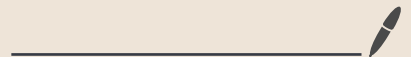


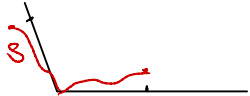
Minimal Cones & AC

Minimal Submanifold.



Lecture 1.

小扰动.

Q: $C^n \subset \mathbb{R}^{n+1}$, $C \cap S^n =: S$. C^∞ 

For small perturb S' of S , what does the M_z hypersurf.

M w body S' looks like?

Is C still tangent cone of M ?

• A. No, Yes " modulo finite dim perturb "

• Suppose $M = \text{graph}_C(u) = \{ x + u(x) \cdot \nu_C(x) \mid x \in C \cap B_1 \}$

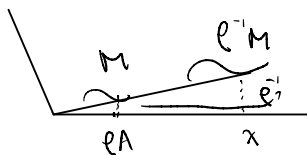
$u \in C^\infty(C \cap B_1)$
is a minimal hypersurf. $\cup$ tangent cone at 0 being C

Then ① $0 = H_M = \underbrace{(\Delta_C + |A_C|^2)}_{:= L_C} u + \text{high order term}$
"Jacobi operator"

② $p^{-1} \cdot M \xrightarrow{p \rightarrow 0} C \Leftrightarrow p^{-1} u^{(p)} \xrightarrow{p \rightarrow 0} 0$

Fact: $p^{-1} M = \text{graph}_C(p^{-1} \cdot u^{(p)})$,

where $u^{(p)} := u(px)$



• Thm 1 (Caffarelli - Hwang - Simon, 1984)

Suppose $\tau \in (\frac{2-n}{2}, +\infty) \setminus \Gamma_C$

• $\exists$ finite dim subspace $W_\tau \subset C^\infty(S) \subset L^2(S)$, s.t.

$$L_{C,\tau} : C_\tau^{2,\alpha}(\underbrace{C \cap B_1}_{C_1}) \longrightarrow C_{\tau-2}^k(C_1) \times (C^{2,\alpha}(S) \cap W_\tau^\perp)$$

$C \neq \emptyset$

is an isomorphism. $u \longmapsto (L_C u, \pi_{W_\tau^\perp}(u|_S))$

(边值不能随意取)

where $\Gamma_C := \{ \gamma \mid \exists \psi \in C^\infty(C), \text{ s.t. } L_C \psi = 0, \}$

$\{ \psi(r,x) = e^{\gamma \psi(x)} \} = \text{"deg of homog Jac field on } C \text{"}$

$$\|u\|_{C_\tau^{k,\alpha}(C_1)} = \sup_{\rho \in (0,1]} \rho^{-\tau} \|u^{(\rho)}\|_{C^{k,\alpha}(\underbrace{A_{1/2,1}}_{\{x \mid 1/2 \leq |x| \leq 1\}} \cap C)}$$

$$C_\tau^{k,\alpha}(C_1) = \{ u \mid \|u\|_{C_\tau^{k,\alpha}} < +\infty \}$$

Rem ① $\|u\|_{C_\tau^{k,\alpha}} \leq K \iff \|u^{(\rho)}\|_{C_\tau^{k,\alpha}(A_{1/2,1})} \leq K \rho^\tau$

" $\iff$ " $|u(r\theta)| \lesssim r^\tau$ in $C^{k,\alpha}$ sense.

② $\|u\|_{C_\tau^k} \sim \sup_{C_1} \rho^{-\tau} (|u| + r \cdot |\nabla u| + r^2 |\nabla^2 u| + \dots + r^k |\nabla^k u|)$

③ If $\tau = 1 + \varepsilon$, $\varepsilon \ll 1$, then

$$\dim W_\tau = \text{ind}(S) = \sup_{V \subset C^\infty(S)} \dim V \quad \leftarrow \text{call}$$

$$\delta^2 |S|(x, x) = 0 \quad \text{"Morse index of } S"$$

$$\forall x \in V \setminus \{0\}$$

Cor $\forall \tau \in (1, +\infty) \setminus T_C$, $\exists \delta(\tau, C) > 0$ s.t. $\forall \varphi \in C^\infty(S)$

$$\|\varphi\|_{C^{2,\alpha}} \leq \delta, \exists \text{ minimal hypersurface } M \subset \mathbb{R}^{n+1},$$

$$\text{s.t. } \partial M = \text{graph}_S(\tilde{\varphi}), \quad \pi_{W_\tau}^\perp(\tilde{\varphi} - \varphi) = 0$$

and tangent cone of M at o is C .

PF: $\forall u \in C_\tau^{2,\alpha}(C_1)$ hope: $0 = H_M = L_C u + Q[u]$
 $(M: \text{graph}_C(u))$

Claim: $\|Q[u_1] - Q[u_2]\|_{C_{\tau-2}^\alpha(C_1)} \stackrel{(\circledast)}{\leq} C(\tau, C) (\|u_1\|_{C_\tau^{2,\alpha}} + \|u_2\|_{C_\tau^{2,\alpha}})$

$\bullet Q[0] = 0$

$\leq C(\tau, C)$

$\bullet \|u_1 - u_2\|_{C_\tau^{2,\alpha}}$

Assuming the claim, $L_{C,\tau}^{-1}(-Q[u], \varphi) \in C_\tau^{2,\alpha}(C_1)$

$T: u \mapsto$

Hope: T is contraction.

Let $K > 0$ TBD

$$1) \forall v_1, v_2 \in C_\tau^{2,\alpha}(C_1), \text{ s.t. } \|v_i\|_{C_\tau^{2,\alpha}} \leq K$$

$$\begin{aligned} \|\mathcal{T}(v_2) - \mathcal{T}(v_1)\|_{C_\tau^{2,\alpha}} &= \|L_{C,\tau}^{-1}(\mathcal{Q}[v_2] - \mathcal{Q}[v_1])\|_{C_\tau^{2,\alpha}} \\ &\leq \overline{C}_{C,\tau} \|L_{C,\tau}^{-1}\|_{\text{op}} \cdot (\|v_1\|_{C_\tau^{2,\alpha}} + \|v_2\|_{C_\tau^{2,\alpha}}) \\ &\quad \cdot \|v_2 - v_1\|_{C_\tau^{2,\alpha}} \\ &\leq \overline{C}_{C,\tau} K \|v_2 - v_1\|_{C_\tau^{2,\alpha}} \end{aligned}$$

take $\overline{C}_{C,\tau} K \leq \frac{1}{2}$
 $\Rightarrow \mathcal{T}$ is contraction

$$2) \mathcal{T}(\overline{B}_K^{C_\tau^{2,\alpha}(C_1)}) \subset \overline{B}_K^{C_\tau^{2,\alpha}}, \quad \forall \|v\|_{C_\tau^{2,\alpha}} \leq K$$

$$\begin{aligned} \|\mathcal{T}(v)\|_{C_\tau^{2,\alpha}} &= \|L_{C,\tau}^{-1}(\mathcal{Q}[v], \varphi)\| \\ &\leq \|L_{C,\tau}^{-1}\|_{\text{op}} \left(\underbrace{\|\mathcal{Q}[v]\|_{C_{\tau-2}^\alpha}}_{\lesssim \|v\|_{C_\tau^{2,\alpha}}^2} + \underbrace{\|\varphi\|_{C_\tau^{2,\alpha}}}_{\delta} \right) \\ &\leq \overline{C}'(K^2 + \delta) < K \end{aligned}$$

Take $\overline{C}'K < \frac{1}{2}$.

$$\overline{C}'\delta < \frac{1}{2}K$$

Banach contraction Thm

$$\Rightarrow \exists! v \in C^2_\tau(C_1), \text{ s.t. } \begin{cases} H_M = 0 \\ \pi_{W_S}(v|_S - \varphi) = 0 \end{cases}$$

