L$

expect: $e \Lambda^n \frac{\int_{B_1^n} |\nabla u|^2}{r^{n-2}} \uparrow$

• Energy minimizing V.S. Stationary

Step 2 $u_{x_0, r_i}(y) = u(x_0 + r_i y) \longrightarrow \phi$

• Geometric & Topological Application (2-d. H.M.)

- $u: D^2 \overset{C^0}{\longrightarrow} (M^n, h)$

1°: E is conformally invariant

$$E(u, g_2) = E(u \circ \phi, g_1)$$

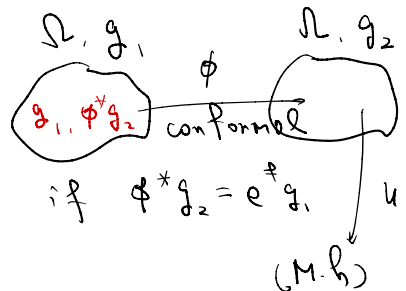
$$\parallel$$

$$E(u, f, f^* g_2)$$

$$E(u, g) \overset{\parallel}{=} \frac{1}{2} \int_{\Omega} g_2^{\alpha\beta} u_\alpha u_\beta \sqrt{\det g_2^{\alpha\beta}} dx^1 dx^2$$

$$\phi^* g_2 = e^f g_1 = \frac{1}{2} \int_{\Omega} e^{-f} g_1^{\alpha\beta} u_\alpha u_\beta e^f \sqrt{\det g_1^{\alpha\beta}}$$

$$= E(u, g_1)$$



$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Rmk: $E(u, [g])$, $u \in W^{1,2}$

$[g] \in \text{Moduli sp on } \Sigma_{g_0}$ or Teichmüller space on Σ_{g_0}

Con: $1^\circ \quad \phi: (\Sigma, g_1) \rightarrow (\Sigma, g_2)$ unq diff.

$u: (\Sigma, g_2) \rightarrow (M, h)$ is harmonic

$\Leftrightarrow u \circ \phi: (\Sigma, g_1) \rightarrow (M, h)$

Prop: $A(u) \leq E(u)$

"=" $\Leftrightarrow$ u is "weakly conformal" $u^*h = e^{\frac{2}{n}} g$

$$\text{Area}(u(\Omega)) = \int_{\Omega} \sqrt{\det(u^*h)} \, dx^1 dx^2 \sim \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\text{PF: } u^*h = \begin{pmatrix} h_{u(x)}(u_{x^1}, u_{x^1}), & h_{u(x)}(u_{x^1}, u_{x^2}) \\ h_{u(x)}(u_{x^2}, u_{x^1}), & h_{u(x)}(u_{x^2}, u_{x^2}) \end{pmatrix}$$

$$A = \sqrt{\det u^*h} = \sqrt{\lambda_1 \lambda_2}$$

$$E(u) = \frac{1}{2} \int |\nabla u|^2 d\text{vol}_g \quad \stackrel{=}{=} \quad dx^1 dx^2$$

$$= \int \frac{\text{tr}_g(u^*h)}{2} d\text{vol}_g$$

$$\stackrel{=}{=} \frac{\lambda_1 + \lambda_2}{2} \geq \sqrt{\lambda_1 \lambda_2}$$

Cor: $u: (\Sigma, g) \rightarrow (N, h)$

weakly conformally & harmonic
 退化点外是 conform.

$\Rightarrow u(\Sigma)$ is a "minimal" surface.

$$\left. \frac{d}{dt} \right|_{t=0} A(u_t) = 0$$

(pf: $\frac{A(u_t) - A(u)}{t} \leq \frac{E(u_t) - E(u)}{t}$)
 $\mathbb{R}^+$ ↗

Def: $u: (\Sigma, g) \rightarrow (N, h)$ map

• Hopf differential

$$\Phi(z) = \phi(z) dz^2$$

$$\phi(z) = |u_x|^2 - |u_y|^2 - 2\sqrt{-1} \langle u_x, u_y \rangle$$

$$z = x + iy \quad (\text{不依赖于复坐标选取}).$$

$$u^*h = \begin{pmatrix} |u_x|^2 & \langle u_x, u_y \rangle \\ - & |u_y|^2 \end{pmatrix}$$

Lemma: u is harmonic $\Rightarrow \Phi$ is holomorphic

pf

C-R equation.

