

- Einstein Manifold

$$(M^n, g), \quad Ric_g = \lambda g, \quad \lambda \text{ is a constant.}$$

- Collapsing / Non-collapsing.

Basic

(X, d) "length" metric space

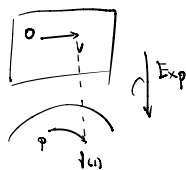
$$d(x, y) = \inf \left\{ \sum_{i=1}^n |t_i - t_{i-1}| \mid t: [0, 1] \rightarrow X, \begin{matrix} t(0) = x \\ t(1) = y \end{matrix} \right\} \quad \begin{matrix} \text{"length"} \\ \text{property} \end{matrix}$$

not definition?

Example $(\mathbb{Z}, d_0)$, $d_0(m, n) = |m - n|$

Lemma On a Riemannian manifold its Riemannian metric is uniquely determined by its length structure.

Exponential map



$$\text{Exp}(v) \equiv t(1), \quad \gamma: [0, 1] \rightarrow M^n$$

$$t(0) = p, \quad \gamma'(0) = v$$

Lemma (Gauss) $\forall p \in M^n$, Exp_p is a locally diffeomorphism

Example $f(x) = \frac{1}{k} \sin(kx)$, $x \in \mathbb{R}$, $k > 0$

$x \in J \subset \mathbb{R}$ (1) $|J| \ll \frac{1}{k}$, At x_0 , f can be well approximated by f_{x_0} .

length

(2) $J \approx \frac{1}{k}$, Bad scales

不精确

于二阶信息 (3) $J \gg \frac{1}{k}$, $f \approx 0$.

下面跳到 2nd Geometry

Curvature

$$R(x, Y)Z = \nabla_{x,Y}^2 Z - \nabla_Y^2 x Z$$

$$\nabla_{x,Y}^2 Z = \nabla_x \nabla_Y Z - \nabla_{\nabla_Y x} Z$$

$$Rm(x, Y, Z, W) = g(R(x, Y)Z, W)$$

Q1: $Rm \text{ Cur.} = 0$ at each pt.

What's the manifold?

Q2: 如何与高维平直流形?

Lemma In Geodesic normal coordinates

$$g_{ij}(p) = \delta_{ij}, \quad T_{ij}^k(p) = 0.$$

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{iklj} x^k x^l + o(|x|^2)$$

局部-所度量几何

是平几的?

Thm (Bieberbach, 1911)

(M^n, g) closed, $Rm_j \equiv 0 \Rightarrow (M^n, g)$ is finitely covered by a Torus.

$$M \simeq \mathbb{R}^n / \pi_1(M^n)$$

Thm* (-) Let $\Gamma \subset \text{Isom}(\mathbb{R}^n)$ be a discrete co-compact.

Then Γ is a finite extension of $\mathbb{Z}^n$, i.e.

$$\text{[例]} \quad 1 \longrightarrow \mathbb{Z}^n \xrightarrow{\text{inj.}} \Gamma \xrightarrow{\text{surj.}} H \longrightarrow 1, \quad \#(H) \leq W, \quad W = W(n)$$

n -free Abell. group.

Here, Γ is called a Bieberbach group.

Cor Given $n \geq 2$, there are at most $O(n)$ isomorphism classes of Bieberbach groups in $\text{Isom}(\mathbb{R}^n)$

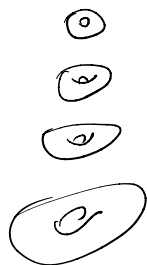
Example $(M^n, g_\epsilon), g_\epsilon = \epsilon^2 g_0$

$$(M^n, g_\epsilon) \xrightarrow{\epsilon \rightarrow \infty} \text{pt.} \quad (|Rm_{g_\epsilon}| \equiv 0 \text{ when } \epsilon \rightarrow \infty)$$

Q: If M^n admits a family of metrics g_ϵ with

$$|Rm_{g_\epsilon}| \leq 1, \quad \text{diam}_{g_\epsilon}(M^n) \rightarrow 0$$

What does M^n look like?