Pf of Claim $\cdot H_{p^{-1}M}|_x \stackrel{E_x}{=} p \cdot H_M|_{p_x} = p(L_C u + Q[u])^{(p)}$
 $\Delta_C + |A_C|^2 \quad //$

$$L_C(p^{-1}u^{(p)}) + Q[p^{-1}u^{(p)}]$$

Expand & compare $\Rightarrow Q[p^{-1}u^{(p)}] = p \cdot Q[u]^{(p)}$

$$\|Q[u_1]^{(p)} - Q[u_2]^{(p)}\|_{C^\alpha(A_{\frac{1}{2},1})} = p^{-1} \cdot \|Q[p^{-1}u_1^{(p)}] - Q[p^{-1}u_2^{(p)}]\|_{C^\alpha(A_1)}$$

$$N \subseteq \mathbb{R}^{m+1}, |A_N| \leq 1$$

$$H_M = L_N u + Q_N[x, u, \partial u, \partial^2 u]$$

$$\|Q_N[u]\|_{C^{k,\alpha}} \leq \|u\|_{C^{k+2,\alpha}}$$

$$\lesssim_C p^{-1} \cdot (p^{-1} \|u_1^{(p)}\|_{C^{2,\alpha}(A_{\frac{1}{2},1})} + \dots)$$

$$\cdot \|p^{-1}(u_1^{(p)} - u_2^{(p)})\|_{C^{2,\alpha}(A_{\frac{1}{2},1})}$$

$$\lesssim_C p^{-3} e^{2\tau} \cdot \kappa$$

$$\sup_{t \in [0,1]} p^{-(t-2)} \dots \leq \dots$$

□

Rmk: $M = \text{graph}_N(u)$; Let (x^i) be normal coord.

$$\{\partial_i(x + u(x) \nu_N(x))\}_i \text{ spans tangent bundle of } M$$

$$= \partial_i + \partial_i u \cdot \nu_N + u \cdot A_N(\partial_i, \partial_j) \partial_k g^{jk} = \partial_i^M$$

$$\langle \nabla_{\partial_i}, \partial_j^M \cdot \nu_M \rangle = g_{ij}^M$$

$$\nu_M = a_0 \nu_N + \sum a_i \partial_i$$

Jacobi Operator on Cones Parametric C by $S \times (0, \infty)$
 $rg \mapsto (0, r)$

$$L_C = \Delta_C + |A_C|^2 \Big|_{rg} \stackrel{E_x}{=} \partial_r^2 + \frac{n-1}{r} \partial_r + \underbrace{\frac{1}{r^2} (\Delta_S + |A_S|^2)}_{\mathcal{L}_S}$$

Rem. In general, $\Sigma^n \subset (M^{n+1}, g)$ minimal,

$$H_{\text{graph}_\Sigma}(u) = \underbrace{(\Delta_\Sigma + |A_\Sigma|^2 + \text{Ric}_M(\nu_\Sigma, \nu_\Sigma))}_{L_\Sigma} u + \mathcal{Q}_\Sigma[u]$$

L_Σ : Jac operator of Σ

$$\begin{aligned} \delta^2 |_\Sigma (\varphi, \varphi) &= \int_\Sigma |\nabla \varphi|^2 - (|A_\Sigma|^2 + \text{Ric}(\nu, \nu)) \varphi^2 \\ &= \int_\Sigma -\varphi L_\Sigma \varphi \end{aligned}$$

$$\mathcal{L}_S = L_S - (n-1)$$

- Let $\lambda_1 < \lambda_2 \leq \dots \nearrow +\infty$ be eigenvalues of $-L_S$,
 $\varphi_1, \varphi_2, \dots$ be L^2 -o.n. eigenfunctions of $-L_S$
 (i.e. $-L_S \varphi_i = \lambda_i \varphi_i$)

Then $\forall r \in C^2(C), \quad V(0) = \sum_{j=1}^{\infty} a_j(r) \cdot \varphi_j(0)$

$$(a_j(r) = \int_S v(r\theta) \varphi_j(\theta) d\theta)$$

$$L_C V = f \Leftrightarrow \underbrace{(D^2 + \frac{n-1}{r} D - \frac{\lambda_j}{r^2})}_{\mathcal{L}_{\lambda_j}} v_j(r) = f_j(r)$$

$$\sum_{j=1}^{\infty} (D^2 + \frac{n-1}{r} D - \frac{\lambda_j}{r^2}) v_j(r) \varphi_j$$

Rmk $\mathcal{L}_{\lambda} w = f$, general sol is

$$w(r) = c_+ r^{\gamma^+} + c_- r^{\gamma^-} + r^{\gamma^+} \int_r^1 f(s) \frac{s^{1-\gamma^+}}{\gamma^- - \gamma^+} ds$$

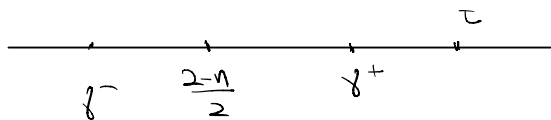
where $\gamma^{\pm}(\lambda) = \frac{2-n}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 - \lambda_j}$, $\lambda_j > -\left(\frac{n-2}{2}\right)^2$

<
=

• If $\lambda = -\left(\frac{n-2}{2}\right)^2$, first two terms are replaced by

$$r^{\frac{2-n}{2}} (c_+ + c_- \log r), \quad \text{third term} \sim$$

$\lambda_j \geq -\left(\frac{n-2}{2}\right)^2$ minimal case
stable



Lemma Suppose. $\tau > \frac{2-n}{2}$. If $\|f\| \leq K \cdot r^{\tau-2}$, then

i) If $\gamma^+(\lambda) > \tau$, then $\forall b \in \mathbb{R}, \exists!$ sol to $\begin{cases} \mathcal{L}_\lambda w = f \\ w(1) = b \end{cases}$

$$|w(r)| \lesssim_c \left(\frac{K}{|\gamma^+ - \gamma|} + |b| \right) r^\tau \text{ on } (0, 1]$$

ii) If $\gamma^+(\lambda) < \tau$ or $\lambda \leq -(\frac{n-2}{2})^2$, then $\exists!$ sol to

$$\mathcal{L}_\lambda w = f \quad \& \quad |w(r)| = O(r^\tau)$$

$$|w(r)| \lesssim K \cdot r^\tau$$

解空间

(decay 的速度与 r^τ 相比, 快的话, 加速值于解自由度)

Recall (Thm 1) $\begin{cases} \mathcal{L}u = f & (f \in C_{\tau-2}^\alpha) \\ \pi_{W_\tau}^\perp u = \pi_{W_\tau}^\perp \varphi \end{cases} \quad (*)$

对应一些 b

pf of Thm 1 Hope find unique sol. to

Define $W_\tau = \bigoplus_{\substack{\text{either } \lambda \leq -(\frac{n-2}{2})^2 \\ \text{or } \gamma^+(\lambda) < \tau}} W_\lambda (\mathcal{L}_\lambda)$

eigenspace of $\mathcal{L}_\lambda$ w eig. value λ

$\forall j$ s.t. $\varphi_j \notin W_\tau$, let $U_j(r)$ be the unique sol of

$$\begin{cases} \mathcal{L}_{\lambda_j} U_j = f_j(r) \\ U_j(1) = \gamma_j \quad (\varphi(1) = \sum \gamma_j \varphi_j(1)) \end{cases}$$