↙ tangent vector

$$\langle \Delta u, u_x \rangle = 0$$

□

↓
 单位法向量方向

Lemma $u: \mathbb{S}^2 \rightarrow (N, h)$ harmonic
 $\Rightarrow u$ is weakly conformal

┌ $u: \sum_{g=1}^2 \rightarrow (N, h)$ H.M. ┐

Application 1 Schoen - Yau

Thm (π^3, h) $\text{Scal}_h \geq 0$

$\Rightarrow h \simeq$ flat metric

Pf: 0) Only need to show No such h exists

if $(*)_1 \text{Scal}_h \geq 0$ $\text{Scal}_h(p) > 0$ for some $p \in \pi^3$

1) $\exists f (*)_1 \checkmark \Rightarrow h \sim \hat{h}$

$\text{Scal}_{\hat{h}} > 0$ everywhere.

Goal $\Rightarrow " \Rightarrow "$

• $u: \pi^2 \rightarrow \pi^3$ incompressible

$u_*: \pi_1(\pi^2) \rightarrow \pi_1(\pi^3)$ inj

$\Sigma = \{ v \in \omega^{1,2}(\pi^2, \pi^3), v_*: \pi_1(\pi^2) \rightarrow \pi_1(\pi^3) = (u_0)_* \}$

- $E_0 = \inf \{ E(v) : v \in \mathcal{E} \}$

$$v_i \longrightarrow u_\infty \in W^{1,2}(\pi^2, \pi^3)$$

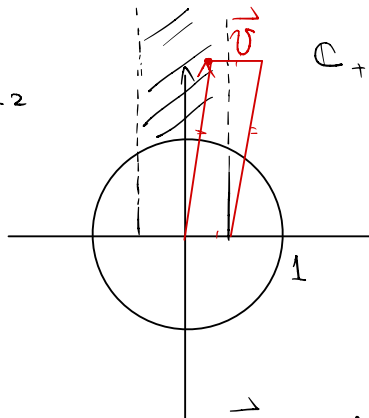
$$\Rightarrow u_\infty \in \mathcal{E}$$

$$\Rightarrow u_\infty \text{ is harmonic, } \pi^2 \rightarrow \pi^3$$

recall, $\delta^2 E_u(v, v) = \int_{\Omega} |\nabla v|^2 - R^N(v, e_\alpha, v, e_\alpha)$

- u_∞ "operation" conformal & Harmonic

Moduli space on π^2



- $\forall c \in M(\pi^2) \sim c \simeq \vec{v} \in \mathbb{H}_+^2$

$$\rightsquigarrow \text{harmonic } u_c : \pi_c^2 \longrightarrow (\pi^3, h)$$

$$E_c = E(u_c) = \inf \{ E(v, c) : v \in \mathcal{E} \}$$

Q: $\inf \{ E_c, c \in M(\pi^2) \}$

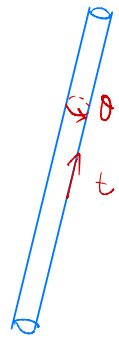
Claim: $\inf \{ E_c, c \in M(\pi^2) \}$ is achieved

pf: if not. $\exists c_j \uparrow \infty$

$$E(u) = \int_0^{L_c} \int_{S'_t} |\nabla u|^2 d\theta dt.$$

Sobolev
→
embedding

Length of $u(S'_{t_0}) \downarrow 0$



" $\Rightarrow \Leftarrow$ "

... $\exists U_{c_0} : \pi_{c_0}^2 \longrightarrow (\pi^3, h)$ conf H.M.

Culliver - Osserman $\Rightarrow U_0$ is immersion.