Torus bundle.

iteration

Thm (Gromov 1978, Ruh 1981) Given $n \geq 2$, $\exists \varepsilon = \varepsilon(n) > 0$.

if (M^n, g) closed, satisfies $(\text{diam}(M^n))^2 \cdot \max_{M^n} |Rm_g| < \varepsilon$

then M^n is a "finite-step" generalization of a Bieberbach mfd.

$$\text{Ric}(X, Y) = \sum_{i=1}^n R_m(E_i, X, Y, E_i)$$

$$R_{ijk\ell} + R_{iklj} + R_{j\ell ki} = 0$$

$$\boxed{R_{ijk\ell}}$$

~~is~~

$$\frac{m(m+1)}{2} - \binom{n}{4}$$

$$m = \frac{n(n+1)}{2}$$

Einstein eq. ~~eds~~

$$\Delta_g(f) = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij} \partial_j f) = g^{ij} \partial_i \partial_j f + \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij}) \partial_j f$$

$$\Delta_{\mathbb{R}^n} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

choose $(x^1, \dots, x^n)$ coordinate system

$$\Delta_g(x_j) = \frac{1}{\sqrt{\det g}} \partial_i (\sqrt{\det g} g^{ij})$$

$$= g^{ij} \partial_i \partial_j f + \Delta_g(x_j) \partial_j f$$

Def (Harmonic coordinates) $\Phi = (u^1, \dots, u^n) : B_r(p) \rightarrow \mathbb{R}^n$ is harmonic

if $\forall 1 \leq j \leq n \quad \Delta_g(u^j) = 0$.

In harmonic coordinates

$$\text{Ric}_{ij} = -\frac{1}{2} \Delta_g(g_{ij}) + Q(\partial g)$$

quasi-linear elliptic operator

Question: Control the size of harmonic coordinates?

(In Einstein Manifold)

Recall: $\Gamma = \pi_1(M^n)$ contains n -linearly independent translations

$$\widetilde{M^n} \cong \mathbb{R}^n$$

Lemma (M^n, g) complete $M^n \stackrel{\text{isom}}{\equiv} \mathbb{R}^n \times N$ iff $\exists$ a C^2 -function

$$f: M^n \rightarrow \mathbb{R} \quad \text{s.t.} \quad |\nabla f| \equiv 1 \quad \nabla^2 f = 0.$$

(In this case, ∇f // the $\mathbb{R}$ -factor)

Bochner formula: $\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla \Delta f, \nabla f \rangle + \text{Ric}(\nabla f, \nabla f)$

Lemma Assume $\text{Ric} \geq 0$, Let f satisfy $|\nabla f| \equiv 1$

Then $\Delta f \equiv 0$ iff $\nabla^2 f \equiv 0$

想造一个函数.

Thm (Cheeger - Croke, 1968) (M^n, g) complete, $\text{Ric}_g \geq 0$.

if M^n admits a line, then $M^n \equiv \mathbb{R} \times N^{n-1}$

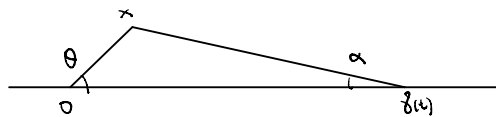
$f: \mathbb{R} \rightarrow M^n$ is isometric embedding

Busermann function

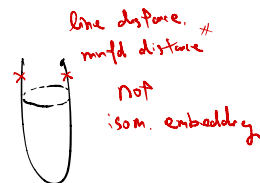


关于 t 单调有界

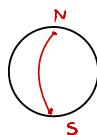
$$b_f(x) = \lim_{t \rightarrow \infty} (d(x, f(t)) - t)$$



$$\begin{aligned} b_f(x) &= \lim_{t \rightarrow \infty} (t - |x - f(t)|) \\ &= -|Ox| \cdot \cos \theta \end{aligned}$$



line distance, $\neq$ manifold distance
not isom. embedding



s is a cut pt of N.
即度量性质在 S 处发生剧烈变化
N 到 s 的测地线到 s 后不能继续延伸

Pf 1. Goal: ∇f is a Killing field. (X is a Killing field if $\mathcal{L}_X g \equiv 0$)