$\forall j$ s.t. $q_j \in W_\tau$. let $v_j(r)$ be the unique sol. of

$$L_{\lambda_j} v_j = f_j, \quad |v_j(r)| = O(r^\tau)$$

$$\|v(r, \cdot)\|_{L^2}^2 = \sum v_j(r)^2 \leq \left(\sum K_j^2 \right) \cdot \|q\|_{L^2}^2 r^\tau \leq \underbrace{\left(\sup r^{2-\tau} \|f(r, \cdot)\|_{L^2}^2 \right)}_{+\|q\|_{L^2}^{2\tau}}$$

$$|f_j|(r) = \left| \int_S f(r, \theta) \varphi_j(\theta) \right| = K_j \cdot r^{\tau-2}$$

$$\lesssim \underbrace{\left(\|f\|_{C_{2-\tau}^\alpha} + \|q\|_{C^{2,\alpha}} \right)^2}_{K} \cdot r^\tau$$

$\Rightarrow v$ solves (*) in weak sense.

$$v^{(p)} \text{ solves } L_C v^{(p)} = \rho^2 (L_C v)^{(p)} = \rho^2 f^{(p)}$$

by elliptic est.

$$\xrightarrow{\text{by elliptic est.}} \|v^{(p)}\|_{C^{2,\alpha}(A_{\frac{1}{2},1})} \leq \|v^{(p)}\|_{L^2(A_{\frac{1}{4},2})} + \|\rho^2 f^{(p)}\|_{C^\alpha(A_{\frac{1}{4},2})}$$

$$\lesssim K \cdot \rho^\tau$$

$$\Rightarrow \|v\|_{C_{\tau}^{2,\alpha}} \lesssim K$$

□

Thm 1' Suppose $\tau \in (\frac{2-\eta}{2}, +\infty)$. Then $\exists$ finite dimension subspace $W_\tau' \subset C^\infty(S)$

$$\text{s.t. } L_{C,\tau}^{-1} : C_{\tau}^{2,\alpha}(C \setminus B_1) \longrightarrow C_{\tau-2}^\alpha(C \setminus B_1) \times C^{2,\alpha}(S) \times W_\tau'$$

$$u \longmapsto (L_C u, \quad u|_S, \quad \underbrace{\|W_\tau'(\partial_r u)|_S\|}_{\text{norm}})$$

is an isomorphism.

L^2 o.n. proj.
onto W_τ

take Lem (1). $\mathcal{D}_\tau \rightarrow \mathcal{D}_\tau^*$ $W(u)=b$.

Rmk: $\exists f \quad u \in C_\tau^{2,\alpha}(C \setminus B_1) \quad \tau=1-\varepsilon$

$$P^{-1} u^{(P)} \xrightarrow{P \rightarrow \infty} 0$$

$$P^{-1} \cdot \text{graph}_C(u) \xrightarrow{P \rightarrow \infty} C$$

Correction $W_{1+\varepsilon}$ in thm 1

has $\dim = \text{ind}(S) + \dim \ker L_S$

$$\Delta + |A_S|^2 + \text{Ric}(v, v)$$

$$C_\tau^{2,\alpha} \rightarrow C_{\tau-2}^\alpha$$

NOT Fredholm if $\tau \in \Gamma_C$.

Proposition

① $L_C u = 0$. if $u(r\theta) = r^\gamma \varphi(\theta) \Rightarrow -(\Delta_S + |A_S|^2) \varphi(\theta) = \lambda \varphi(\theta)$

$$\gamma(\gamma + n - 2) - \lambda = 0$$

$$\gamma \in \{ \gamma^\pm(\lambda) \}$$

$$\Rightarrow \Gamma_C = \{ \gamma^\pm(\lambda_j) \mid \lambda_j \text{ eigenvalue of } L_S \}$$

② C stable $\Leftrightarrow \lambda_1 \geq -(\frac{n-1}{2})^2$ (stability $\Leftrightarrow \int |\nabla \varphi|^2 - |A_C|^2 \varphi^2$
 $\Leftrightarrow \exists w > 0$ $C \geq 0$
 solving $L_C w = 0$ $\forall \varphi \in C_c^\infty(C)$)

pf: " $\Leftarrow$ " $r^{\gamma^+(\pi_1)} \varphi_1(0)$ is a positive Jacobi field
 $\stackrel{\text{by (4)}}{\Rightarrow} C$ stable.

$$\Rightarrow \gamma_1^+ = \frac{2-\eta}{2} + i\beta, \quad \beta > 0$$

$$\psi := \operatorname{Re}(r^{\gamma_1^+} \varphi_1) = r^{\frac{2-\eta}{2}} \cos(\beta \log r) \varphi_1(0) \text{ Jac. field.}$$

$$\Rightarrow \exists 0 < R_1 < R_2 < +\infty$$

$$\text{s.t. } \psi|_{\partial A_{R_1, R_2}} = 0$$

if C stable, then $\exists w > 0, L_C w = 0$

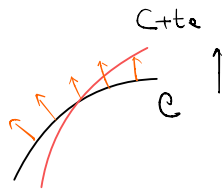
$$\Rightarrow \exists \Lambda > 0 \text{ s.t. } \begin{cases} \Lambda w \geq \psi & \text{on } A_{R_1, R_2} \\ \exists p \in \dot{A}_{R_1, R_2} \text{ s.t. } \Lambda w(p) = \psi(p) \end{cases}$$

contradiction

$\frac{1}{2}$ to strong max. principle.

$$(L_C(\underbrace{\Lambda w - \psi}_{\neq 0}) = 0)$$

③ Consider $e \in \mathbb{R}^{n+1}$ unit vector,



$$C + te \xrightarrow{t \rightarrow 0} C, \quad t^{-1} u_t \rightarrow e \cdot \nu_C = \psi_e$$

$$\parallel \text{graph}_C(u_t)$$

$$\Rightarrow L_C(\psi_e) = 0, \quad \psi_e(\lambda A) = \lambda^\alpha \psi_e(A)$$

$$\Rightarrow (\Delta_S + |A_S|^2) (\psi_e|_S) = 0$$

$$L_S = \mathcal{L}_S + (n-1) \Rightarrow -L_S (\psi_e|_S) = -(n-1) \psi_e|_S$$

$$\Rightarrow W_{-(n-1)}(-L_S) \supset \{ \psi_e | e \in \mathbb{R}^{n+1} \}$$

$$\Rightarrow \dim W_{-(n-1)}(-L_S) \geq n+1.$$

④ Recall $\lambda_1 \left(\frac{-\Delta_S - |A_S|^2}{-\mathcal{L}_S} \right) \leq -(n-1)$

$$\begin{aligned} \text{ind}(-\mathcal{L}_S) &\geq \dim W_{-(n-1)}(-\mathcal{L}_S) + 1 \\ &\geq n+2 \end{aligned}$$

~~equivalently~~
 $\nexists S \subset \mathbb{S}^n$
 minimal
 hypersurface

Rem [Urbano, '90] $S \subset \mathbb{S}^3$ $\{ (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid |x| = |y| = \frac{1}{\sqrt{2}} \}$
 minimal hypersurface up to not.

if $\text{ind}(S) = 5$, then $S = \text{Clifford torus}$

• Prelim Facts Suppose $M^n \subset (\Omega^{n+1}, g)$ minimal,

ν_M unit normal. $\forall u \in C^\infty(M), \|u\|_{C^1} \ll 1$

$$|A_M| \leq 1$$

$\forall u \in C^\infty(M), \|u\|_{C^1} \ll 1, M_u = \text{graph}_M(u) = \{ \exp_x^\nu(u(x) \cdot \nu_M(x)) \mid x \in M \}$

$$M_u \text{ minimal} \iff 0 = H_{M_u} = \Delta_M u + \underbrace{\left(|A_M|^2 + \text{Ric}_g(\nu_M, \nu_M) \right) u + Q[u]}_{L_M u}$$

where $Q[u] = \vec{a}_1 \cdot \nabla^2 u + \vec{a}_2 \cdot \nabla u + a_3 u$ $\|L_M u\|$

$$|\nabla^k a_i| \lesssim \sum_{j=0}^{k+1} |\nabla^j u|$$

• when $\|u\|_{C^1} \ll 1$, then $\mathcal{L}u$ is uniform ellip.