$\Sigma_0 = U_{c_0}(\pi_{c_0}^2) \hookrightarrow (\pi^3, h)$ minimal

$$S^2 \text{ Area}(f, f) = \int_{\Sigma_0} |\nabla f|^2 - (\text{Ric}^h(v, v) + |A^{\Sigma_0}|^2) f^2 d\Sigma$$

≥ 0

How to deal

$$\downarrow -\frac{1}{2} \int (R^N + |A|^2) f^2 d\Sigma$$

< 0

$$+ \frac{1}{2} \int R^2 f^2 d\Sigma$$

take " $f \equiv 1$ ". $S^2 \text{ Area}(f, f) < 0$. contradiction

Schoen - Yau



Gauss-Bonnet

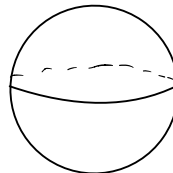
$$= \int_{\Sigma} K = 4\pi(1-g)$$

$= 0$.

$$-\text{Ric}^N(v, v) R^N = R^{\Sigma} + |A|^2 - |H|^2$$

$\Downarrow$

$$|A|^2 + \text{Ric}(v, v) = \frac{1}{2} (R^N - R^{\Sigma} + |A|^2)$$



$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = 2$$

- $u: \mathbb{S}^2 \rightarrow (N, h)$ H.M.

Sacks - Uhlenbeck

$$\text{Thm } \left. \begin{array}{l} \pi_k(N) \neq \{0\} \\ \Rightarrow \exists \text{ nontrivial harmonic} \end{array} \right\} \begin{array}{l} k=2 \\ k \geq 3 \end{array}$$

$$u: \mathbb{S}^2 \rightarrow (N, h)$$

$$k=2 \quad N = \mathbb{S}^2 \times \bar{N}$$

$$k=3 \quad N = (\mathbb{S}^k, h)$$

$$\mathbb{S}^k = \bigwedge_{k-2} \mathbb{S}^2 = [0,1]^{k-2} \times \mathbb{S}^2 / \sim$$

$$u: [0,1]^{k-2} \times \mathbb{S}^2 \rightarrow \mathbb{S}^k$$

- $\inf_{u \sim u_0} \max_{\bar{t} \in [0,1]^{k-2}} \mathbb{E}(u(\bar{t}, \cdot))$

Thm (Micallef - Moore)

$$(M^n, g) \quad \exists k: M \rightarrow (0, \infty) \text{ s.t.}$$

$$k(p) < \sec_g(p) \leq 4k(p)$$

- $\pi_1(M) = \{0\}$

$$\Rightarrow M \stackrel{\text{hom}}{\simeq} \mathbb{S}^n$$

Thm (M-M)

$$(M^n, g) \cdot \mathcal{Z}_1(M) = \{0\}.$$

• Positive Isotropic Curv. (PIC)

$$\Rightarrow M \stackrel{h_{0,0}}{\simeq} S^n$$

• 2nd variation of E

$$\delta^2 E_{u, S^2-N}(w, w) = \int_{S^2} |\nabla w|^2 - R^N(w, e_\alpha, w, e_\alpha)$$

$$w \in T(u^*TN)$$

$$\left\{ \begin{array}{l} \text{free} \end{array} \right.$$

$$= 8 \int_{\Sigma} |\nabla_{\frac{\partial}{\partial z}} w|^2 - \left\langle \left\langle R(w \wedge \frac{\partial w}{\partial z}), (w \wedge \frac{\partial w}{\partial z}) \right\rangle \right\rangle$$

$$\mathbb{E} = u^*TN \otimes \mathbb{C}$$

• PIC (M^n, g)

• Curvature operator R

$$R: \Lambda^2 T_p M \rightarrow \Lambda^2 T_p M$$

$$\langle R(x \wedge y), u \wedge v \rangle = R(x, y, u, v) \quad \forall x, y, u, v \in T_p M$$

$$\begin{array}{l} \cdot \left\{ \begin{array}{l} \langle \cdot, \cdot \rangle \\ \downarrow \\ T_p M \otimes \mathbb{C} \end{array} \right. \begin{array}{l} \xrightarrow{u \wedge v \quad x \wedge y} \left(\cdot, \cdot \right) \text{ complex bilinear} \\ \xrightarrow{x \wedge y} \langle \cdot, \cdot \rangle \text{ Hermitian} \end{array} \end{array}$$

