

Introduce minimal surfaces eg. "weak" notion $S \rightarrow T$ v.

Motivation Q: Plateau Problem: given closed $S^{n-1} \subset \mathbb{R}^{n+m}$

$\exists$? surface M^n with boundary

$\partial M = S$, s.t. M has least area

among surfaces with boundary S ?

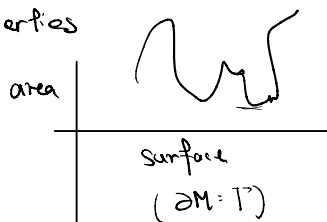
physical model: Soap film/bubbles because energy $\propto$ area.

how to analyze M (of least area)?

$\hookrightarrow$ can we critically/variational properties

$\rightarrow C$ minimal for area

should have $\frac{d}{dt} \Big|_{t=0} \text{area}(\varphi_t(M))$



def. M is stationary (or minimal) if M is critical for area in sense.

$$\frac{d}{dt} \Big|_{t=0} \text{area}(\varphi_t(M)) = 0$$

$\forall$ family of diffe. φ_t

s.t. $\varphi_0 = \text{id}$.

$\varphi_t = \text{id}$ outside compact set

φ_t leaves ∂M unchanged.



phm if $M^n \subset \mathbb{R}^{n+m}$ smooth. $X = C^1_c$ vector field. $\varphi_t(x)$ = family of diffeos. $\varphi_0 = \text{id}$, $\partial_t \varphi|_{t=0} = X$

$$\text{then } \left. \frac{d}{dt} \right|_{t=0} \text{area}(\varphi_t(M)) = \int_M \text{div}_M(X) dV_M$$

$$= - \int_M H \cdot X dV_M + \int_{\partial M} g \cdot X dV_{\partial M}$$

$$\text{where } \text{div}_M(X) = \sum_{i=1}^n D_{e_i} X \cdot e^i$$

$\{e_i\}$ = ON basis of $T_x M$

$$\text{and } \vec{H} = \left(\sum_{i=1}^n D_{e_i} e_i \right)^2 = \text{mean curvature vector}$$

$$= \text{"negative gradient of area"}$$

$$= \Delta_M \vec{x}$$

$$\Rightarrow M \text{ minimal} \Leftrightarrow \vec{H} = 0 \Leftrightarrow \Delta_M \vec{x} = 0$$

Pf of First variation formula.

use area function if $F: M \rightarrow \mathbb{R}^{n+m}$, C^1 and are _____

$$\text{then } \text{area}(F(M)) = \int_M J_M F dV_M$$

$$\text{where } J_M F = \sqrt{\det D_F^T D_F} = \sqrt{\det DF(e_i) \cdot DF(e_j)}$$

$\{e_i\}$ = ON-basis of $T_x M$.

$$\text{area}(\varphi_t(M)) = \int_M J_M \varphi_t dV_M \rightsquigarrow \text{EJS} \quad \left. \frac{\partial}{\partial t} \right|_{t=0} J_M \varphi_t = \text{div}_M(X)$$

(take $F = \varphi_t|_M$)

$$\text{recall } \det(I + tA) = 1 + t \text{tr} A + O(t^2)$$

$$\text{now } \varphi_t(y) = y + tX|_y + O(t^2), \text{ so } D_{\varphi_t}(e_i) = e_i + tD_{e_i}X + O(t^2)$$

$$\Rightarrow D_{\varphi_t}(e_i) \cdot D_{\varphi_t}(e_j) = \delta_{ij} + t(e_i \cdot D_{e_j}X + e_j \cdot D_{e_i}X) + O(t^2)$$

$$\Rightarrow \det(\psi) = 1 + 2t \operatorname{div}_M(x) + O(t^2)$$

$$\begin{aligned} \text{So } J_M \varphi_t &= \sqrt{\det(\psi)} = \sqrt{1 + 2t \operatorname{div}_M(x) + O(t^2)} \\ &= 1 + t \operatorname{div}_M(x) + O(t^2) \end{aligned}$$

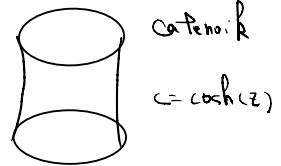
$$\Rightarrow \operatorname{area}(\varphi_t(M)) = \int_M 1 + t \operatorname{div}_M(x) + O(t^2) dV$$

$$\begin{aligned} \Rightarrow \left. \frac{d}{dt} \right|_{t=0} \operatorname{area}(\varphi_t(M)) &= \int_M \operatorname{div}_M(x) dV \\ &\quad \swarrow \text{normal, part} \quad \searrow \text{tangent?} \\ &= \int_M \operatorname{div}_M(x^\perp) dV + \int_M \operatorname{div}_M(x^\top) dV \\ &\quad \parallel \text{(divergence thm)} \\ &\quad \int_M \sum_{i=1}^n D_{e_i} \tilde{x} \cdot e_i dV \quad \int_{\partial M} x \cdot \nu dV_{\partial M} \\ &= \int_M \sum_i e_i (\underbrace{\tilde{x} \cdot e_i}_{=0}) - \sum_i \tilde{x} \cdot D_{e_i} e_i \\ &= - \int_M x \cdot \sum_i (D_{e_i} e_i)^\perp = - \int_M x \cdot H dV \quad \square \end{aligned}$$

e.g. plane



e.g.



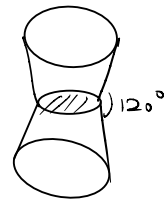
e.g. helicoid



e.g.

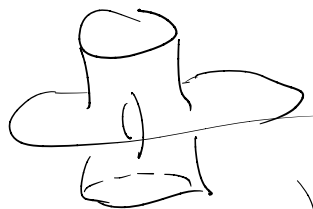


e.g.



e.g. Costa - Hoffman - Meeks

(3 ends, genus 1)



in general, if $M^3 \subseteq \mathbb{R}^3$ is minimal

$\leadsto$ Weierstrass representation

$$F(z) = \int (\varphi_1(z), \varphi_2(z), \varphi_3(z)) dz$$

$z_0 \rightarrow z \in \Sigma = \text{Riem surface}$

where $\sum_{i=1}^3 \varphi_i^2 = 0$, $\varphi_i = \text{meromorphic func.}$

(but need to solve period problem)

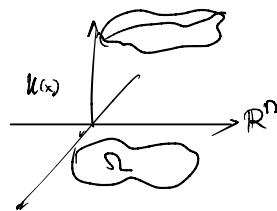
(why? $\Delta_M x_i = 0 \Leftrightarrow \Delta_\Sigma x_i = 0$
 $\parallel$ $F(\Sigma)$ F conformal.)

e.g. $M = \text{graph}_\Omega(u)$, for $u: \Omega^n \rightarrow \mathbb{R}$

= minimal

$$\Leftrightarrow u \text{ solves } \sum_{i=1}^n D_i \left(\frac{D_i u}{1 + |Du|^2} \right) = 0$$

$$\Leftrightarrow \sum_{i,j=1}^n \left(\delta_{ij} - \frac{D_i u D_j u}{1 + |Du|^2} \right) = 0$$



(MSE)

if $u: \Omega \rightarrow \mathbb{R}$ is Lipschitz solve to MSE $\Rightarrow u$ smooth

but if $u: \Omega \rightarrow \mathbb{R}^m$, then $\text{graph}(u)$ minimal $\Leftrightarrow u$ solves

$$\sum_{i,j=1}^n g^{ij} D_{ij} u^\alpha = 0, \quad g^{ij} = \text{inverse of } g_{ij} = \delta_{ij} + D_i u \cdot D_j u$$

Let u be Lipschitz but not smooth

e.g. $u: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by $u(x) = \frac{\sqrt{5}}{2} |x| \sigma\left(\frac{x}{|x|}\right)$, $\sigma: S^3 \rightarrow S^2$
 = Hopf - fibration

eg. holomorphic subvty of $\mathbb{C}^k$ (or Kähler manifold)

is area-minimizing in holom. class

$$M^l = \{ f_1(z_1, \dots, z_n) = \dots = f_{k-l}(z_1, \dots, z_n) = 0 \} \text{ for } f_i \text{ holomorphic.}$$

why? M is calibrated by (products of) Kähler form.