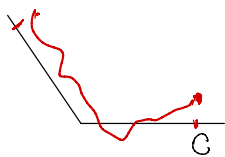
• (strong maximum) $\mathcal{L}_M u = 0, u \geq 0$ on M & $\exists x \in M$

$$\text{s.t. } u(x) = 0 \implies u \equiv 0$$

• (Harnack inequality) $\mathcal{L}u = 0, u > 0$ on $M \implies \forall \Omega' \ll \Omega$

$$\sup_{\Omega' \cap M} u \lesssim \inf_{\Omega' \cap M} u$$

• (Ellip. est.) $\|u\|_{C^{k,\alpha}(\Omega')} \lesssim \|u\|_{L^2(\Omega)} + \|\mathcal{L}u\|_{C^{k-2,\alpha}(\Omega)}$



• Thm (Hardt-Simon) Let $C = \partial E \subset \mathbb{R}^{n+1}$, minimizing hypercone, $\text{Sing}(C) = \emptyset$ $\exists!$ $H \subset E$, minimizing bdy. E open
 call C a "regular cone"

s.t. $\text{dist}(0, \text{supp}(H)) = 1$ $\{\lambda H\}_{\lambda > 0}$ foliate E .

Moreover (1) $\text{Sing}(H) = \emptyset$ $\Downarrow$

2) H is a radial graph: $\exists! w \in C^\infty(E \cap S^n)$

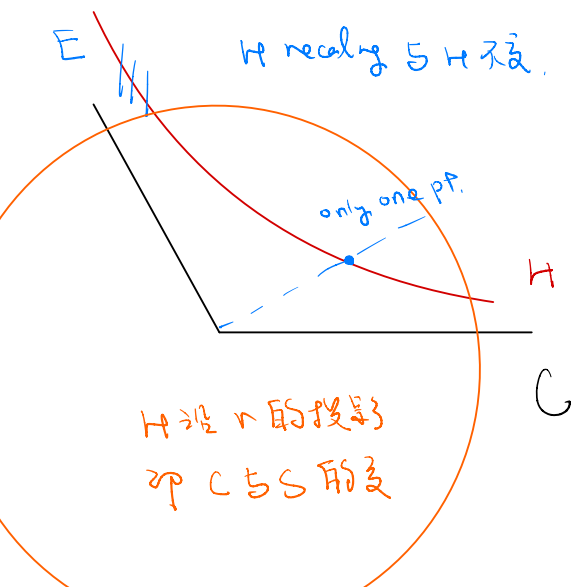
s.t. $H = \{w(\theta), \theta \mid \theta \in E \cap S^n\} \stackrel{\text{diffe.}}{\sim} E \cap S^n$

3) outside some B_R , $H = \text{graph}_C(u)$ w.o.f.t.:

either $u(r\theta) = r^{\lambda_1^-} \varphi_1(\theta) + O(r^{\lambda_1^- - \varepsilon})$

or $u(r\theta) = \begin{cases} r^{\lambda_1^+} \varphi_1(\theta) + O(r^{\lambda_1^+ - \varepsilon}) & \text{if } \lambda_1(s) > -\left(\frac{n-2}{2}\right)^2 \\ r^{\frac{2-n}{2}} (c_1 + c_2 \log r) \varphi_1(\theta) + O(r^{\frac{2-n}{2} - \varepsilon}) & \text{if } \lambda_1(s) = -\left(\frac{n-2}{2}\right)^2 \end{cases}$

H radial graph H radial. if $\lambda_1(s) = -\left(\frac{n-2}{2}\right)^2$

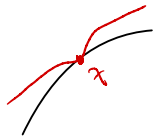


$$L_C u = 0$$

$$u = \sum (c_j^+ r^{\lambda_j^+} + c_j^- r^{\lambda_j^-}) \varphi_j(\theta)$$

• Preparations ① (Strong Maximum Principle)

Let $\Omega \subset_{\text{open}} \mathbb{R}^{n+1}$



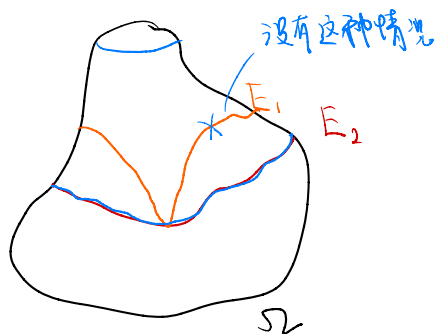
$E_1 \subset E_2 \subset \Omega$, Caccioppoli sets w minimizing bodies

iff $\exists x \in \text{spt } \partial[E_1] \cap \text{spt } \partial[E_2]$

$$\Rightarrow M_1 = M_2$$

where M_i is the connect component of $\text{spt } \partial[E_i]$

containing x .



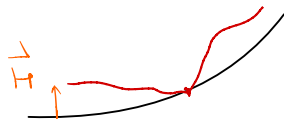
Rem (Toy version) iff $E_1 \subset E_2$,

$\partial E_1, \partial E_2$ are minimal cones

$$\Rightarrow E_1 = E_2$$

Frankel property $\forall$ minimal hypersurface $S_1, S_2 \subset \mathbb{B}^n$
must intersect!

[Solomon-White '89]



② Thm (Simon '08) Let $M \subset \Omega$, minimizing bdy, $u \in C^\infty(M)$
 $L_M u \leq 0$, then $\forall P_j \rightarrow P \in \text{Sing}(M)$

$$u(P_j) \xrightarrow{j \rightarrow \infty} +\infty$$

$\text{Reg}(M)$

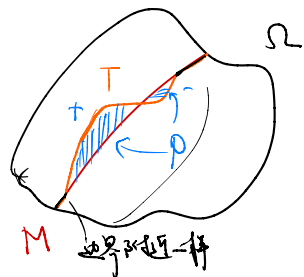
(Next time: $\forall P \in \text{Sing}(M)$, $\exists \tilde{P}_j \rightarrow P$ s.t. $u(\tilde{P}_j) \rightarrow +\infty$)

③ (Unique Minimizing Property) Suppose $M \subset \Omega$ be a minimizing bdy w finite mass.

Then $\forall T \in I_n(\Omega)$ s.t. $T - M = \partial P$ for some $\text{spt } P \subset \text{cpt } \Omega$
 either $T = M$; or $\|T\|(\Omega) > \|M\|(\Omega)$

pf of ③ Suppose $\|T\|(\Omega) = \|M\|(\Omega)$

$\Rightarrow T$ is also a minimizing bdy



$\text{spt } T$ is C^∞ away from a closed set of $\dim \leq n-1$

$\Rightarrow$ By unique continuation property $T \equiv M$

on the conn. comp. contain $M \setminus \text{spt } P$
 of $\text{spt } M$

$\Rightarrow \text{spt}(T-M) \subset \Omega$ & $T-M$ is also a minimizing bdy

$$\partial[P]$$

$$\Rightarrow T-M=0$$

□

Pf of Existence

Let $\varepsilon = E \cap S^n$, $\varepsilon_j = \{x \in \varepsilon \mid d(x, \partial S) > \frac{1}{j}\}$

$$\text{let } E_j = G(\mathcal{E}_j) \quad \xrightarrow{j \rightarrow \infty} \mathcal{E}$$

$$= \{ r\theta \mid r > 0, \theta \in \varepsilon_j \}$$

Minimize perimeter among $\{F \mid F \setminus B_1 = E_j \setminus B_1\}$ gives a F_j
in B_2

s.t. $F_j \setminus B_i = E_j \setminus B_i$; F_j has minimizing body in B_i ;