$$\begin{aligned} (\mathcal{L}_{\nabla f} g)(x, Y) &= (\nabla f)(g(x, Y)) - g([\nabla f, x], Y) - g(x, [\nabla f, Y]) \\ &= g(\nabla_{\nabla f} x, Y) + g(x, \nabla_{\nabla f} Y) - g(\nabla_x \nabla f, Y) + g(\nabla_Y \nabla f, x) \end{aligned}$$

$$-g(x, \nabla_g f, Y) + g(x, \nabla_Y \nabla f) \\ = 2(\nabla^2 f)(x, Y) \equiv 0 \quad \square$$

$$M^n \cong \mathbb{R} \times N$$

$$P_1 = (t_1, w_1)$$

$$P_2 = (t_2, w_2)$$

$$d^2(P_1, P_2) = |t_1 - t_2|^2 + d_N^2(w_1, w_2)$$

(对分裂即刻态, 有极定理)

$$d^2(w, z) = 2 \int_0^d s \, ds = 2 \int_0^d (f(\sigma(s)) - f(\sigma(0))) \, ds$$

$$= 2 \int_0^d (f(\sigma(s)) - f(w)) \, ds$$

$$= 2 \int_0^d \left(\int_0^{l_s} \langle \nabla f, \tau'_s \rangle (\tau_s(w)) \, dt \right) ds$$

$$\text{Let } t = l_s$$

$$\therefore \sigma' = 0 \quad ?$$

$$= 2 \int_0^d \left(\int_0^{l_s} \langle \nabla f, \tau'_s \rangle (\sigma(s)) \, dt \right) ds$$

$$= 2 \int_0^d l_s \cdot \left(\langle \nabla f, \tau'_s \rangle (\sigma(s)) \right) ds = 2 \int_0^d (l_s \cdot l'_s) ds = l_s^2 \Big|_0^d \\ \parallel \\ \langle \sigma', \tau'_s \rangle (\sigma(s))$$

$$\Gamma V : [0, 1] \times (-\epsilon, \epsilon) \rightarrow M^n, \quad V(t, 0) = f(t), \quad f: \text{geodesic} \\ (t, s)$$

$$X(t) = \frac{\partial}{\partial s} \Big|_{s=0} V(t, s) \text{ along } f$$

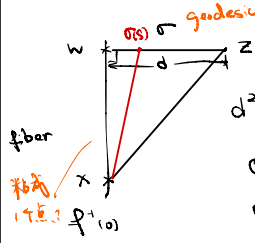


$$f_s(t) = V(t, s)$$

$$\frac{d}{ds} \Big|_{s=0} L(t_s) = \langle X, f'(t) \rangle - \langle X, f'(t_0) \rangle$$

]

$$f: M^n \rightarrow \mathbb{R} \quad \text{Surjective}$$



WTS:

$$d^2(w, z) = d^2(x, z) - d^2(x, w)$$

$$\sigma: [0, d] \rightarrow M^n \text{ nh. geod.}$$

$$\sigma(0) = w, \quad \sigma(d) = z$$

$$\tau_s: [0, l_s] \rightarrow M^n$$

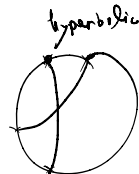
$$\tau_s(0) = x, \quad \tau_s(l_s) = \sigma(s)$$

Lecture 2

Thm (Cheeger - Colding, 1968) (M^n, g) complete, $\text{Ric}_g \geq 0$.

if M^n admits a line then $M^n \cong \mathbb{R} \times N^{n-1}$

$f: \mathbb{R} \rightarrow M^n$ is isometric embedding



$$g = dt^2 + e^{2t} d\theta^2$$

Key: Find a function $R: M^n \rightarrow \mathbb{R}$

trivial bundle map

① distance $|\nabla R| \equiv 1$

非退化性

→ 3-个全洲地子流形 (从子流形角度)

② harmonic $\Delta R = 0$

$\subset \mathbb{R}^n$

Thm (Rademacher) Let $f: B_1(0) \rightarrow \mathbb{R}$, Lipschitz, $\text{Lips}(f) \leq 1$. Then f is differentiable almost everywhere.

cheeger (1999)

$$\lim_{r \rightarrow 0} \int_{B_r(x)} f dx = f(x) \quad x \text{ Lebesgue point}$$

对函数本身做 Blow-up.