$\hookrightarrow$ if $z_i = x_i + \sqrt{-1} y_i$ and $\omega = dx_1 \wedge dy_1 + \dots + dx_k \wedge dy_k$

then $\sigma = \frac{1}{l!} \underbrace{\omega \wedge \omega \wedge \dots \wedge \omega}_{l\text{-times}}$ has properties:

$$① \sigma|_M = dV|_M.$$

$$② \sigma \leq 1, \text{ (i.e. } \sigma|_p \leq dV_p \text{)} \quad (\text{Wick's ineq.})$$

$$③ d\sigma = 0$$

" σ is calibration for M "

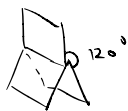
now if $M' - M = \partial u$ for spt u cpt.

then $\text{area}(M) = \int_M \sigma$ ← could ask $\partial u = M \cup M'$



$$= \int_{M'} \sigma + \int_M d\sigma = \int_{M'} \sigma \leq \int_{M'} dV_{M'} = \text{area}(M')$$

eg. Y and T cones



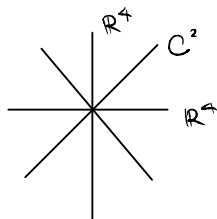
fix.

T -cone = cone of regular tetrahedron.

minimizing among Lips. deformations

(not homological minimize).

eg. Simon's cone. $C^7 \subset \mathbb{R}^8 = \{x, y \in \mathbb{R}^4 \times \mathbb{R}^4, |x|=|y|\}$



= cone over $S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}})$

= minimizing in homology

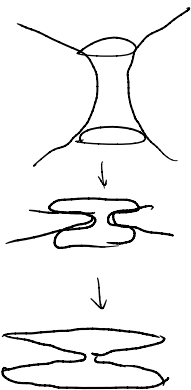
want "weak" notion of (minimal) surface

which ① allows for "singularities"

② gives us good compactness.

(analogy = Sobolev space)

eg.



scale down catenoid.

limit
→

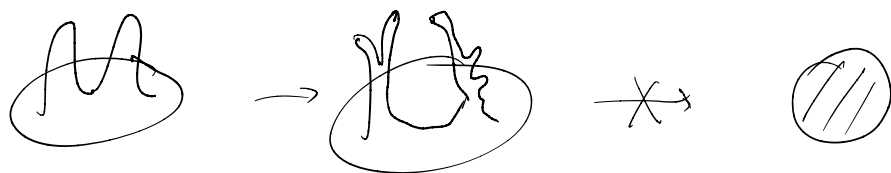


e.g. could try to solve Plateau by direct minimizing

↳ choose M_i with $\text{area}(M_i) \rightarrow \inf \{ \text{area}(M) : \partial M = P \}$
 $\partial M = P$

→ ??? take a limit of M_i ?

No! (for "classical" notions of limit)



Soon → rectifiable sets, varifolds.

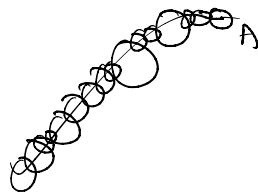
(assume "basic" measure theory)

Hausdorff measure: α -dim Hausdorff measure $\mathcal{H}^\alpha$ of set $A \subset \mathbb{R}^{n+m}$
 $= \mathcal{H}^\alpha(A) = "$ α -dim volume of A "
 $= \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^\alpha(A)$

where $\mathcal{H}_\delta^\alpha(A) = \inf \left\{ \sum_{j \in \mathbb{N}} w_\alpha \left(\frac{\text{diam } G_j}{2} \right)^\alpha : A \subset \bigcup_{j \in \mathbb{N}} G_j, \text{diam } G_j < \delta \right\}$

$w_\alpha = \text{volume of unit ball if } \alpha \in \mathbb{N}$

$=$ Borel regular measure on $\mathbb{R}^{n+m}$



Some basic properties ① $\mathcal{H}^{n+m} = \mathcal{L}^{n+m} = \text{Lebesgue measure}$

② for any $A \subset \mathbb{R}^{n+m}$, $\exists d = d(A) = "$ Hausdorff dimension of A "

s.t. $\mathcal{H}^\alpha(A) = 0$, $\alpha > d$ and $\mathcal{H}^\alpha(A) = \infty$ if $\alpha < d$

e.g. Cantor set $C \subset [0,1]$ has Hausdorff dim = $\frac{\log 2}{\log 3}$

③ if $F: \Omega \xrightarrow{\subset \mathbb{R}^n} \mathbb{R}^{n+m}$ C^1 , one-to-one.

($F(\Omega)$ = "submanifold")

then $\mathcal{H}^n(F(\Omega)) = \int_\Omega |JF| dV$, (area Formula)

rectifiable set (basic building block)

def'n $M \subset \mathbb{R}^{n+m}$ is countably n-rectifiable if $M = M_0 \cup \bigcup_{j=1}^{\infty} M_j$

where $\mathcal{H}^n(M_0) = 0$, and $M_j \subset N_j = C^1$ n-dim submanifold.

e.g. any C^1 -submanifold

could define $\tilde{M}_j = M_j \setminus (M_0 \cup M_1 \cup \dots \cup M_{j-1})$

e.g. any (subset) of a countable union of C^1 submanifolds.

$\hookrightarrow$ e.g. $M = \bigcup_{q_i \in \mathbb{Q}^m} q_i + S^n(\varepsilon^i)$

equivalent define (by Whitney extension thm) $M = M_0 \cup \bigcup_{j \in \mathbb{N}} F_j$ ($K_j \subset \mathbb{R}^n$)

where $F_j: \mathbb{R}^n \rightarrow \mathbb{R}^{n+m}$ Lipschitz. $K_j \subset \mathbb{R}^n$. $\mathcal{H}^n(M_0) = 0$.

intuition: rectifiable set $\sim$ Lipschitz submanifold + $\mathcal{H}^n$ -measure = 0
"weakest regularity to still do first-order analysis"

basic facts: ① Given $M \subset \mathbb{R}^{n+m}$ and if $\mathcal{H}^n \llcorner M$ locally-finite

(i.e., $\mathcal{H}^n(M \cap K) < \infty$ $\forall$ cpt K)

analogy Rade machess
thm

if

P

(Random meas. = dual to continous func. $\rightarrow$ weak $\ast$ convergence)

then $T_x M = T_x N_j \subset \mathcal{H}^2$ a.e. $x \in M_j$

↳ if $f: \mathbb{R}^{n+m} \rightarrow \mathbb{R}$, Lipschitz $(\Rightarrow f$ differentiable $\mathbb{L}^{n+m}$ -a.e.)

$\leadsto$ if $f: \mathbb{R}^m \rightarrow \mathbb{R}^k$ is Lipschitz.

$$U \in T_x M, \text{ so } D_x f: T_x M \rightarrow \mathbb{R}^k$$
$$(f: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^k \text{ Lips})$$

$$= \int \det Df(e_i) Df(e_j) \quad \{e_i\} - \text{ON basis of } T_x M$$

→ area formula: if $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}^{k(=n)}$ Lips and one-to-one,

then $f(M)$ is n -rectifiable and $\mathcal{H}^n(f(M)) = \int_M J_n f \, d\mathcal{H}^n$

aside: if f is not one-to-one

$$\text{then } \int_M J_n f \, d\mathcal{H}^n = \int_{f(M)} \mathcal{H}^0(f^{-1}(y) \cap M) \, d\mathcal{H}^n(y)$$

↑
multiplicity function
= # points in $f^{-1}(y) \cap M$

③ $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}^k$ Lips. then $f^{-1}(y) \cap M$ is $(n-k)$ -rectifiable for $\mathcal{H}^k$ a.e. $y \in f(M)$

Useful to allow for multiplicity

Vari-folds

defn integral varifold = "rectifiable set + multiplicity"

↳ precisely. $V = \underline{v}(M, \theta)$ = integral varifold if

M = countably n -rectifiable set

$$\theta \in L^1_{\text{loc}}(\mathcal{H}^n \llcorner M, \mathbb{Z}_+)$$

i.e. $\theta: M \rightarrow \mathbb{Z}_+$ s.t. $\int_{M \cap K} |\theta| \, d\mathcal{H}^n < \infty$, $\forall$ cpt K

e.g. $M = C^1$ -submanifold. $\theta \in \mathbb{Z}^+$ constant

$\hookrightarrow \underline{V}(M, \theta)$ "M except area is not θ -times original area"

can interpret V as Radon measure in Grassmannian bundle
 $= \mathbb{R}^{n+m} \times G(n, n+m)$