$$\text{Sp}^1(\partial[F_i]) \cap S^n \subset \partial E_i$$

Claim 1) $\text{sp}^A(F_j) \subset \bar{E} \quad \left(\begin{array}{c} \text{strong max.} \\ \Rightarrow \\ \text{possible} \end{array} \right) \quad \text{sp}^A(F_j) \subset E$

pf of claim We have $\| \partial[F_i \sqcup E] \| (B_2) + \| \partial[F_i \cap E] \| (B_2)$

$$\star \leq \| \partial [F_j] \| (B_2) + \| \partial [F] \| (B_2)$$

Maggi: Set of Finite Perimeter --- Chapter 16

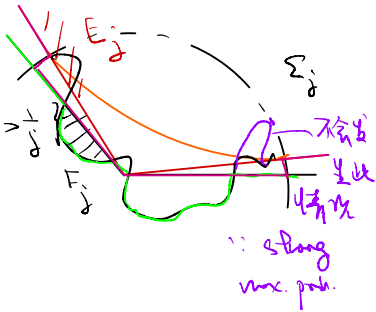
If $(\text{spa } \partial[F_j]) \setminus E \neq \emptyset$ then $\geq \|\partial[E]\|(\mathcal{B}_2)$

$$\Rightarrow \| \partial [F_j \cap E] \| (B_2) < \| \partial [F_j] \| (B_2),$$

Contradicts to choice of F_j

- By the Glavin 1), $d_j := \dim(x, \text{spt } F_j) > 0$

Claim 2) $d_j \xrightarrow{j \rightarrow \infty} 0$ (if suffices: $F_j \xrightarrow{j \rightarrow \infty} E$)



Pf: $F_j \setminus B_1 = E_j \setminus B_1 \xrightarrow{j \rightarrow \infty} E \Rightarrow F_j \xrightarrow{j \rightarrow \infty} F_\infty$

Cauchy set ~~is~~ not convergence.

with $F_\infty \setminus B_1 = E \setminus B_1$

$$\|\partial[F_\infty]\|(B_2) \leq \lim_{j \rightarrow \infty} \|\partial[F_j]\|(B_2)$$

$$\leq \|\partial[E]\|(B_2) \Rightarrow F_\infty = E \quad \square$$

• Now consider $d_j^{-1} \cdot F_j \xrightarrow{j \rightarrow \infty} \hat{F}_\infty$ has minimizing bdy in

$$\mathbb{R}^{n+1} \quad \& \quad \text{dist}(0, \text{spt}[\hat{F}_\infty]) = 1, \quad \hat{F}_\infty \neq E, \quad \hat{F}_\infty \subset \bar{E}$$

strong
 $\Rightarrow$
max. principle

$$\hat{F}_\infty \subset E$$

$H := \partial \hat{F}_\infty$ is a desired minimizing bdy. $\square$

From now on, Suppose $H = \partial \hat{F}$ is a general minimizing

bdy. $\hat{F} \subset E$

blow down

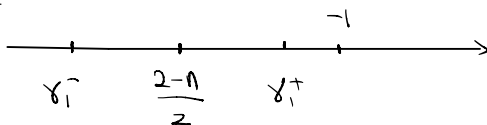
* Step 1 $\rho^{-1} H \xrightarrow{\rho \rightarrow \infty} C$

$\& \quad H = \text{graph}_C(u)$ outside some $B_{\bar{R}}$

$$\text{s.t. } \exists r = \left\{ r_i^\pm := \frac{2-n}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1(S)} \right\} \quad \text{s.t.}$$

$$u(r\theta) = \begin{cases} \geq r^{\delta-2} \\ \leq r^{\delta+2} \end{cases}$$

$$n \geq 7$$



Step 2 Recall u solves $L_C u = -Q[u]$.

Recall yesterday $\|Q[u]^{(p)}\|_{C^\alpha(A_{1,2})} \lesssim \rho^{-1} \|\rho^{-1} u^{(p)}\|_{C^{2,\alpha}(A_{1,2})}^2$

$$\lesssim \rho^{-3} \cdot (\rho^{1+\varepsilon})^2$$

$$\lesssim \rho^{2\delta-3+2\varepsilon}$$

Take $\sigma \in \mathbb{R} \setminus \mathbb{P}(C)$ s.t. (when $\varepsilon \ll 1$)

$$2\delta-3+2\varepsilon \leq \sigma-2 \leq \delta-\varepsilon-2$$

By discussion yesterday, $\exists w$ solves $\begin{cases} L_C w = Q[u] \\ \|w^{(p)}\|_{C^{2,\alpha}(A_{1,2})} \lesssim \rho^\sigma \end{cases}$

$$\Rightarrow U := u + w, \quad L_C U = 0$$

$$U(r\theta) \gtrsim r^{\delta-\varepsilon} - C \cdot r^\sigma \gtrsim r^{\delta-\varepsilon} \quad \text{for } r \gg 1.$$

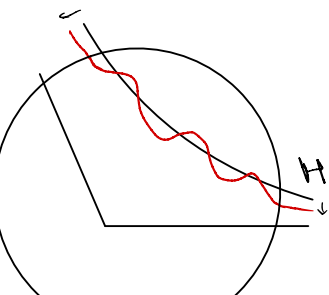
$$U(r\theta) \lesssim r^{\delta+\varepsilon}$$

Ex. Such U satisfies the desired asymp in H-S's thm

Step 3 Recall $\rho^{-1} H = \text{graph}_C(\rho^{-1} u^{(p)})$ outside large ball.

$$\Rightarrow \rho^{-1} u^{(p)}(r\theta) = \rho^{-1} \left[(\rho r)^{\delta_1^-} \varphi_1(\theta) + O((\rho r)^{\delta_1^- - \varepsilon}) \right]$$

$$= \underbrace{\rho^{-1+\delta_1^-}}_{< 1} r^{\delta_1^-} \varphi_1(\theta) + \text{h.o.t.}$$



Hence, Outside some B_{R_p} , $(\rho^{-1} \hat{F}) \setminus B_{R_p} \supset \hat{F} \setminus B_{R_p}$

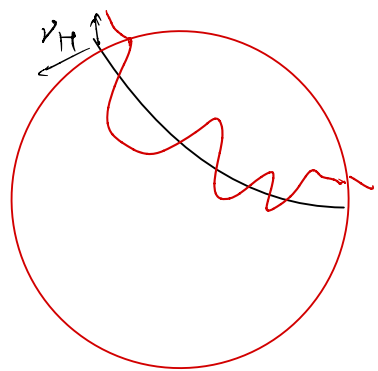
same argument

$$\implies \rho^{-1} \hat{F} \supset \hat{F} \quad \forall \rho > 1.$$

as in proof of

Existence

$$\implies \forall \rho > 1, \quad \rho^{-1} H \cap H = \emptyset$$



Step 4 Consider $\psi(x) := x \cdot \nu_H$

Jacobi field induced by scaling

(Ex: if write $\rho^{-1} H = \text{graph}_H(\psi_\rho)$ in regular pos, then

$$(\rho^{-1})^\top \cdot \psi_\rho \xrightarrow{\rho \rightarrow 1} \psi)$$

$$\text{Hence } \begin{cases} \psi > 0 \\ L_H \psi = 0. \end{cases}$$

Recall if $\exists p \in \text{Sing}(H)$, then $\exists p_j \rightarrow p$ s.t.