定量微分理论.

$$f_{x_0, r}(y) \equiv \frac{f(x_0 + ry) - f(x_0)}{r}$$

r 越小, y 可取越多,

$$r \rightarrow 0, f_{x_0}: \mathbb{R}^n \rightarrow \mathbb{R} \quad \text{still Lips. func.}$$

↑
Tangent map. $|\nabla f_{x_0}| \equiv \text{const}$ if x_0 is a Lebesgue point.

(对 $|\nabla f|$ 的 Lebesgue thm).

$\Rightarrow$ Tangent map is harmonic and distance function.

Thm (M^n, g) complete, $\text{Ric} \geq 0$, $p \in M^n$, $r_p(\cdot) \equiv \text{dist}_g(p, \cdot)$

$$\Delta r_p \leq \frac{n-1}{r_p}$$

"=" iff $B_1(p) \equiv B_1(0) \subseteq \mathbb{R}^n$
 $\uparrow$
 no cut point

Lemma (M^n, g) complete $g = dr^2 + f^2(r)h$ iff $\exists$ smooth

functions F, H s.t. $\nabla^2 F \equiv H \cdot g$
 $\uparrow$
 matrix

(if $g = dr^2 + f^2(r)h$, $F(r) = \int_{r_0}^r f(t) dt$

"=" $g_0 = dr^2 + r^2 g_{S^{n-1}}$ why?

$$\Delta(r^2) = 2r \Delta r + 2 = 2n$$

$$\nabla r^2 = 2r \nabla r$$

$$\frac{1}{2} \Delta |\nabla(r^2)|^2 = |\nabla^2(r^2)|^2 + \langle \nabla \Delta(r^2), \nabla r^2 \rangle + \text{Ric}(\nabla r^2, \nabla r^2)$$

$$\frac{1}{2} \Delta (4r^2) = |\nabla^2(r^2)|^2 + \text{Ric}(\nabla r^2, \nabla r^2)$$

$$4n \geq |\nabla^2(r^2)|^2 = \sum_{i=1}^n \lambda_i^2$$

WTS 常数矩阵

$$\sum_{i=1}^n (\lambda_i - 2)^2 \leq 4n + 4n - 4 \sum_{i=1}^n \lambda_i = 0$$

$$\Rightarrow \nabla^2(r^2) \equiv 2g$$

hence we have $g_0 = dr^2 + r^2 g_{S^{n-1}}$

Cor 1 $\text{Ric} \geq 0$. $\text{Vol}_g(B_r(x)) \leq \text{Vol}_0(B_r)$

"=" iff $B_r(x) \equiv B_r \subseteq \mathbb{R}^n$

Cor 2 $\text{Ric} \geq 0$. $Q_x(r) \equiv \frac{\text{Vol}_g(B_r(x))}{\text{Vol}_0(B_r)}$ (as r)

Further, if $Q_x(r_1) = Q_x(r_2)$ for some $0 < r_1 < r_2$
~~两种尺度~~ then $B_{r_2}(x) \equiv B_{r_2}(0) \subseteq \mathbb{R}^n$

we can get a cone structure

Thm (Cheeger - Colding) $\text{Ric} \geq 0$. $\text{Vol}_g(B_r(x)) \geq V > 0, \forall \varepsilon > 0$,

$\exists N = N(\varepsilon, n, V) < \infty$, s.t. except for at most N bad scales

$$d_{\text{GH}}(B_r(x), B_r(Z^*)) < \varepsilon: B_r(Z^*) \subset \underset{\substack{\uparrow \\ \text{cone}}}{C(Z)}$$

Define $W_x(r) \equiv -\ln \frac{\text{Vol}_g(B_r(x))}{\text{Vol}_0(B_r)}$ $\uparrow$

$$C(n, V) \geq W_x(r_0) = \overbrace{W_x(r_0) - W_x(r_1) + W_x(r_1) - \dots}$$

$$r_j \equiv 2^{-j}$$

$$\begin{aligned} &= (W_x(r_0) - W_x(r_1)) + (W_x(r_1) - W_x(r_2)) + \dots \\ &\geq \sum_{k=0}^{\infty} W_x(r_k) - W_x(r_{k+1}) \end{aligned}$$

单调性 连接了小尺度与大尺度

体积 non-collapsing 给了能量有上界的条件.