$\hookrightarrow$ if $\varphi(x, S) \in C_c^\circ(\mathbb{R}^{n+m} \times G(n, n+m))$

then $V(\varphi) \stackrel{\text{def}}{=} \int_M \varphi(x, T_x M) \theta(x) d\mathcal{H}^n$

$\uparrow$
 unoriented?
 $n = \text{planes in } \mathbb{R}^{n+m}$

OR $V = (\theta \mathcal{H}^n \llcorner M) \otimes S_{T_x M}$

(general varifold = Radon measure on $\mathbb{R}^{n+m} \times G(n, n+m)$)

useful because gives notion of convergence and compactness

def. (integral) varifolds $V_i \rightarrow V \Leftrightarrow V_i \rightarrow V$ as Radon

measure in Grassmannian $\Leftrightarrow \int \varphi(x, S) dV_i(x, S)$

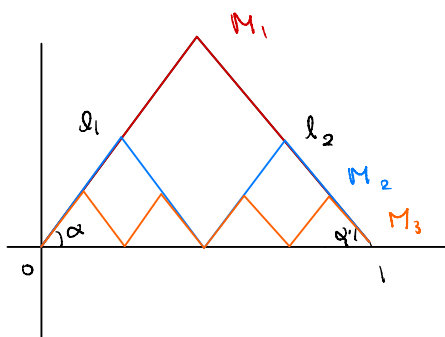
$$\downarrow$$

$$\int \varphi(x, S) dV(x, S) \quad \forall \varphi(x, S)$$

can also interpret as Radon measure on $\mathbb{R}^{n+m}$

$\hookrightarrow$ "mass measure" $\|V\|(\mathcal{F}(x)) = \int_M f(x) \theta(x) d\mathcal{H}^n$

OR $\|V\|(A) = \int_{M \cap A} \theta(x) d\mathcal{H}^n(x)$



sequence $M_i = 1$ -rectifiable sets

$$\longrightarrow V_i = |M_i| = \int (M_i, 1) = \text{integral}$$

1-varifolds ($\theta \equiv 1$)

$$\rightsquigarrow \|V_i\|(A) = \mathcal{H}'(M_i \cap A) \\ = \text{length of curve in } A$$

$$\|V_i\|(\mathbb{R}^2) = \text{length}(l_1) + \text{length}(l_2) \\ = \frac{1}{\cos \alpha}$$

def. if M = rectifiable
then $|M| \stackrel{\text{def}}{=} \int (M, 1)$

$$V_i = (\mathcal{H}' \llcorner M_i) \otimes \delta_{T_x M_i} \quad \left(\text{So } V_i(\varphi(x, S)) = \int_M \varphi(x, T_x M_i) d\mathcal{H}' \right)$$

what happened if $i \rightarrow \infty$?

$$\|V_i\|(\varphi(x)) = \int_M \varphi(x) d\mathcal{H}' = \int_0^1 \varphi(\gamma_i(t)) |\dot{\gamma}_i| dt$$

$$i \rightarrow \infty \downarrow \quad \gamma_i(t) : [0, 1] \rightarrow \mathbb{R}^2 = \text{peano for } M_i$$

$$\int_0^1 \varphi(x) \frac{1}{\cos \alpha} d\mathcal{H}'$$

$$\text{so } \|V_i\| \longrightarrow \frac{1}{\cos \alpha} \mathcal{H}' \llcorner [0, 1]$$

$$\text{varifold measure } V_i(\varphi(x, S)) = \int_M \varphi(x, T_x M_i) d\mathcal{H}'$$

$$\longrightarrow \int_0^1 \frac{\varphi(x, l_1) + \varphi(x, l_2)}{2 \cos \alpha} d\mathcal{H}'$$

$$\text{i.e. } V_i \longrightarrow \mathcal{H}' \llcorner [0, 1] \otimes (\delta_{l_1} + \delta_{l_2}) / 2 \cos \alpha \quad \text{not an}$$

integral varifold

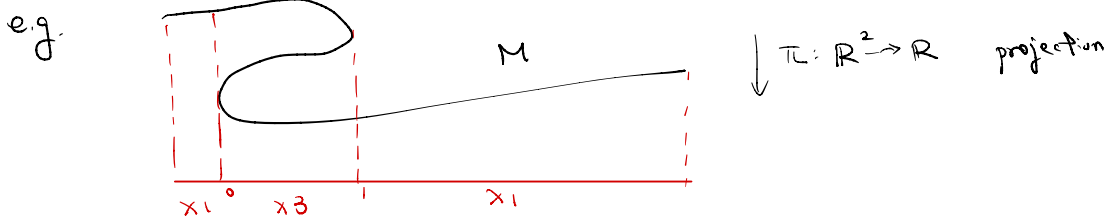
varifolds useful to understand variational problem.

pushforward $f_{\#} V$ of integral varifold = $V(f(M), \theta_f(y) = \sum_{x \in f^{-1}(y)} \theta(x))$

($f: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^k$ Lips. $f|_{\text{spt } V}$ proper)

$$(f_{\#} V)(\varphi(x, S)) = \int \varphi(f(x), Df|_x T_x M) J_M f \theta(x) d\mathcal{H}^n$$

(if f is one-to-one)



$$\pi_{\#} |M| = \mathcal{H}^1 \llcorner (-\infty, 0] + 3 \mathcal{H}^1 \llcorner [0, 1] + \mathcal{H}^1 \llcorner [1, \infty)$$

"multiplicity is preserved under pushforward"

def. first variation of varifold V in direction X ($= C_c^1$ vector field in $U \subset \mathbb{R}^{n+m}$)

$$= \delta V(X) = \left. \frac{d}{dt} \right|_{t=0} \| \varphi_t \# V \| (U)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \int_{M \cap U} J_M \varphi_t \theta d\mathcal{H}^n = \int_{M \cap U} \text{div}_{T_x M}(X) \theta(y) d\mathcal{H}^n(y)$$

(some computation)

$$= \int \text{div}_V(X) d\|V\|$$

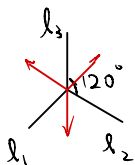
$\varphi_t = t$ -param family

iff diffeos.

$$\varphi_0 = \text{id}, \quad \left. \frac{\partial}{\partial t} \right|_{t=0} \varphi_t = X$$

Lecture 2.

V is stationary if $\delta V = 0$,

e.g.  if $V = |M|$ is smooth $\Rightarrow \delta V(x) = \int_M \operatorname{div}_{T_x M}(x) d\mathcal{H}^n$

$$= - \int_M H_M \cdot x d\mathcal{H}^1 + \int_{\partial M} y \cdot x d\mathcal{H}^{n-1}$$

$$\delta V(x) = \sum_{i=1}^3 - \int_{\ell_i} H_{\ell_i} \cdot x d\mathcal{H}^1$$

Conormal
(one word)

$$+ \sum_{i=1}^3 \int_{\partial \ell_i} y \cdot x d\mathcal{H}^0$$

" since $y_1 + y_2 + y_3 = 0$

e.g. if M smooth, stationary, then $\delta |M|(x) = \int_{\partial M} x \cdot y d\mathcal{H}^{n-1}$

stationary away from ∂M

e.g. if V n -dim in $\mathbb{R}^n$, then $\delta V = 0 \Leftrightarrow V = \theta |\mathbb{R}^n|$

$$\hookrightarrow \|V\|(A) = \int_A \theta d\mathcal{H}^n = \int_A \theta d\mathcal{L}^n$$

(basically just $\theta: \mathbb{R}^n \rightarrow \mathbb{Z}^+$)

b/c $\delta V(x) = \int \operatorname{div}(x) \theta d\mathcal{L}^n$. choose $x = \varphi e_i$

$$\Rightarrow \int (D_i \varphi) \theta = 0 \quad \forall \varphi \in C_c^1(\mathbb{R}^n)$$