- $\mathbb{R}$ cpl linear extention.

$$R: \Lambda^2 TM \otimes \mathbb{C} \longrightarrow \Lambda^2 TM \otimes \mathbb{C}$$

- cpl sectional cur.

$$K(\sigma) = \frac{\langle R(Z \wedge W), Z \wedge W \rangle}{\|Z \wedge W\|^2}$$

$$\sigma = \{\sigma, w\} \subset \underset{\text{2-plane}}{T_p M \otimes \mathbb{C}}$$

- $Z \in T_p M \otimes \mathbb{C}$ is "Isotropic" if $(Z, Z) = 0$

- $V \subset T_p M \otimes \mathbb{C}$ is "Is₋" $(Z, Z) = 0, \forall Z \in V$.

$T_p M$ on basis $\{e_i\}$

$$Z = e_1 + \sqrt{-1}e_2 \quad \rightsquigarrow (Z, Z) = |e_1|^2 - |e_2|^2 - 2\sqrt{-1}\langle e_1, e_2 \rangle = 0$$

- Is₋ $V \subset T_p M \otimes \mathbb{C}$

$$\dim V \leq \lceil \frac{n}{2} \rceil$$

Def g has PIC if

$$K(\sigma) > 0, \quad \forall \text{ Isotropic 2-plane } \sigma \subset T_p M \otimes \mathbb{C}$$

Rmk: ① $n \geq 4$

② "PIC" $\sim$ "stable and H.M."

"~" "Scal" ~ "stable min. hypersuf."

• Computation

$T_p M$ o.n. $\{e_i\}$.

σ . Iso. $\{Z, W\}$. $Z = e_1 + \sqrt{-1}e_2$, $W = e_3 + \sqrt{-1}e_4$

$$\langle\langle R(Z \wedge W), Z \wedge W \rangle\rangle$$

$$= R_{1313} + R_{2424} + R_{1414} + R_{2323}$$

$$\underline{-2R_{1324} + 2R_{1423}}$$

// Bianchi's identity.

$$-2R_{1234}.$$

Section cur.

Cor $R > 0 \Rightarrow PIC$

Prop: $\frac{1}{4}$ -pinching $\Rightarrow PIC$

Pf: "Berger" identity

$$\text{If } R^M \in [a, b] \Rightarrow |R_{1234}| \leq \frac{2}{3}(b-a)$$

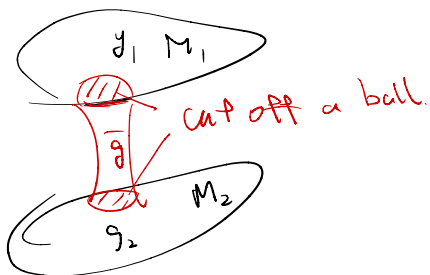
$$\langle\langle R(Z \wedge W), Z \wedge W \rangle\rangle > 4K(p) - \frac{2 \times 2}{3}(4K - K) = 0$$

• $S^{n-1} \times S^1$ PIC.

• $(M_1, g_1) (M_2, g_2)$ PIC.

$M_1 \# M_2$ PIC.

product, $M_1 \times M_2 \neq$ PIC.



$S^4 \times S^4$, contradiction.