$$\psi(p_j) \rightarrow +\infty \quad \text{as } j \rightarrow \infty$$

$$\implies \text{Sing}(H) = \emptyset.$$

Step 5 (Uniqueness) $H_1, H_2 \subset E$ minimizing bodies

$$\text{dist}(o, H_1) = \text{dist}(o, H_2) = 1$$

$$\text{Hope } H_1 = H_2$$

$$C r^{\frac{n}{2}}$$

$$r_1^{\frac{n}{2}} < r_2^{\frac{n}{2}}$$

$$r^{\frac{n-2}{2}} (C_1 + C_2 \log r) =$$

Recall $H_i \setminus B_R = \text{graph}_C(u_i)$, $u_i(r\theta) = C_i r^{\delta_i^-} \varphi_i(\theta)$

$$p^{-1} F_1 \subset F_2 \quad \text{if } p^{-1+\delta_i^-} C_1 > C_2 + O(r^{\delta_i^- - \varepsilon})$$

$$\text{" } \text{graph}_C(p^{-1} u_i^{(p)})$$

$$\text{" } p^{-1+\delta_i^-} C_1 \dots$$

$$\supset \quad \text{if } \leq$$

$$\Rightarrow \text{Take } \bar{p} < -1. \quad \bar{p}^{-1+\delta_i^-} C_1 = C_2$$

$$\text{Then } \bar{p}^{-1} F_1 = F_2 \Rightarrow \bar{p} = 1$$

since

$$\text{dist}(0, F_1) = \text{dist}(0, F_2)$$

□

Cor Suppose $C = \partial E \subset \mathbb{R}^{n+1}$ is a minimizing bdy

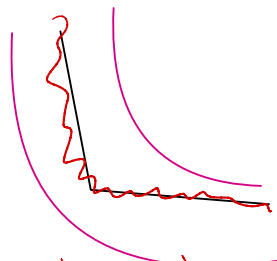
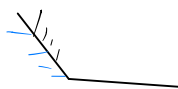
$H \subset E$ be C^∞ minimizing bdy determined by H-S. Then

called a "leaf of Martin-Simon foliation"

Also $\exists H' \subset \mathbb{R}^{n+1} \setminus \bar{E}$ C^∞ minimizing bdy

$$H \setminus B_R = \text{graph}_C(u) \setminus B_R$$

$$H' \dots (u')$$



$$\text{Then } u = C r^{\delta_i^-} \varphi_i(\theta)$$

$$u' = \dots$$

$$\text{or } u = \int \dots$$

$$\text{or } \dots$$

it's impossible
to have cone $\frac{1}{2}$

If M is a minimizing bdy in $\mathbb{R}^{n+1}$ s.t. $M \setminus B_R = \text{graph}_C(0) \setminus B_1$

s.t. $\lim_{r \rightarrow \infty} \|u^{-1}v\|_{C^0(\partial B_r)} + \|u'^{-1}v\|_{C^0(\partial B_r)} = 0$

Then $M = \pm C$.

call C "strictly minimizing" if H & H' satisfies these asymptotic.

$$C^2 \supset C = \{z=0\} \cup \{w=0\}$$

note. $\frac{2n}{n-2}$ $\swarrow$ $H = \{wz=1\}$
 ~~$\frac{2n}{n-2}$~~
 ~~$\frac{2n}{n-2}$~~

Thm [Chen '97] Let $C \subset \mathbb{R}^{n+1}$ minimizing strictly hypersurface ($\text{Sig}(C)=0$)

$\forall T \subset \mathbb{R}^{n+1} \setminus B_1$ minimizing bdy s.t. $\|U^{(p)}\|_{C^2(A_{1,2})} \lesssim p^{1-\varepsilon}$
 $\text{"graph}_C(U)$ $(\Rightarrow p^{-1}T \xrightarrow{p \rightarrow \infty} C)$

Then $\exists \Sigma \subset \mathbb{R}^{n+1}$, minimizing bdy

s.t. $\Sigma \setminus B_R = \text{graph}_C(u)$

$$\|(u-v)^{(p)}\|_{C^2(A_{1,2})} \lesssim \ell^{\gamma_1^-}$$

In particular, $\exists \{r_q\}_{q \in B_2^{\mathbb{I}}}$ $\checkmark$ minimizing bdy

in $\mathbb{R}^{n+1} \setminus B_1$ s.t. $\forall q \neq q'$ $\text{"graph}_C(U_q)$

$$\ell^{-\gamma_1^-} (U_q - U_{q'})^{(p)} \xrightarrow{\ell \rightarrow +\infty} +\infty$$

$$I = \text{ind}(C \cap S^n) = \dim W'_{1-2} \quad \Sigma \leq 1$$

pf (Sketch) ① $\forall R > 1$

Let Σ_R be minimizing hypersurf w.

$$\partial \Sigma_R = \Gamma \cap \partial B_R$$

Take $d(R) := \inf \{ R' \mid \Sigma_R \cap A_{R', R} = \text{graph}_C(u_R) \cap A_{R', R} \}$
 $\| \rho^{-1} u_R^{(e)} \|_{C^2(A_{1,2})} \leq \delta \}$

Hope $\lim_{R \rightarrow \infty} d(R) < +\infty$

② Since $R^{-1} \Sigma_R \longrightarrow C \cap B_1$ $\lim_{R \rightarrow \infty} R^{-1} d(R) = 0$

③ By constructing foliations with fast decay

$$\sup_{\partial B_{r_2}} |u_R - v| \lesssim_C \left(\frac{r_2}{r_1} \right)^{\gamma_1} \sup_{\partial B_{r_1}} |u_R - v|, \forall d(R) \leq r_1 < r_2 \leq R$$

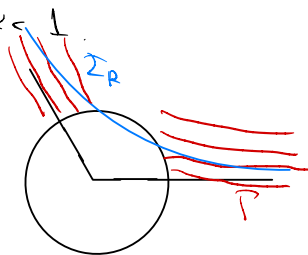
④ If $d(R_j) \rightarrow +\infty$, we have $\| \rho^{-1} u_{R_j}^{(e)} \|_{C^2(A_{1,2})} \geq K(\delta) > 0$
 if $\rho = d(R)$

$d(R_j) \cdot \Sigma_{R_j} \xrightarrow{j \rightarrow \infty} \hat{\Sigma} \subset \mathbb{R}^{n+1}$ minimizing body $\hat{\Sigma} = \text{graph}_C(\hat{u})$ in $\mathbb{R}^n \setminus B_2$

$d(R_j)^{-1} \Gamma \xrightarrow{j \rightarrow \infty} C$ $\sup_{\partial B_r} |\hat{u} - v| \lesssim \left(\frac{r}{2} \right)^{\gamma_1} \sup_{\partial B_r} |u - v|$

$\Rightarrow \hat{\Sigma} = C$ ⚡

□



Lecture 3

Growth of Positive Jacobi fields

For $E \subset \Omega \subset \mathbb{R}^m$, a Cacciopp set.

u. minimizing bdy $M = \text{Reg}(\partial[E])$

Define $r_M: \bar{M} \rightarrow [0, +\infty)$ (Called "reg scale" of M)

by $r_M(x) := \begin{cases} 0 & \text{if } x \in \text{Sing}(M) \\ \sup \{r > 0 \mid B_r(x) \cap \text{Sing}(M) = \emptyset, |A_M|_{B_r(x)} \leq \frac{1}{2026r}\} & \text{if } x \notin \text{Sing}(M) \end{cases}$

Prop 1) if $B_{r'}(x) \subset B_{r_M(x)}(x)$,

then $r_M(x) \geq r'$. In particular $r_C(\lambda x) = \lambda r_C(x)$

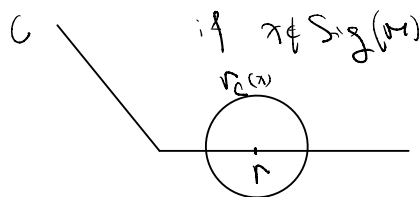
$x \mapsto r_M(x)$ is 1-Lip.