$\Rightarrow D_i \theta = 0$ distributionally

$\Rightarrow \theta = \text{const.}$

in general, δV locally bounded if $\delta V(x) \leq C(w)|x|_{C^0(U)}$
in U

$\forall X$ cptd in W , $\forall W \subset \subset U$

$$\Rightarrow \delta V \in C_c^0(U, \mathbb{R}^{n+m})^*$$

$\leadsto$ Radon-Nikodym
$$\delta V(x) = - \int H_v \cdot x \, d\|V\| + \int x \cdot \eta_v \, d\sigma_v$$

$$\sigma_v \perp \|V\|,$$

$$|\eta_v| = 1$$

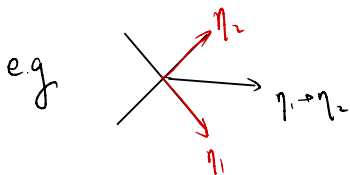
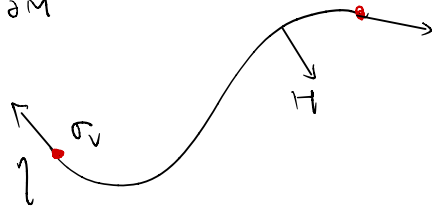
generalized
mean curvature

generalized
conormal

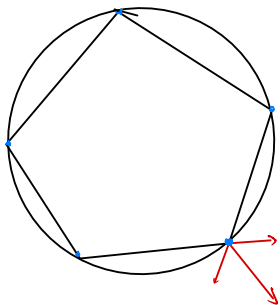
generalized
boundary

e.g. $M = \text{smooth}$, $V = |M|$

$$\Rightarrow \delta |M|(x) = - \int_M H \cdot x \, d\mathcal{H}^n + \int_{\partial M} H \cdot \eta \, d\mathcal{H}^{n-1}$$



e.g.



$M_N = \text{regular } N\text{-gen in } B_1$

$$V_N = |M_N|, \quad H_V \equiv 0$$

$\eta_V \perp S'$ "V has free boundary in S' "

$$\sigma_V = \left(2\pi^2 \sum_{\substack{\text{vertices} \\ P}} P \right) 2 \sinh\left(\frac{\pi}{N}\right)$$

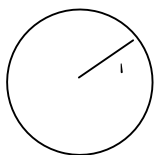
$$N \rightarrow \infty$$

$$V_N \rightarrow |S'|$$

$$\begin{aligned} H &\neq 0 \\ \sigma &= 0 \end{aligned}$$

$$(\text{but } \delta V_i \rightarrow \delta V)$$

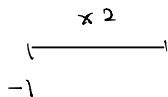
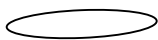
e.g.



$$\begin{aligned} H &\neq 0 \\ \sigma &= 0 \end{aligned}$$

$$M_i$$

$$V_i = |M_i| \longrightarrow 2|[1,1]|$$



$$\begin{aligned} H &= 0 \\ \sigma &\neq 0 \end{aligned}$$

Allard's compact thm if V_i sequence of integral varifolds

and $\|V_i\|$, δV_i are uniformly locally bounded.

then (subseq.) $\exists$ integral varifold V

$$\text{s.t. } V_i \rightarrow V \quad (\text{as varifold})$$

$$(\Rightarrow \|V_i\| \rightarrow \|V\|)$$

naïves



$\sigma_{V_i} \rightarrow \infty$, so δV_i unbdd.

Varifolds good for variational problems

but ① notion of boundary not super useful

② measure theoretic convergence can be "too strong"



$$\text{area}(M_i) \rightarrow \inf \{ \text{area } M \}$$

$$\partial M_i^n := \Gamma$$

$$\partial M = T^n \}$$

could arrange $|M_i| \rightarrow c \mathcal{L}^{m+n} \llcorner U$.

Currents

def integral-multiplicity rectifiable n -current

= "integral varifold + orientation"

precisely

$$T = \llcorner (M, \theta, \vec{e})$$

n -rectifiable
set

$\mathbb{Z}^+$ multiplicity
locally $\mathcal{H}^n$ LM
integral-function

$\mathcal{H}^n \llcorner M$ -measure
choice of orienting
 n -vector for
 $T_x M$

e.g. $M^n = \mathbb{C}^1$ oriented submanifold $\rightsquigarrow [M] = \tau (M, \theta \equiv 1, \underbrace{dV_M}_{e_1 \wedge \dots \wedge e_n})$

T can act on cply sp'd n -forms $w \in \mathcal{D}^n(U)$

space of cply sp'd n -form in U .

$$\text{by } T(w) = \int_M \langle w, \vec{\tau} \rangle \theta d\mathcal{H}^n = \int_M w(e_1, \dots, e_n) \theta d\mathcal{H}^n$$

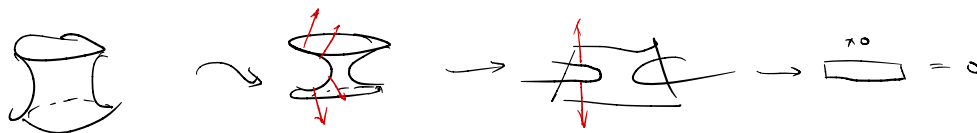
$$\text{if } \vec{\tau} = e_1 \wedge \dots \wedge e_n$$

general current = dual to $\mathcal{D}^n(U)$ (or $\mathcal{D}^n(\mathbb{R}^{n+m})$)

↓
gives convergence + compactness.

def: $T_i \rightarrow T$ as currents $\Leftrightarrow T_i(w) \rightarrow T(w) \quad \forall w \in \mathcal{D}^n(U)$
(in U) $\Leftrightarrow T_i \rightarrow T$ in weak- $*$
(dual to $\mathcal{D}^n(U)$)

e.g. scale down oriented cotenoid. $T_i = \left[\frac{\text{oriented cotenoid}}{r_i} \right] \rightarrow 0$



mass not continuous under limit

cancellation is possible!

given integer-mult rectifiable current T , $|T|$ = underlying integral varifold

$$= \nu \llcorner (M, \theta)$$

can also express mass

$$\|T\|(A) = \int_A \theta \, d\mathcal{H}^n = \text{mass measure}$$

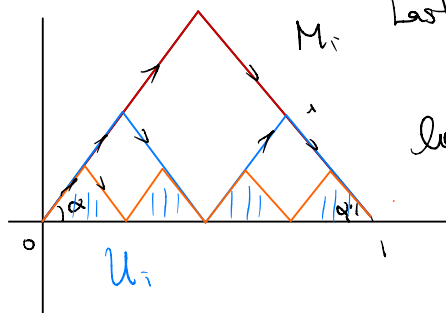
$$\text{as } \|T\|(w) = \sup \{ T(w), |w| \leq 1, \text{ spt } w \subset W \}$$

(also works for general currents)

$$\Rightarrow \text{if } T_i \rightarrow T, \text{ then } \|T\| \leq \liminf \|T_i\|$$

"linear semicontinuous"

eg.



$$\text{Last time: } \|M_i\| \rightarrow \frac{\mathcal{H}^1 \llcorner [0, 1]}{\cos \alpha} \text{ as measure}$$

$$\text{let } [M_i] \rightarrow [0, 1] \text{ as current}$$

$$\text{total mass} = 1 < \frac{1}{\cos \alpha} = \text{total mass of } M_i$$

nice way to see the convergence, $-M_i + [0, 1] = \partial U_i$

$$\text{and area } (U_i) \rightarrow 0.$$

$$[M_i](w) = \int_{M_i} \langle w, \tau \rangle \, d\mathcal{H}^1 = \int_{M_i} w = \int_{[0, 1]} w + \int_{U_i} dw$$

$\uparrow$ 1-form $\downarrow$ 0

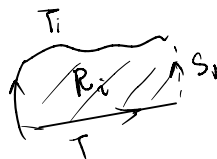
$$\rightarrow \int_{[0, 1]} w = [0, 1](w)$$

general principle ("flat norm")

(if T_i, T reasonable)

$$T_i \rightarrow T \Leftrightarrow T_i - T = \partial R_i + S_i$$

where $\|R_i\| + \|S_i\| \rightarrow 0$



def. $T = n$ -current $\leadsto \partial T = (n-1)$ -current

defined by $\partial T(w) \stackrel{\text{def}}{=} T(dw)$

good: if $T_i \rightarrow T$ as currents, then $\partial T_i \rightarrow \partial T$ as currents
(since $T_i(dw) \rightarrow T(dw)$.)