$f: \Sigma^2 \rightarrow M^n \quad C^\infty$

$TE = (f^*TM) \otimes \mathbb{C} \quad TM \otimes \mathbb{C} \quad (\dots)$
 $\downarrow \quad \downarrow$
 $\Sigma \longrightarrow M$
 $\ll \dots \gg$

cpl rank = n

$\nabla: \sim$ cpl linear

$A^{p,q}(E)$ (p,q) -form with volume in E on Σ

$A^{1,0}(E) = \left\{ dz \otimes f_* \frac{\partial}{\partial x^i} \right\}$

$A^{0,1}(E) = \left\{ d\bar{z} \otimes f_* \frac{\partial}{\partial x^i} \right\}$

$\nabla \sim \nabla': T(E) \rightarrow A^{1,0}(E)$

$\nabla'': T(E) \rightarrow A^{0,1}(E)$

Claim : $\exists!$ holomorphic structure on E

$$\text{s.t. } \nabla'' = \partial_{\bar{z}}$$

$$\Leftrightarrow W \in \Gamma(E) \text{ holomorphic } (\nabla_{\partial \bar{z}} W = 0)$$

$$\Leftrightarrow \nabla'' W = 0 \quad / \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

Lemma f is harmonic $\Rightarrow \nabla_{\partial \bar{z}} \frac{\partial f}{\partial \bar{z}} = 0$

$$f \text{ is conformal} \Rightarrow \frac{\partial f}{\partial \bar{z}} \text{ is Isotropic}$$

$$\left(\frac{\partial f}{\partial \bar{z}}, \frac{\partial f}{\partial \bar{z}} \right) = 0$$

pf $\nabla_{\frac{\partial f}{\partial \bar{z}}} \frac{\partial f}{\partial \bar{z}} = \frac{1}{4} (\Delta f)^T = 0.$

pf of $\int_{\Sigma} E_f(w, w)$

$$\int_{\Sigma} |\nabla_{\partial \bar{z}} w|^2 = \frac{1}{4} \int_{\Sigma} (|\nabla_x w|^2 + |\nabla_y w|^2) \quad \begin{array}{l} -\sqrt{-1} \langle \partial_x w, \partial_y w \rangle \\ +\sqrt{-1} \langle \partial_y w, \partial_x w \rangle \end{array}$$

$$\longrightarrow -\sqrt{-1} \int_{\Sigma} \langle R(\partial_x \partial_y) w, w \rangle.$$

$$\int_{\Sigma} E_f(w, w) = 2 \int_{\Sigma} |w|^2 - R(w, \partial_x, w, \partial_x) - R(w, \partial_y, w, \partial_y)$$

$$I(w, w) := S^2 E_f(w, w) = 8 \int_{\Sigma} |\nabla_{\partial \bar{\partial}} w|^2 \ll R(w, \frac{\partial f}{\partial \bar{z}}), w, \frac{\partial f}{\partial \bar{z}} \gg$$

$$\underline{w = \Gamma(E)}$$

• $f: \Sigma \rightarrow (N, h)$ conformal & Harmonic

$\Rightarrow \frac{\partial f}{\partial \bar{z}} \begin{cases} \text{holo} & \Gamma(E) \\ \text{isotropic} \end{cases}$

Def: • Index of $f := \max \dim$ of $P \subset \Gamma(E)$ s.t. $I|_P < 0$.

• Jacobi field of $f: w \in \Gamma(E), I(w, v) = 0 \quad \forall v \in \Gamma(E)$

— Rmk: $\frac{\partial f}{\partial \bar{z}}$ Jacobi field.

closed • Nullity of $f = \dim$ of P of Jacobi field.

$\text{thm}(M-M) (M^n, g)$ PIC + $\pi_1(M) = 0 \Rightarrow \pi_k(M) = 0$
 $\Rightarrow$ Any Harmonic $f: S^2 \rightarrow (M^n, g)$ $2 \leq k \leq n-1$

has index $\geq \frac{n}{2} - \frac{3}{2}$

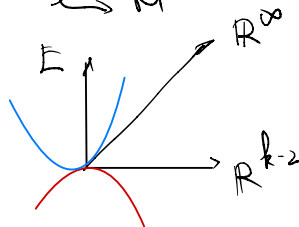
" $\Rightarrow$ Sphere thm"