Z : compact metric space. $A, B \subset Z$ compact.

$$d_H(A, B) \equiv \inf \{ r > 0 : T_r(A) \supseteq B, T_r(B) \supseteq A \}$$

Lemma. $d_H(A, B) = 0 \Rightarrow A = B$ 所有紧集的集合
↓

Thm (Blaschke) if Z is compact, then $(K(Z), d_H)$ is compact.

Def (Gromov-Hausdorff Distance)

X, Y compact metric space

$$d_{GH}(X, Y) = \inf_{Z, \varphi, \psi} \{ d_H(\varphi(X), \psi(Y)) \mid \begin{array}{l} \varphi: X \hookrightarrow Z \text{ isometric} \\ \psi: Y \hookrightarrow Z \text{ embeddings} \end{array} \}$$

$\text{Met} \equiv$ set of all compact metric spaces

Lemma (Met, d_{GH}) is a complete metric space.

Theorem (Gromov) $\mathcal{C} \subset \text{Met}$ is precompact iff

(1) $\exists D > 0$ (large enough) s.t. $\text{diam}(X) \leq D, \forall X \in \mathcal{C}$

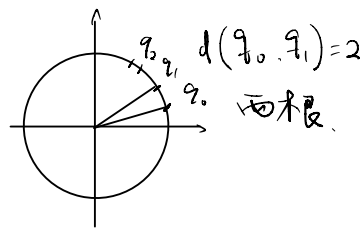
(2) $\forall \epsilon > 0, \exists N = N(\epsilon, \mathcal{C}) < \infty$ if any $X \in \mathcal{C}$

has an ϵ -net $X(\epsilon)$ that satisfies

$$\#(X(\epsilon)) \leq N$$

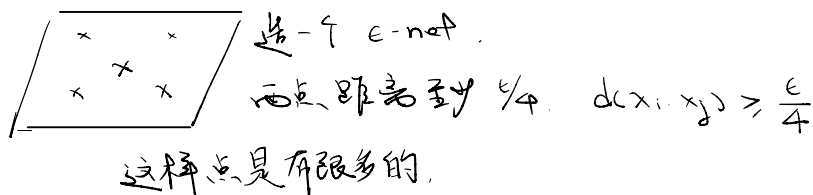
某种一致的维数上界

$\{x_i\}_{i \geq 0}$



Cor $\mathcal{M}(n, D, K) \equiv \left\{ (M^n, g) : \text{Ric}_g \geq K, \text{diam}_g(M^n) \leq D \right\}$

is precompact in (Met, d_{GH}) .



Let $x_0 \in M^n(\epsilon)$ be a point that satisfies $N = \# M^n(\epsilon)$

$$\text{Vol}_g(B_{\frac{\epsilon}{10}}(x_0)) \equiv \min \left\{ \text{Vol}_g(B_{\frac{\epsilon}{10}}(x_i)) ; 1 \leq i \leq N \right\}.$$

$$\text{Vol}_g(B_D(x_0)) = \text{Vol}_g(M^n) \geq N \text{Vol}_g(B_{\frac{\epsilon}{10}}(x_0))$$

$$\therefore N \leq \frac{\text{Vol}_g(B_D(x_0))}{\text{Vol}_g(B_{\frac{\epsilon}{10}}(x_0))} \leq C(D, \epsilon, n, K) \quad \square$$

Q: In what sense is X_∞ closed to a manifold?