bad: ∂T need not be rectifiable / integer-multiplicity.

e.g., if $T = \text{integer-multiplicity (rectifiable) } n\text{-current in } \mathbb{R}^n$

$$\begin{aligned} \text{then } T(w = f dx^1 \wedge \dots \wedge dx^n) &= \int_{\mathbb{R}^n} \frac{\langle dx^1 \wedge \dots \wedge dx^n, e_1 \wedge \dots \wedge e_n \rangle}{\pm 1} f \, d\mathcal{L}^n \\ &= \int_{\mathbb{R}^n} f \tilde{\theta} \, d\mathcal{L}^n, \text{ where } \tilde{\theta} = \begin{cases} +1 \\ -1 \end{cases} \end{aligned}$$

depending on orientation of $\mathbb{R}^n$

$$\begin{aligned} \partial T(w = *X) &= T(dw = d(*X)) = T(d \lrcorner X \, dx^1 \wedge \dots \wedge dx^n) \\ &= \int_{\mathbb{R}^n} d \lrcorner X \, \tilde{\theta} \, d\mathcal{L}^n \end{aligned}$$

$\uparrow$ $\nwarrow$
 $(n-1)$ -form vector field

$$\partial T = 0 \Leftrightarrow \int d \lrcorner X \, \tilde{\theta} \, d\mathcal{L}^n = 0 \quad \forall X$$

$$\Leftrightarrow (\text{take } X = \varphi e_i) \quad \tilde{\theta} = \text{const}$$

$$\Leftrightarrow T = \tilde{\theta} [\mathbb{R}^n] \quad (\text{constancy thm for currents})$$

~~$$\| \partial T \| < \infty \Leftrightarrow \int \text{div}(X) \tilde{\theta} \, d\mathcal{L}^n \leq C \|X\|_{\infty}$$~~

boundary rectifiability if $T = \text{integer-mult. rect. } n\text{-current in } \mathbb{R}^{n+m}$

if $\| \partial T \|$ locally finite $\Rightarrow \partial T = \text{integer-mult. rect. } (n-1)\text{-current}$

e.g. (could) (if $T = \text{integer-mult. } n\text{-current in } \mathbb{R}^n$)

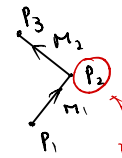
$$\| \partial T \| < \infty \Leftrightarrow \int \text{div}(X) \hat{\theta} d\mathcal{L}^n \leq c(\omega) |X|_{C^0(\omega)} \quad \forall X \text{ sptd in } W$$

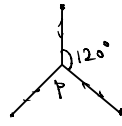
$$\Leftrightarrow \hat{\theta} \text{ has loc. bounded variation "BV}_{loc}" \quad \forall W \subset \subset \mathbb{R}^n$$

e.g. if $M = \text{oriented } C^1 \text{ submanifold}$, then $\partial[M] = [\partial M]$

$$(\text{b/c } \int_M d\omega = \int_{\partial M} \omega)$$

e.g.  $\partial[M] = [q] - [p]$

e.g.  $\partial[M_1 \cup M_2] = [p_3] - [p_1] + [p_2] - [p_1] = [p_3] - [p_1]$
not boundary point !!!

e.g.  (as current)
as boundary, p has multiplicity
 $\Rightarrow$ always a boundary point.

def. $T = \text{integral } n\text{-current}$ if both T and ∂T are $n\text{-mult.}$ and $(n-1)\text{-mult.}$ respectively

aside $\text{spt } T \stackrel{\text{def}}{=} \text{spt } \|T\|$ as measures

= smallest closed set C s.t. $\|T\|(\mathbb{R}^{n+m} \setminus C) = 0$

if $T = \tau(M, \theta, \vec{\tau}) \rightsquigarrow$ many choice for M (can change on set

of measure 0)

bad e.g. $\rightarrow$ if $M = \bigcup_{q_i \in \mathbb{Q}^{n-m}} q_i + 2^{-i} S^n \Rightarrow \mathcal{H}^n(M \cap K) < \infty$
 $\nrightarrow \text{cpt } K$

$\leadsto [M] = \text{integer-mult. rect. } n\text{-current}$

But! $\text{spt } T = \bar{M} = \mathbb{R}^{nm}$

(maybe up to slightly shifting spheres) $\downarrow$

Compactness thm if $T_i = \text{integral } n\text{-currents}$ and $\|T_i\|$

and $\|\partial T_i\|$ are loc uniformly bdd. then (subsequence) $\exists$

integral n -currents T , s.t. $T_i \rightarrow T$ as currents

($\Rightarrow \partial T_i \rightarrow \partial T$)

"Solution" of Plateau in class of integral currents

fix $P^{n-1} = \text{closed (oriented) submanf} \subset \mathbb{R}^{n+m}$

define $\mathcal{I} = \{ \text{integral } n\text{-currents } T, \text{ s.t. } \partial T = [P] \} \neq \emptyset$

choose seq $T_i \in \mathcal{I}$ with $\|T_i\|(\mathbb{R}^{n+m}) \rightarrow \inf \{ \|T\|(\mathbb{R}^{n+m}) : T \in \mathcal{I} \}$

$\Rightarrow \|T_i\|$ unif bdd

$\|\partial T_i\| = \|\partial T\|$ uniformly bdd.

($< \infty$ since can take one
one P)



(subseq) $\exists$ integral n -current T s.t. $T_i \rightarrow T$ as currents

$$(\Rightarrow \partial T_i \rightarrow \partial T)$$

so $T \in I$

$$\begin{array}{c} \parallel \\ [T] \end{array} \quad \begin{array}{c} \parallel \\ [T] \end{array}$$

$$\text{and } \|T\| < \liminf_{i \rightarrow \infty} \|T_i\|$$

$$\Rightarrow \|T\|(\mathbb{R}^{n+m}) \leq \inf \{ \|T\|(\mathbb{R}^{n+m}), T \in I \}$$

also $\geq$ since $T \in I$

$\Rightarrow \|T\|(\mathbb{R}^{n+m})$ has least possible mass for $\partial T = [P]$

note T need not be unique!

(generically T is unique, or if T calibrated)

Q: how smooth is minimizer T ?

↳ "regularity theory" $\rightarrow$ next lecture

Sets of locally-finite perimeter (Caccioppoli sets)

↳ special case of current (codim=1, multiplicity=1)

def $E \subset \mathbb{R}^n$ = set of loc. finite perimeter in $U \subset \mathbb{R}^n$

if $\sup \left\{ \int_E \operatorname{div} X \, d\mathcal{L}^n; |X|_{\mathcal{L}^0} \leq 1, \operatorname{spt} X \subset W \subset U \right\} < \infty$

if ∂E is smooth $\Rightarrow \int_E \operatorname{div}(X) d\mathcal{L}^n = \int_{\partial E} X \cdot \eta d\mathcal{H}^{n-1}$

$\nwarrow$ conormal

(E to have reasonable " ∂E ") $\leq \mathcal{H}^{n-1}(\partial E \cap W) |X|_C$

$\Leftrightarrow \|\partial[E]\|$ loc. finite.

because (b/c) $\partial[E](W + X) = [E](dW = \operatorname{div} X dx^1 \wedge \dots \wedge dx^n)$

$$= \int_E \operatorname{div}(X) d\mathcal{L}^n$$

$$\sup_{|W| \leq 1} \partial[E](W) < \infty \Leftrightarrow \sup_{|X| \leq 1} \int_E \operatorname{div} X < \infty$$

" $\partial[E] = [\partial^* E] = \subseteq (\partial^* E, \cdot, \eta)$ "