2) if minimizing bdies $M_j \xrightarrow{j \rightarrow \infty} M$ in Ω ,

then for $x_j \xrightarrow{j \rightarrow \infty} x_\infty$,

we have $r_{M_j}(x_j) \rightarrow r_M(x_\infty)$

($r_M \sim$ dist to Sing set of all possible "nearby minimizing bdies")



$$3) \quad \forall \delta > 0, \quad B_\delta(\text{Sing}(M)) \subset \{r_M(x) < \delta\}.$$

[Chaegeer - Naber '13]

$$\|M\|(\{r_M \leq \delta\} \cap \Omega') \lesssim_{n, \Omega, \Omega', \varepsilon} \delta^{7-\varepsilon} \quad \forall \Omega' \subset \subset \Omega.$$

pf: $\square$.

$\cdot R_{\geq \delta}(M) := \{x \in M \mid r_M \geq \delta\}$ "effective part. of $\text{reg}(M)$ "

Key Lemma $\exists K_n^* > 0$ s.t. $\forall C \subset \mathbb{R}^{n+1}$ minimally hypercone

$$1) R_{> K_n^*}(C) \cap S^n \neq \emptyset$$

2) conn. comp. of $\text{Reg}(M)$ containing $R_{> K_n^*}(C) \cap S^n$ is dense in C .

3) $\forall u > 0$ Jac. Field on G , we have $\forall \kappa \in [0, K_n^*]$

$$\left(\underbrace{(\Delta_C + |A_C|^2)}_{L_C} u = 0 \right)$$

$$\sup_{R_{> \kappa r}(C) \cap \partial B_r} u \leq \bar{C}(n, \kappa) \left(\inf_{R_{> \kappa}(C) \cap \partial B_1} u \right) \cdot r^{\gamma_n}, \quad r > 1$$

$$\text{let } \gamma_n = \frac{2-n}{2} + \sqrt{\left(\frac{2-n}{2}\right)^2 - (n-1)}$$

$$\lambda_1(S) \leq -(n-1)$$

App. Cor A Let $u \in C^2(\mathbb{R}^n)$ solve (MSE)

$$\operatorname{div} \left(\frac{\nabla u}{(1+|\nabla u|^2)^{\frac{1}{2}}} \right) = 0 \text{ be nonlinear}$$

then $T_u := \{ (x, u(x)) \mid x \in \mathbb{R}^n \} \subset \mathbb{R}^{n+1}$ has every tangent

cone at ∞ splits in $\mathbb{R}^{n+1}$ (i.e. $e_j^{-1} T_u \xrightarrow{e_j \rightarrow \infty} C = C_0 \times \mathbb{R}$)

In particular, $n \geq 8$

Pf. Let $\psi_{r_0}(x) := e_{n+1} \cdot \nu_{T_u} = \frac{1}{\sqrt{1+|\nabla u|^2}} > 0$, Jac. field on T_u

$$\psi_C(x) = \lim_{j \rightarrow \infty} \psi_{e_j^{-1} T_u} \geq 0$$

$$\frac{d}{dt} \int_C f(x, y+t) dx \equiv 0.$$

We must have $\psi_C \equiv 0$

$$\Rightarrow e_{n+1} \parallel TC \Rightarrow C = C_0 \times \mathbb{R}$$

nonlinear

Cor B Let $u \in C^2(\mathbb{R}^n)$ solve (MSE). $\forall \delta > \delta_0$

$$\lim_{\rho \rightarrow \infty} \rho^\delta \sup_{B_\rho} |\nabla u| = +\infty$$

Equiv. if $\lesssim o(1/x)$ $\checkmark$
 $|\nabla u|(x) \lesssim |x|^{-\delta_0 - \epsilon}$, $\forall x \in \mathbb{R}^n$, then u is linear

Pf. Recall $\psi(x) = e_{n+1} \cdot \nu_{T_u} > 0$ Fix $\delta' \in (\delta_0, \delta)$

Claim $\exists \Lambda_{\gamma'} > 1$ & $R_{\gamma'} > 1$ s.t. $\forall \rho > R_{\gamma'}$

$$\sup_{B_{\geq \kappa \Lambda \rho}(P_u) \cap \partial B_{\Lambda \rho}} \psi \leq \left(\inf_{B_{\geq \kappa \rho}(T_u) \cap \partial B_{\rho}} \psi \right) \Lambda^{\gamma'}$$

With the Claim $m(\rho)$

$$m(\rho) \leq m(\rho) \Lambda^{\gamma'} \quad \forall \rho \geq R_{\gamma'}$$

iteratively for $\rho \geq R_{\gamma'}$ $m(\rho) \leq \rho^{\gamma'} \cdot C(u, \delta')$

$$\inf_{B_{\geq \kappa \rho}(T_u) \cap \partial B_{\rho}} \psi \geq \inf_{(B_{\rho}^n \times \mathbb{R}) \cap T_u} \psi = \frac{1}{\sup_{B_{\rho}^n} \sqrt{1 + |\nabla u|^2}} \quad \square$$

Yau's Conj. $\forall u \in C^\infty(\mathbb{R}^n)$ solving (MSE) has polynomial growth near ∞ .

Cor C If $M \subset \Omega$ is a minimizing bdy, $\forall p \in \text{Sing}(M)$
then $\nabla u > 0$. Jac. field on $\text{Reg}(M)$

We have, $\exists p_i \rightarrow p$ s.t. $u(p_i) \nearrow +\infty$

Pf: $\rho \mapsto \sup_{B_{\geq \kappa \rho}(M) \cap \partial B_{\rho}} u =: M(\rho)$ satisfies

$M(\lambda \rho) \geq \lambda^{n-\varepsilon} M(\rho)$ for some $0 < \lambda(\varepsilon) \ll 1$ & $\rho \ll 1$.

In particular, $M(p) \xrightarrow{p \downarrow 0} +\infty$

□

Cor D (Special Case of SMP) Suppose $E_1 \subset E_2 \subset \Omega$

are minimizing bodies, $p \in \text{spt}(\partial[E_1]) \cap \text{spt}(\partial[E_2]) \cap \Omega$

Suppose $\forall r_j \rightarrow 0$, we have $r_j^{-1}(\partial E_i - p) \xrightarrow{j \rightarrow \infty} C_i$

$\Rightarrow C_1 = C_2$

then, we have $\exists \rho > 0$ s.t. $E_1 \cap B_\rho(p) = E_2 \cap B_\rho(p)$

PF: $\forall \lambda > \lambda_n$, $\exists \rho_{n,\lambda} > 0$ s.t. $d(p) := \sup_{R \gg \rho_{n,\lambda}} \text{dist}(\cdot, \text{spt} M_2)$

Let $M_i = \partial[E_i]$

$\lambda(\lambda, n) \in (0, 1)$

$R \gg \rho_{n,\lambda} \cap \partial B_\rho(p)$

$\forall \ell < \rho_{n,\lambda}$, $d(\lambda p) \geq \lambda^\lambda d(p)$

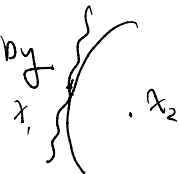
□

Fact. If S_1, S_2 be $(n-1)$ -dim currents w mlt: 1 tangent

in (M, g) s.t. $\text{spt}(S_1) \cap \text{spt}(S_2) = \emptyset$

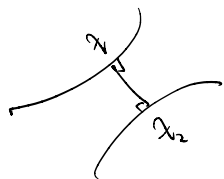
then, $d = \text{dist}(\text{spt}(S_1), \text{spt}(S_2))$ is realized, by

$x_i \in \text{Reg}(S_i)$



PF: Suppose $x_i \in \text{spt}(S_i)$ realizes dist.