$$M^n, \pi_k(M) = 0 \iff S^k \hookrightarrow M^n$$

$$[S-U] : u_0: [0, 1]^{k-2} \times S^2 \longrightarrow S^k \hookrightarrow M^n$$

$\downarrow$ min-max

$f: S^2 \rightarrow M$ Harmonic
 $\text{ind}(f) \leq k-2$



$$\Rightarrow k-2 \geq \frac{n}{2} - \frac{3}{2} \Rightarrow k \geq \frac{n}{2} + \frac{1}{2}$$

pf: $E = T^*(M) \otimes \mathbb{C}$ cplx-rank: n

$$\downarrow$$

$$\Sigma \simeq S^2$$

① Groth... thm

$$\underline{L_1 \oplus \dots \oplus L_p \oplus L_{p+1} \oplus \dots \oplus L_r \oplus L_{r+1} \oplus \dots \oplus L_n}$$

$$C_1(L_i) \left\{ \begin{array}{l} > 0 \quad 1 \leq i \leq p \\ = 0 \quad p+1 \leq i \leq r \\ < 0 \quad r+1 \leq i \leq n. \end{array} \right. \downarrow S^2$$

② $\forall 1 \leq i \leq r \exists$ hol. section $S_i \in T(L_i)$ (R-R)

$$\partial_{\bar{z}}(S_i, S_j) = (\partial_{\bar{z}} S_i, S_j) + (S_i, \partial_{\bar{z}} S_j) = 0$$

if $1 \leq i \leq p \Rightarrow S_i$ has at least 1 zero

$$\Rightarrow (S_i, S_j) \equiv 0$$

$$\text{span} \{S_1, \dots, S_p\} \subset \text{span} \{S_1, \dots, S_r\}^\perp$$

$$\Rightarrow p \leq n-r$$

$\forall p+1 \leq i \leq n, \exists$ meromorphic section S_i

(S_i, S_j) meromorphic func.

By R-R. $r+1 \leq i \leq n$ S_i has ≥ 1 pole.

$$\Rightarrow (S_i, S_j) = 0 \quad \forall r+1 \leq j \leq n, \quad p+1 \leq i \leq n$$

$$\Rightarrow \text{span} \{ S_{r+1}, \dots, S_n \} \subset \text{span} \{ S_{p+1}, \dots, S_n \}$$

$$\Rightarrow n-r \leq p$$

$$\Rightarrow p = n-r$$

$$\Rightarrow \bullet \frac{\partial f}{\partial z} \in \text{Span} \{ S_1, \dots, S_p \} \text{ is Isotropic.}$$

$$\bullet S_{p+1}, \dots, S_r$$

$$\bar{e}_1 = S_{p+1} + \sqrt{-1} S_{p+2}$$

$$\bar{e}_2 = S_{p+3} + \sqrt{-1} S_{p+4}$$

⋮

$$\bar{e}_{\lceil \frac{r-p}{2} \rceil}$$

$$\Rightarrow p + \lceil \frac{r-p}{2} \rceil = p + \lceil \frac{n-2p}{2} \rceil = \left\lceil \frac{n}{2} \right\rceil - 1$$

□

Back to S-U.

$$\gamma: S^2 \rightarrow \mathbb{R}^L$$

$$u \in W^{1,2\alpha}(\Sigma, N) \quad \alpha > 1. \quad + \gamma \cdot u$$

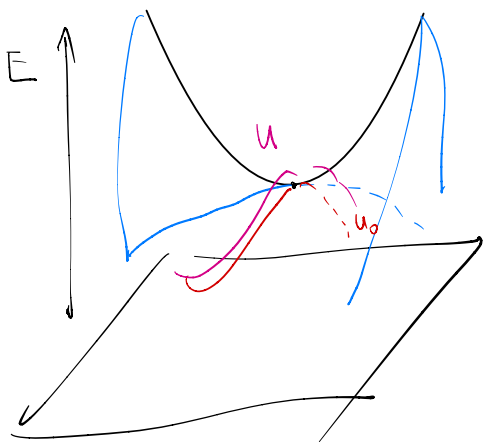
$$E_\alpha(u) = \int \left[(1 + |\nabla u|^2)^\alpha - 1 \right] d\text{vol}_g$$

1° $E_\alpha(u)$ satisfies Palais-Smale.