De Giorgi's structure thm if $E \subset \mathbb{R}^n$ has locally finite perimeter

then $\exists$ set $\partial^* E =$ "reduced boundary of E "

and $\mathcal{H}^{n-1} \llcorner \partial^* E$ - measurable choice of "conormal" $\eta + \bar{\tau}_x \partial^* E$

so that $\int_E \operatorname{div}(X) d\mathcal{L}^n = \int_{\partial^* E} X \cdot \eta d\mathcal{H}^{n-1}$

perimeter if E in $A = P(E, A) = \mathcal{H}^{n-1}(\partial^* E \cap A)$

$$\equiv \|\partial[E]\|(A)$$

cptness thm if $E_i =$ seq. of sets of loc. finite perimeter,
and $P(E_i)$ local unif bdd then (subseq) $\exists E = \text{loc.}$

finite perimeter s.p. $E_i \rightarrow E$ in $L^1_{loc}(\mathbb{R}^n)$

"multi: - one property is preserved"

i.e. $1_{E_i} \rightarrow 1_E$ in $L^1_{loc}(\mathbb{R}^n)$

i.e. $\int (\chi_E \Delta E_i) \cap W \rightarrow 0 \quad \forall W \subset\subset U$

$\Rightarrow [E_i] \rightarrow [E]$ as currents
 $\partial[E_i] \rightarrow \partial[E]$

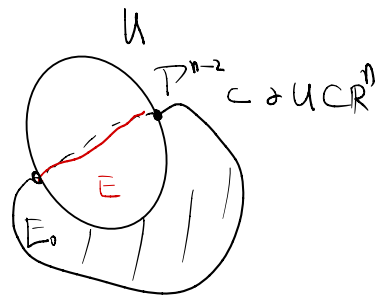
could "solve" Plateau in countable steps

$\hookrightarrow$ find E in $I = \{E \text{ of loc. finite perimeter, s.t. } E|_U = E_0|_U\}$

$\leadsto$ minimize perimeter in I

but if $T = \text{codim-1 current}$ $\partial T = 0$
 integral

$\Rightarrow T = \sum \partial[E_i]$ $E_i = \text{local finite perimeter}$



Lecture 3

(mass-minz integral n -current)

spoile: if T is $\text{codim}=1$, then $T = \text{smooth}$, embedded
away from "singular set" $\text{sing } T$ closed.

$$\dim_{\mathbb{H}}(\text{sing } T) \leq n-1 \quad \text{eg. Simon's cone } C \subset \mathbb{R}^8$$

in higher codim , $\dim_{\mathbb{H}}(\text{sing } T) \leq n-2$.

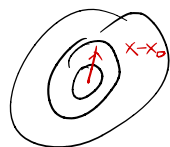
(embedded, smooth away from $\text{sing } T$) e.g. $\{w^3 = z^2\} \subset \mathbb{C}^2$, $z = \pm w^{3/2}$

aside: $\exists$ other solutions to Plateau (Douglas - Radó '30s,

Caccioppoli '30s, Federer - Fleming / Alogoskoufis (currents) '70s)
(De Giorgi)

finds an immersed minimal disk
bounding on S^1 .

Regularity of stationary varifolds



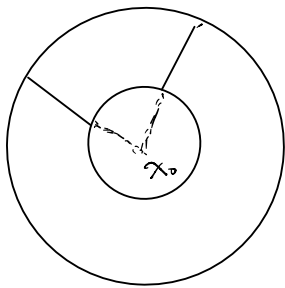
monotonicity formula if $V =$ stationary integral n -varifold
in $U \subset \mathbb{R}^{n+m}$ and $B_s(x_0) \subset B_r(x_0) \subset U$

$$\text{then } \frac{\|V\|_{B_r(x_0)}}{\omega_n r^n} - \frac{\|V\|_{B_s(x_0)}}{\omega_n r^n} = \frac{1}{\omega_n} \int_{B_r(x_0) \setminus \overline{B_s(x_0)}} \frac{|\pi_{T_x V}^{\perp}(x-x_0)|^2}{|x-x_0|^{n+2}} d\|V\|_x \geq 0$$

hence $(H)_V(x_0, r) = \frac{\|V\|_{B_r(x_0)}}{\omega_n r^n} \nearrow \text{ in } r$

and $(H)_V$ constant in $[s, r] \iff x - x_0 \in T_x V$, for $\|V\|$ -a.e.
 $x \in B_r(x_0) \setminus \overline{B_s(x_0)}$

$\iff V$ in $B_r(x_0) \setminus \overline{B_s(x_0)}$ = cone centered @ x_0 .
(i.e. dilation-invariant)



Proof: Take $x_0 = 0$, $V = |M| =$ smooth minimal surface.
just show $\frac{\|V\|_{B_r(0)}}{\omega_n r^n} \nearrow$

plug in vector field $X(\vec{x}) = \vec{x}$ into $\int_M \operatorname{div} X \, d\mathcal{H}^n = \int_{\partial M} \langle X, \nu \rangle \, d\mathcal{H}^{n-1}$

$$\hookrightarrow \int_{M \cap B_r(0)} \operatorname{div}_M(\vec{x}) \, d\mathcal{H}^n = \int_{M \cap \partial B_r(0)} \vec{x} \cdot \eta \, d\mathcal{H}^{n-1} \leq r \cdot \operatorname{area}(M \cap \partial B_r)$$

$$n \operatorname{area}(M \cap B_r) = \int_{M \cap B_r} \sum D_{e_i} \vec{x} \cdot e_i$$

$$\Rightarrow \operatorname{area}(M \cap B_r) \leq \frac{r}{n} \operatorname{area}(M \cap \partial B_r)$$

= area of cone over minimize,



OTOM: coarea formula $\operatorname{area}(M \cap B_r) = \int_0^r \int_{M \cap \partial B_s} \frac{1}{|\nabla |x||} \, d\mathcal{H}^{n-1} \, ds$

$$\Rightarrow \frac{d}{dr} \operatorname{area}(M \cap B_r) = \int_{M \cap \partial B_r} \frac{1}{|\nabla |x||} \, d\mathcal{H}^{n-1}$$

$$|\nabla |x|| = 1$$

$$\geq \operatorname{area}(M \cap \partial B_r)$$

$$\geq \frac{n}{r} \operatorname{area}(M \cap B_r)$$

$$\Rightarrow \frac{d}{dr} \left(\frac{\operatorname{area}(M \cap B_r)}{r^n} \right) \geq 0.$$

"basic" consequences of monotonicity

① $\mathcal{H}_V(x) := \lim_{r \rightarrow 0} \mathcal{H}_V(x, r)$ both exist $\forall x$

$$\mathcal{H}_V(\infty) := \lim_{r \rightarrow \infty} \mathcal{H}_V(x, r)$$

② $\mathcal{H}_V(x) = \mathcal{O}(x) \in \|V\|$ -a.e. x recall $V = \underline{v}(M, \theta)$

③ $x \mapsto \mathcal{H}_V(x)$ upper semi-continuous.

④ $\text{spt } V = \text{spt } \|V\| = \{ \mathcal{H}_V(x) \geq 1 \} = \text{closed}$

⑤ $V = \underline{v}(\text{spt } V, \mathcal{H}_V)$ "spt V looks n -dim"

⑥ if $V_i = \text{stationary integral } n\text{-varifold}$,

and $V_i \rightarrow V$ (as varifolds)

then $\text{spt } V_i \rightarrow \text{spt } V$ in loc Hausdorff distance

tangent cones

fact: $V = \text{stationary integral } n\text{-varifold}$, $x \in \text{spt } V$

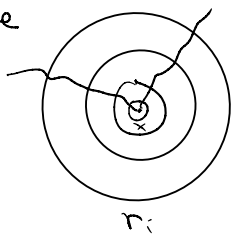
$r_i \rightarrow 0$, then (subseq)

$$\frac{V - x}{r_i} \longrightarrow V' = \text{stationary, integral,}$$

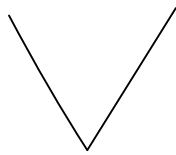
dilation invariant.

correct notation is $(\eta_{x, r_i})_{\#} V$ where $\eta_{x, r}(y) = \frac{y-x}{r}$

picture



$$V \xrightarrow[\text{(subseq)}]{\frac{V-x}{r_i}} V'$$



"tangent cone of V at x "

note: V' need not be unique (ind of r_i)