A: In general, X_∞ may have no manifold point. maximum
↓

Thm (Colding - Naber, 2010) If X_∞ is a Ricci-limit space, $K \in \mathbb{R}$
 then $\exists S \subset X_\infty$ with $\underline{v}_\infty(S) = 0$ s.t. 可想象

取 limit 前 球的体积都是单位球

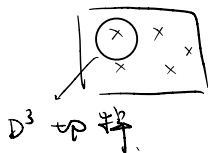
已无体边界作例子.

$\forall x \in X_\infty \setminus S$, its tangent cone is isometric to $\mathbb{R}^k$

$K = \dim_{\text{ess}}(X_\infty)$ Remark: Pan-Wei $\dim_{\mathcal{H}}(x_\infty) \in (2, 3)$.

Example: $X_\infty \equiv T^3$

平坦环面挖掉一些点



$D^3 \times S^1$ Schwarzschild
 $\downarrow$ 可边界一样

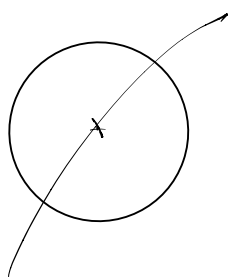
$\frac{D^2 \times S^2}{\text{Ric} \equiv 0}$ $\mathbb{R}^3 \times S^1$
 远处

$$(S^1 \times (T^3 \setminus \bigcup_{i=1}^N D_i)) \cup \bigcup_i \underline{O_i}$$

Topology: $S^1 \times S^2$ $D^2 \times S^2$

$$|\text{Ric}_{g_\epsilon}| \leq K$$

即单从小只及(收敛后)看不出之前状况. 因为几行信息损失掉了!



Thm (Cheeger-Colding) Given $n \geq 2, \epsilon > 0$.

$\exists \delta = \delta(n, \epsilon) > 0$, s.t. if $\text{Ric}_g \geq -\delta$,

$$d_{GH}(B_{\delta^{-1}}(p), B_{\delta^{-1}}(q)) < \delta$$

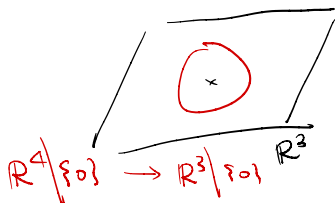
$\leq \mathbb{R}^k$

$\uparrow$
 相当小

then $\exists$ a harmonic map $\Phi = (h_1, \dots, h_k): B_2(p) \rightarrow \mathbb{R}^k$
 s.t. (1) $|\nabla h_\alpha| \leq 1 + \epsilon \quad \forall n \leq \alpha \leq k$ in $B_1(p)$.

$$(2) \sum_{\alpha \neq \beta} \int_{B_1(p)} |\langle \nabla h_\alpha, \nabla h_\beta \rangle - \delta_{\alpha\beta}| + \sum_{\alpha=1}^k \int_{B_1(p)} |\nabla^2 h_\alpha|^2 \leq \epsilon$$

Example (Taub-NUT) $X \stackrel{\text{diffs}}{\simeq} \mathbb{R}^4$, $\text{Ric}_{g_{TN}} \equiv 0$
 $(\simeq \mathbb{C}^2)$



$$(1) \mathbb{R}^4 \xrightarrow[\mathbb{R}^2 \rightarrow \infty]{GH} \mathbb{R}^3$$

(2) $X^4 \setminus K$ circle bundle over a flat base.

$X \perp \pi$ geodesic

if has a line
by Cheeger-Gromoll split thm

$\mathbb{R}^4 \cong \mathbb{R} \times \mathbb{R}^3$
metric is flat.

But it's not!

$$(3) |Rm| \not\equiv 0. \quad |Rm| = O(r^{-5})$$

$$\int_{X^4} |Rm|^2 = 8\pi^2$$

(1988) collapsing fibre thm

Consider $(M_j^n, g_j) \quad |Rm_{g_j}| \leq \epsilon$

$$M_j^n \xrightarrow{GH} \mathbb{R} \quad \text{diffs}$$

N : Simply connected

Then $B_1(x_j) \approx I \times \underbrace{(\Gamma \backslash N)}_{\text{nilpotent Lie group}}$

in fibril manifold $\Gamma \subset \text{Aut}(N) \propto N$

$$1 \rightarrow L \rightarrow \Gamma \rightarrow H \rightarrow 1$$

$$\Delta_{g_j} h = 0 \quad \underbrace{B_2(x_j)}_{\substack{\mathbb{R}^n \\ G_1}}$$

$$\Delta_g u = g^{ij} u_{ij} + \Delta_g(x_j) u_j$$

$$\text{Exp}_{x_j} : B_2(0) \longrightarrow B_2(x_j)$$

Schroeder estimate.