$E: M_{\text{Banach}} \rightarrow [0, \infty)$ satisfies P.S. and if

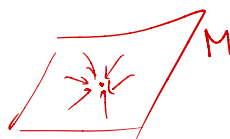
$$\forall \{u_i\} \begin{cases} E(u_i) \leq C \\ \|DE(u_i)\| \rightarrow 0 \end{cases}$$

$$\Rightarrow u_i \rightarrow u_\infty \in M \quad \& \quad DE(u_\infty) = 0.$$



$$H = \inf_{u \sim [u_0]} \max_{t \in \bar{X}} E(u(t)) > 0.$$

$$\Rightarrow H = E(u_0) \quad u_0 \in M, DE(u_0) = 0$$



2° $\Rightarrow \exists \alpha$ -harmonic map

$$u_\alpha: \Sigma_g^2 \rightarrow (N, h)$$

$$E_\alpha - \text{Euler-Lagrange eqn. } 0 = \Delta u + (\alpha-1) \frac{\text{Hess}(\nabla u, \nabla u)}{(1 + |\nabla u|^2)^{\alpha-1}} + A(u)(u, u)$$

Thm. u_α is C^∞

3° $\alpha_i \rightarrow 1$, $u_i := u_{\alpha_i}$

A priori Estimates

$$1^\circ \exists \varepsilon = \varepsilon(N, h) > 0, \text{ s.t. } E(u, D) < \varepsilon$$

$$\Rightarrow \|\nabla u\|_{W^{1,p}(D_{\frac{1}{2}})} \leq C E(u), \quad p > 1$$

2° either $u_\alpha = \text{const}$ or $E_\alpha(u_\alpha) \geq \varepsilon$.

4° Bubbling - Analysis. $\nearrow$ radon measure.

$$\|\nabla u_i\|^2 d\text{vol}_g \rightarrow \nu \in \mathcal{R}(\mathbb{S}^2)$$

$$\forall p \in \mathbb{S}^2 \text{ case ① } \nu(\{p\}) < \varepsilon_0.$$

$$\Rightarrow E(u_i, D_{r_i}(p)) < \varepsilon_0.$$

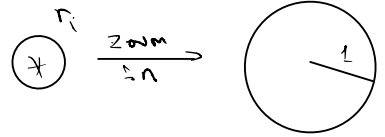
$$\Rightarrow u_i' \rightarrow u_\infty$$

$$\Rightarrow u_\infty \in C^\infty(\Sigma \setminus \mathcal{S}). \quad \mathcal{S} = \{p \in \Sigma, \nu(\{p\}) \geq \varepsilon_0\}.$$

$$\star \#(\mathcal{S}) < +\infty$$

Case ② $p \in \Sigma, \nu(\{p\}) \geq \varepsilon_0$

$$E(u_i, D_{r_i}(p)) \geq \varepsilon_0, \quad r_i \downarrow 0.$$



$$\bar{u}_i(y) = u_i(p + r_i y) : \Omega_{r_i} \rightarrow N.$$

$$E(\bar{u}_i, D_i) = \varepsilon_0 \quad \downarrow \quad \mathbb{R}^2 = \mathbb{C}$$

$$0 \rightarrow (N, h) \quad E(u_\infty) < +\infty$$

$$\Rightarrow \bar{u}_i \rightarrow u_\infty \text{ nontrivial.}$$

$$u_\infty : \mathbb{S}^2 \setminus \{+\infty\} \rightarrow (N, h)$$

Thm $u_\infty : \mathbb{S}^2 \rightarrow (N, h) \in C^\infty$

$$\begin{array}{ccccccc} \pi_1 = 0 & \pi_2 = 0 & \pi_k = 0 \\ \downarrow & \downarrow & \downarrow \\ H_1 = 0 & H_2 = 0 & \dots & H_k = 0 \\ & \searrow & & \swarrow \\ & H_1 \otimes h_{\text{hor}} & \dots & H_k \otimes h_{\text{hor}} \end{array}$$