(no counterexamples through)

moral: "singularities modelled on cones"

why $\exists$ tangent cones? $V_i := (\gamma_{x, r_i})_{\#} V = \text{stationary i.e. } \delta V_i = 0$

$$\text{and } \|V_i\|_{B_R(0)} = \frac{1}{r_i^n} \|V\|_{B_{R r_i}(x)}$$

$$J_\gamma = \frac{1}{r^n} \quad \gamma^{-1}(B_R)$$

Allord compactness

$$\Rightarrow (\text{subseq}) \quad V_i \rightarrow V' \text{ and } \delta V_i \rightarrow \delta V' \\ \text{so } \delta V' = 0.$$

$$= \mathbb{H}_V(x, R r_i) \mathbb{R}^n \\ \leq \mathbb{R}^n \mathbb{H}_V(x, 1)$$

local uniformly bounded.

aside: since $V_i \rightarrow V$ as measures
on Grassmannian bundle

$\Rightarrow$ first-order quantities converge.

$$\Rightarrow \delta V_i(x) \rightarrow \delta V(x)$$

since $(x, S) \mapsto \text{div}_S(X)|_x$ is C^0 func. on Grassmannian.

$$\frac{\|V_i\|_{B_R}}{\omega_n R^n} = \mathbb{H}_V(x, Rr_i) \xrightarrow{r_i \rightarrow 0} \mathbb{H}_V(x)$$



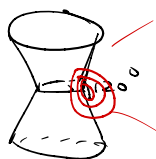
$$\mathbb{H}_{V'}(0, R), \quad \text{so } V' \text{ satisfies } \mathbb{H}_{V'}(0, R) = \mathbb{H}_V(x) \quad \forall R.$$

$$\Rightarrow V' = \text{cone center } \mathcal{C} \, o.$$

e.g. $V = |M| = \text{smooth}$, then every tangent cone $\mathcal{C}_x = |T_x M|$



e.g.



γ -cone

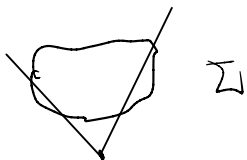
e.g. for $\|V\|$ -a.e. $x \in \text{supp } V$

$$\text{target cone} = \mathcal{O}(x) |T_x M|$$

e.g. of minimal cone

if $M = \text{cone over smooth link } \Sigma^{n-1} \subset S^{n+m-1}$

then $M = \text{minimal in } \mathbb{R}^{n+m} \Leftrightarrow \Sigma^{n-1} \text{ minimal in } S^{n+m-1}$



Σ

eg cone are geodesic net

eg cone are $S^k(\frac{1}{\sqrt{2}}) \times S^k(\frac{1}{\sqrt{2}})$

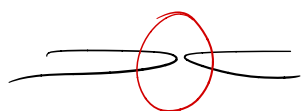
many examples!!!

Allard regularity thm if $V =$ stationary integral
 n -varifold in B_1^{n+m} ,

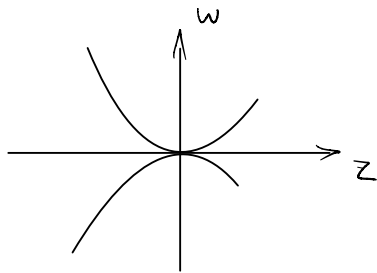
if V is "close" to n -plane $|P|$ with mult-one
 then in $B_{1/2}$, $V =$ smooth, mult-one, graph over P

i.e. $\exists$ smooth $u: P \cap B_{1/2} \rightarrow P^\perp$

s.t. $V \llcorner B_{1/2} = |\text{graph}(u)| \llcorner B_{1/2}$

noneg  catenoid

noneg $\{w^3 = z^2\} \subset \mathbb{C}^2$



"close" ? one option $0 \in \text{spt } V, \int_{B_1} \text{dist}(x, p)^2 d\|V\|(x) < \varepsilon^2$
 and $(H)_V(0,1) \leq \frac{3}{2}$

another option $0 \in \text{spt } V, (H)_V(0,1) \leq 1 + \varepsilon$

↑
 intuition: if $0 \in \text{spt } V$

and $(H)_V(0,1) = 1$

$\Rightarrow 1 \leq (H)_V(0) \leq (H)_V(0,1) \leq 1$

$\Rightarrow V = \text{cone}, \text{ with } (H)_V(\infty) = 1$

but $(H)_V(x) \geq 1 \quad \forall x \in \text{spt } V$

$\Rightarrow V$ cone over every $x \in \text{spt } V$.

$\Rightarrow V = \text{plane (with mult-one)}$

"just need understood cones"

consequence

① V stationary integral, $x \in \text{spt } V$

if any tangent cone @ $x = \text{mult-one plane}$

$\Rightarrow V = \text{smooth mult-one near } x$.

② if $V_i \rightarrow V$ all stationary integral

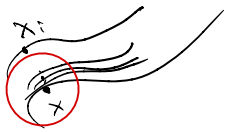
$x_i \in \text{spt } V_i \longrightarrow x \in \text{spt } V$

if V regular, mult-one near x



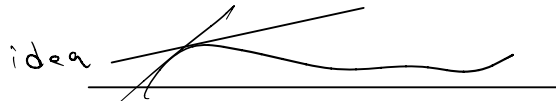
"singular points limit
 singular points"

$\Rightarrow V_i$ regular near x_i for $i \gg 1$



$$\textcircled{H}_V(x, r) < +\infty \Rightarrow \textcircled{H}_{V_i}(x_i, r) < +\infty \quad (i \gg 1)$$

why Allard?



approx $\text{spt } V$ by $\text{graph}(u)$

$u: \mathbb{R}^n \rightarrow \mathbb{R}^m$ basically solves $g^{ij} D_{ij}^2 u = 0$

but $g^{ij} D_{ij}^2 u = \Delta u + D u \times D^2 u \approx 0$

$u \approx 0 \Rightarrow u \approx v$ for $\Delta v = 0$.

$\Rightarrow v = \text{"very regular"} = \sum C_k r^k \varphi_k^{(0)}$

$\Rightarrow$ good decay for u

$\Rightarrow$ good decay for v

key estimate: "excess decay"

$\forall x \in \text{spt } V, \exists \text{ plane } P_x$

s.t.
$$r^{n-2} \int_{B_r(x)} \text{dist}(y, x, P_x)^2 d\|V\|(y) \approx r^{2\alpha} \quad (r \rightarrow 0)$$

$B_r(x)$ " $C^{1,\alpha}$ decay "

for general stationary integral varifold.

$\left\{ \begin{array}{l} \text{Allard} \Rightarrow \text{cpt } V = \text{smooth on open, relatively dense set} \\ \rightarrow (\text{if assume } V \text{ has multiplicity} = 1) \end{array} \right.$

$$\Downarrow \\ H^n(\text{sing } V) = 0$$

Regularity of minimizes

$\rightarrow$ can rule out non-planar cones in certain dimensions.

def "integral n -current T is (mass-) minimizing in U if $\|T\|(W) \leq \|T + \partial S\|(W) \quad \forall \text{ sp } S \subset W, \quad + W \subset U.$

"minimizing in homology.

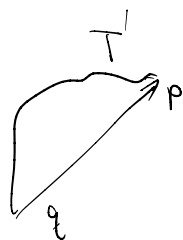
eg any 1-d minimizing integral current = union of disjoint.

lines or line segments or rays

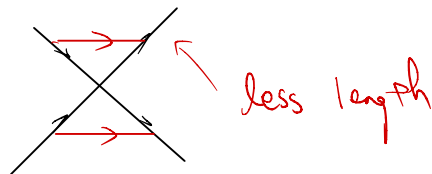
why?

easy case $\partial T = [p] - [q]$

$\rightarrow V = \frac{p-q}{|p-q|}$ calibration



why not intersect?



compactness of minimize: if $T_i = \text{mass-minimize}$ and

$\|T_i\|, \|\partial T_i\|$ loc unif. bdd.

then (subseq) $\nexists$ minimize T , s.t.

$$T_i \rightarrow T$$

"no mass d"

$$\|T_i\| \rightarrow \|T\|$$

(and $\text{spt } T_i \rightarrow \text{spt } T$ in loc. Hausdorff distance away from $\text{spt } \partial T$).