但不直接

Thm (Cheeger) (2) ✓.

What about $\mathbb{R}^k$? smooth mfd?

Thm (Fukaya) $M_j^n \xrightarrow{GH} M_\infty^k, |Rm_{g_j}| \leq 1.$

Then $\exists$ a fiber bundle map $F_j: M_j^n \rightarrow M_\infty^k$ s.t.

(1) $\text{diam}(\text{fiber}) \rightarrow 0$

(2) $|\Pi_{\text{fiber}}| \leq C(n, M_\infty^k) \rightsquigarrow$ Each fiber is an infrarigid

$Ex(S^3, g_\epsilon) \xrightarrow[\epsilon \rightarrow 0]{GH} S^2(\frac{1}{2})$

$S^1 \rightarrow S^3 (\subset \mathbb{C}^2)$

$S^3 \approx SU(2)$

$\downarrow$
 S^2

invariant metric $g_\epsilon = \epsilon^2 \sigma_1^2 + \sigma_2^2 + \sigma_3^2$

Lie-algebra 结构常数可知.

联络即知 $\Rightarrow \text{sec}_{g_\epsilon} \checkmark$.

$0 \leq \text{sec}_{g_\epsilon} \leq 4$

$\swarrow$ S^1 作用

$|z_1|^2 + |z_2|^2 = 1$

$\theta \cdot (z_1, z_2) = (e^{\sqrt{-1}\theta} z_1, e^{-\sqrt{-1}\theta} z_2)$

$\pi \cdot (z_1, z_2) \mapsto (u_1, u_2, u_3)$

$u_1 = \text{Re}(z_1, z_2)$

$u_1^2 + u_2^2 + u_3^2 = \frac{1}{4}$

$u_2 = \text{Im}(z_1, z_2)$

$u_3 = \frac{|z_1|^2 - |z_2|^2}{2}$

Lecture 3

Thm (Fukaya) $(M_j^n, g_j) \xrightarrow{GH} (M_\infty^k, g_\infty)$, $|Rm_{g_j}| \leq 1$.

for any large j , $\exists$ a fiber bundle map

$$F^{n-k} \rightarrow M_j^n \xrightarrow{F_j} M_\infty^k \hookrightarrow \mathbb{R}^k$$

纤维丛
↓

(1) $\text{diam}_{g_j}(F^{n-k}) \rightarrow 0$

$\Gamma \subset \text{Aut}(N) \curvearrowright N$

(2) $|\Pi_{F^{n-k}}| \leq C(n, M_\infty^k) \xRightarrow{\text{Gromov-Rub}} F^{n-k} \approx \Gamma \backslash N$

(3) F_j is an "almost" Riemannian submersion.

Ex (S^3, g_ϵ) $0 \leq \sec_{g_\epsilon} \leq 4$, $S^3 \approx SU(2)$

$$(S^3, g_\epsilon) \xrightarrow{GH} (S^2(\frac{1}{2}), g_0)$$

~~取~~ $SU(2)$ 3个不变向量 $\sigma_1, \sigma_2, \sigma_3$

$$g_\epsilon = \epsilon^2 \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$

Lie group action no fixed point

↓
limit space no singularity.

cf to Moser, iteration
need Sobolev const.

but $j \rightarrow \infty$, C may
be to zero or infinity

covering space $\uparrow$
不变 $\uparrow$
纤维丛 $\uparrow$
Ex' $(S^3, T_k) \xrightarrow{\text{equal GH}} (S^3, S')$