Then $\text{spt} S_1 \cap B_d(x_2) = \emptyset$.



pf of strong max principle

$$p \in \text{Spt}(\partial E_1) \cap \text{Spt}(\partial E_2)$$

$$\text{s.t. } \exists r_j \searrow 0 \text{ s.t. } r_j^{-1} (\partial E_i - p) \xrightarrow{j \rightarrow \infty} C_i \text{ but } C_1 \neq C_2,$$

Then either $(\text{spt } C_1) \cap \mathbb{S}^n \cap (\text{spt } C_2) = \emptyset$ $\frac{1}{2}$ to Frankel property,
 or $\exists q^{(0)} \in (\text{spt } C_1) \cap \mathbb{S}^n \cap (\text{spt } C_2)$

$$\& C_1 \cap B_r(q^{(0)}) \neq C_2 \cap B_r(q^{(0)}) \quad \forall r > 0$$

$\Rightarrow \exists$ tangent cones $\underline{C_1^{(1)}} \neq \underline{C_2^{(1)}}$ of C_1, C_2 resp.

split in $q^{(0)}$ -direction

Individually $\exists C_1^{(k)} \neq C_2^{(k)}$, one lies on one side of another.

splits in $\mathbb{R}^k$ -direction, $k = n - 1$



□

Cor E $M_s \subset \Omega \subset \mathbb{R}^{n+1}$ minimizing bodies, $s \in [-1, 1]$ s.t.

$$\omega \text{ disjoint support, } \& \text{dist}(\text{spt } M_s, \text{spt } M_{s'}) \geq \beta |s - s'|$$

$$\forall s, s' \in [-1, 1]$$

Then let $\mathcal{J} := \bigcup_{s \in [-1, 1]} \mathcal{J}_s(M_s)$, $\mathcal{J}: \mathcal{J} \rightarrow [-1, 1]$

is α -Hölder, $\forall \alpha \in (0, \underbrace{1 - \gamma_0}_{\geq 2})$ $\alpha \mapsto s$ s.t. $x \in \mathcal{J}_s(M_s)$

$$\forall x, x' \in \mathcal{J} \quad |\mathcal{J}(x) - \mathcal{J}(x')| \lesssim |x - x'|^\alpha$$

Rem (Ex) You can prove $\dim_{\mathcal{H}}(\mathcal{S}) \leq n-1$

& covering Lemma $\Rightarrow \dim_{\mathcal{H}} \mathcal{S} \leq \frac{n-1}{\alpha} < 1$ if $n+1 \leq 10$

this prove generic reg of minimizing bdy in $\mathbb{R}^{n+1} \leq 10$

Rmk $\dim_{\mathcal{H}} \mathcal{S} \leq \sup_{C \subset \mathbb{R}^{n+1}} \frac{\dim(\text{Sphere}(C))}{1 - \gamma(C^0)}$
 $n+1=11$.
 minimizing hypersurface, $\leq \gamma_7 = -2$

$\dim(\text{Sphere}(C^0)) = 3 \Rightarrow C^0 = C^T \times \mathbb{R}^3$
 $\Rightarrow C = C^{P,q} \times \mathbb{R}^3$

Pf: 1) 2) follows by a direct / contradiction arguments.

• $S := \text{Reg}(C) \cap \mathbb{S}^n$, $\forall \Omega \subset_{\text{open}} S$, $\lambda_1(\Omega) = 1$

s.t. Dirichlet eig. value of $-(\Delta_S + |A_S|^2)$

$$= \inf_{0 \neq \varphi \in C_c^\infty(\Omega)} \frac{\int_{\Omega} |\nabla \varphi|^2 - |A_S|^2 \varphi^2}{\int_{\Omega} \varphi^2} \quad (\text{decreasing in } \Omega)$$

$$\frac{|\lambda|^2}{p} = \frac{|\gamma|^2}{q} \}$$

Fact [Simons '68, Lawson '69, Zhu '18]

$$\lambda_1(S) \leq -(n-1) \quad "=" \text{ iff } C = C^{P,q} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^{q+1}\}$$

Cor $\exists p_n > 0$, s.t. $\forall$ minimizing hypercone $C \subset \mathbb{R}^{n+1}$
 $\text{plane} \not\subset C$

We have $\lambda_1(R_{>2\rho_n}(C) \cap S^n) \leq -(n-1)$

Pf. Suppose not. $\exists C_j \subset \mathbb{R}^{n+1}$ minimizing hypercones.
 plane

$$\text{s.t. } \lambda_1(R_{>\frac{1}{j}}(C_j) \cap S^n) > -(n-1)$$

Up to subseq. $C_j \rightarrow C_\infty \quad \forall \delta > 0, \lambda_1(R_{>\delta}(C_\infty) \cap S^n)$

$$\geq \limsup_{j \rightarrow \infty} \lambda_1(R_{>\delta}(C_j) \cap S^n) \geq -(n-1)$$

$$\Rightarrow \lambda_1(\text{Reg}(C_\infty) \cap S^n) \geq -(n-1)$$

$$\Rightarrow C_\infty = C^{\text{p.t.}} \text{ is a regular cone.}$$

by Alford Reg.

$$\Rightarrow \text{for } j \gg 1, \text{Ang}(C_j) = 0.$$

$$\exists \delta > 0, \text{ s.t. } R_{>\delta}(C_j) \cap S^n = C_j \cap S^n \text{ for } j \gg 1.$$

$$\Rightarrow \lambda_1(R_{>\frac{1}{j}}(C_j)) \leq \lambda_1(R_{>\delta}(C_j) \cap S^n)$$

$$\stackrel{L}{-(n-1)} = \lambda_1(C_j \cap S^n)$$

$$\leq -(n-1)$$

⚡ $\square$

Lem: $C \subset \mathbb{R}^{n+1}$ minimizing hypercone

$$\text{Let } R_{>\rho_n}(C) \cap S^n \subset \Omega \subset C_\infty \subset \text{Reg}(C) \cap S^n$$

Let φ_Ω be the 1st function of $\underbrace{-(\Delta_S + |\Lambda_S|^2)}_{L_S}$ on Ω

$$\begin{cases} -L_S \varphi_\Omega = \lambda_1(\Omega) \varphi_\Omega \\ \varphi_\Omega|_{\partial\Omega} = 0. \end{cases}$$

(We have $\lambda_1(\Omega) \leq -(n-1)$)

Then $\forall u > 0$ Jac. field on C :

$$r \mapsto \frac{r^{-\gamma} \int_{\Omega} \varphi_\Omega(\theta) u(r\theta) d\theta}{V(r)}, \text{ is nonincreasing in } r.$$

pf:

$$V''(r) + \frac{n-1}{r} V'(r) = \int_{\Omega} \varphi_\Omega(\theta) \underbrace{\left(\partial_r^2 + \frac{n-1}{r} \partial_r \right) u(r\theta) d\theta}_{-\frac{1}{r^2} L_S u}.$$

$$= -\frac{1}{r^2} \int_{\Omega} \varphi_\Omega(\theta) L_S u(r\theta) d\theta.$$

$$= -\frac{1}{r^2} \int_{\Omega} \underbrace{(L_S \varphi_\Omega)}_{-\lambda_1(\Omega) \varphi_\Omega} \cdot u(r\theta) d\theta + \frac{1}{r^2} \int_{\partial\Omega} \underbrace{\partial_\nu \varphi \cdot u}_{\leq 0}.$$

$$\leq \frac{n-1}{r^2} \int_{\Omega} \varphi_\Omega(\theta) u(r\theta) d\theta$$

$$= \frac{n-1}{r^2} V \stackrel{\frac{1}{S^\beta}}{\leq} \stackrel{S^{-\gamma/\beta}}{S^{-\gamma/\beta}}$$

Let $W(s) = V(s^{-1/\beta}) \cdot S^{2n/\beta}$, where $\beta = 2 \sqrt{(\frac{n-2}{2})^2 - (n-1)}$

Easy to check: $W''(s) \leq 0, \forall s > 0$

Since $W(s) > 0$ for all s .

We have $W'(s) \geq 0$.

□,

this Lemma + Harnack ineq for u & $\int_{\Omega}$
proves the Key lem.