consequence $\nexists$ minz tangent cones

$$\frac{T - x}{r_i} \xrightarrow{\text{(subseq)}} T' = \text{minz integral, conical, } n\text{-current}$$

goal: show only minz cones (in certain dim/codim) = multi-planes $\Rightarrow$ regularity by Allard

codim-1 special: if $T = \text{codim-1, integral } n\text{-current}$,

$\partial T = 0$ (in B_1) then $\exists E_i \subset E_{i+1} \subset \dots$ sets

of loc finite perimeter s.t. $T = \sum \partial[E_i]$ and $\|T\| = \sum \|\partial[E_i]\|$
 in B_1 (proof: take cone on T .
 $\Rightarrow$ get low-dim current)

$\hookrightarrow$ if T minz then each $\partial[E_i]$ minz also

why? if T' _____ to $\partial[E_i]$

$$\text{then } \sum_j \|\partial[E_j]\|(\omega) = \|T\|(\omega)$$

$$\leq \|T - \partial[E_i] + T'\|(\omega)$$

$$\leq \sum_{j \neq i} \|\partial[E_j]\|(\omega) + \|T'\|(\omega)$$

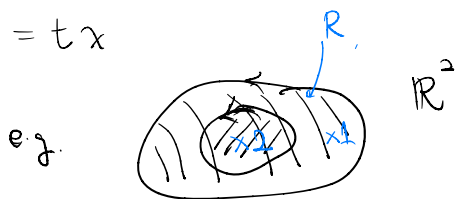
$$x \in \text{supp } T \Leftrightarrow x \in \text{supp } \partial[E_i] \text{ for some } i$$

upshot: in codim-1, WLOG $T = \partial[E]$ is boundary
 of set of loc finite perimeter

$\rightarrow$ always multi-one.

Why sheeting? let $R = \text{"cone" of } T_i = h_{\#} [0,1] \times T$

where $h(t, x) = tx$



$R = \text{integral } (n+1)\text{-current in } \mathbb{R}^{n+1}$
with $\partial R = T$.

but $R(\omega \equiv f dx^1 \wedge \dots \wedge dx^n)$

$$= \int f \tilde{\theta} d\mathbb{L}^{n+1}, \quad \tilde{\theta}: \mathbb{R}^{n+1} \rightarrow \mathbb{Z}$$

$$\text{let } E_i = \{ \tilde{\theta} \geq i \} \quad (i \geq 1)$$

$$E_i = \{ \tilde{\theta} \leq -1+i \} \quad (i \leq 0)$$

$$\Rightarrow R = \sum_{i \geq 1} [E_i] - \sum_{i \leq 0} [\mathbb{R}^{n+1} \setminus E_i]$$

fails in $\text{codim} \geq 2$ $\{W^3 = Z^2\}$.

(tangent cone $C_0 = 2[W=0]$
but also singular C_0)

Classification of low-dim cones (Simons)

if $T = \text{minimizing, integral, } n\text{-current in } \mathbb{R}^{n+1}$,

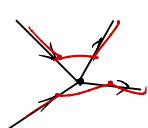
conical, smooth away from 0. mult-one smooth

then for $n \leq 6, \Rightarrow T = \text{planar}$ ($T = [M]$)

Sharp: Simon's $C^7 = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4, |x| = |y|\}$ is
minimizing $\mathbb{R}^8$. (Barbier ~ De Giorgi - Giusti)

why Simon's + ρ_M

$$\rho = 1$$



must be line.

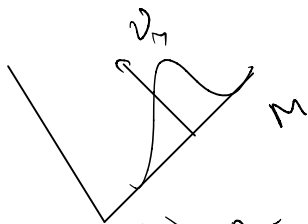
$$2 \leq n \leq 6$$

use stability $\leadsto$ if φ_t - 1-param family of

diffgeom

$$\partial_t \varphi|_{t=0} = X = f \nu_M \text{ on } M.$$

ν_M unit normal field on M .



$$\text{area}(\varphi_t(M)) \geq \text{area}(M)$$

$$\Rightarrow 0 \leq \left. \frac{d^2}{dt^2} \right|_{t=0} \text{area}(\varphi_t(M))$$

$$= \int_M |\nabla f|^2 - |A|^2 f^2 dV$$

$\uparrow$ second fundamental form.

(write $\Sigma = \text{link of cone } M$) stability of M

$$\Leftrightarrow \lambda_1(-\Delta_\Sigma - |A_\Sigma|^2) \geq -\left(\frac{n-2}{2}\right)^2$$

$$\Leftrightarrow -\left(\frac{n-2}{2}\right)^2 \int_\Sigma \varphi^2 \leq - \int_\Sigma \varphi (\Delta_\Sigma \varphi + |A_\Sigma|^2 \varphi) \quad \forall \varphi \geq 0$$

OTOH: (Simon's identity)

$$|A_\Sigma| (\Delta + |A|^2) |A| = (n-1) |A|^2 + |\nabla A|^2 - |\nabla |A||^2$$

take $\varphi = |A|$

$$0 \leq \int_{\Sigma} |\nabla A|^2 - |\nabla |A||^2 \leq \underbrace{\left(\left(\frac{n-2}{2} \right)^2 - (n-1) \right)}_{< 0 \text{ if } n \leq 6} \int_{\Sigma} |A|^2$$

so if $n \leq 6$ then $|A_{\Sigma}| = 0$

$\Rightarrow \Sigma = \text{totally geodesic}$

$\Rightarrow \text{cone } M = \text{plane.}$

□

thm if $T = \partial[E]$ is minz n -current and $n \leq 6$.

$\Rightarrow T = \text{smooth (mult-one)}$

proof: $x \in \text{supp } T \rightsquigarrow \frac{T-x}{r_i} \equiv \partial \left[\frac{E-x}{r_i} \right] \rightarrow T' = \partial[E']$
= tangent
(one)

PAUSE

dimension reduction Suppose every smooth (away from o),

conical minimize of $\dim < k$, is planar + mult-one.

then $\dim(\text{sig } T) \leq n-k$ (and discrete if $n=k$)

(in particular, if $T = \partial[E]$ minimizing, then $\dim(\text{sig } T) \leq n-7$)

proof: suppose $H^{\alpha}(\text{sig } T) > 0$ for $\alpha > n-k$

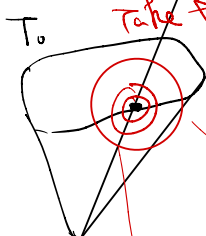
(to words contradiction)

$\Rightarrow \exists$ tangent cone T_0 of T with $\mathcal{H}^\alpha(\text{sig } T') > 0$ also

Take tangent cone at this pt.

(b/c. singular pts $\rightarrow$ singular pts.)

$\exists x \in \text{sig } T_0 \setminus \{0\}$. (since $\alpha > 0$)



$\rightarrow$ any tangent cone T_1 of T_0 $\notin x \neq 0$

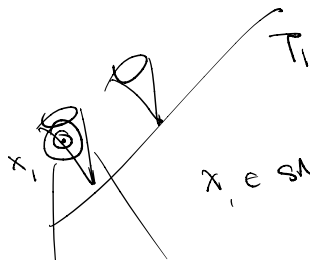
has form $T_1 = \mathbb{R} \times T'_1$

(gained symmetry)

$\exists x \in \text{sig } T_1 \setminus \mathbb{R} \times \{0\}$

also can assume $\mathcal{H}^\alpha(\text{sig } T'_1) > 0$.

if $\alpha > 1$.



$x_1 \in \text{sig } T_1 \setminus \mathbb{R} \times \{0\}$

any tangent cone T_2 of T_1 $\in x_1$ has form $T_2 = \mathbb{R}^2 \times T'_2$
(gain symmetry since $x_1 \notin \mathbb{R} \times \{0\}$)

can assume $\mathcal{H}^\alpha(\text{sig } T'_2) > 0$

$\mathbb{R}^2$

if $\alpha > 2 \leadsto$ repeat

iterate. $T_{n-k} = \mathbb{R}^{n-k} \times T'_{n-k}$ but $\alpha > n-k$

$\leadsto T_{n-l} = \mathbb{R}^{n-l} \times T'_{n-l}$ where $\alpha \leq n-l$ low-dim.

but then $T' = l$ -current, singular

but smooth away from 0

and $l < k$ by our assumption $\alpha > n-k$

by assumption, $T' = \text{planar} \Rightarrow T_{n-l} = \mathbb{R}^n$ but T' singular