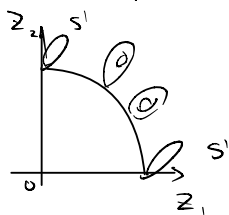
$$\begin{array}{ccc} T_k \subset S^1 & S^3/T_k & \xrightarrow{GH} S^2 \\ \downarrow & & \downarrow \end{array}$$

$(k \rightarrow \infty, \text{dense in } S^1) \uparrow$

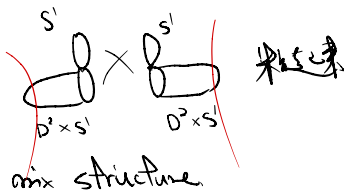
limit space (连续空间)

Ex (S^3, g_ϵ) , $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$.

$(\theta, \varphi) \cdot (z_1, z_2) = (e^{\sqrt{-1}\theta} z_1, e^{\sqrt{-1}\varphi} z_2)$



$S^3 \rightarrow [0, \frac{\pi}{2}]$



Lemma (M^n, g) , T^k : effective acts on M^n s.t. each orbit is positive dimensional

Then M^n admits T^k -invariant metrics g_ϵ s.t.

(1) $|Rm_{g_\epsilon}| \leq C$

(2) $(M^n, g_\epsilon) \xrightarrow{\epsilon \rightarrow 0} (M^n / T^k, d_0)$

(~~并非~~ 双边角等 collapsing, fiber must positive dim)

Pf: 取单参数子群 $H \subset T^k$ dense

$g_\epsilon = \epsilon^2 g'' \oplus g^\perp$

$\epsilon \rightarrow 0$

$|Rm_{g_\epsilon}| \leq C$

T^2 act free.

$O(1)$ frame. /

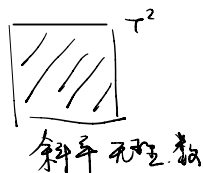
$(F(S^3), T^2) \rightarrow \underline{O(4)/T^2}$ 为流形

$\downarrow$
 S^3

$\searrow$

$\downarrow$

S^3/T^2 T^2 act not free has singularity.



if has fixed point

$t \cdot A = A$
 $\cap$
 $T^2 \quad O(4)$
 $\downarrow$
 $t = 1d$

$$\begin{array}{ccc} (F(S^3), O(3)) & \longrightarrow & (O(4)/T^2, O(3)) \\ \downarrow S^3 & & \downarrow \\ S^3 & \longrightarrow & (O(4)/T^2)/O(3) \end{array} \quad S^2 \times S^2$$

Thm (Fukaya) $(M_j^n, g_j) \xrightarrow{GH} (X_\infty^k, g_\infty)$, $|Rm_{g_j}| \leq 1$.
 for any large j , $\exists$ a (possibly singular) fibration

$$M_j^n \xrightarrow{F_j} X_\infty^k \rightarrow \mathbb{A}^1$$

(1) $\text{diam}_{g_j}(F^{n-k}) \rightarrow 0$

(2) Each fiber is an infranil

曲线一族有界 +

$$(F(B_{r_0}(0)), \tilde{g}_j) \longrightarrow (Z_\infty, \tilde{g}_\infty)$$

非塌缩 $\Rightarrow$ smooth
 + $\tilde{g}_\infty$ 有统一
 regularity.

$C^{1,\alpha}$?

曲线
 至为
 有覆盖空间

$$\begin{array}{ccc} \downarrow & & \downarrow \\ B_{r_0}(P_j) & \longrightarrow & B_{r_0}(P_\infty) \end{array}$$

$$(F(B_{r_0}(0)), \Gamma) \xrightarrow{eq GH} (Z_\infty, \Gamma_\infty)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ F(B_{r_0}(P_j)) & \longrightarrow & \underbrace{Y_\infty}_{\text{smooth}} \end{array} \quad (g_j, \nabla_{g_j, \epsilon}^{Lc})$$

(事实上, 是从上面 quote 下来)

Furthermore, F_j lifts to an $O(n)$ -equivariant fiber

bundle map $\hat{F}_j: F(M^n) \rightarrow Y_\infty$

s.t. the following diagram commutes

$$\begin{array}{ccc}
 (F(M_j^n), O(n)) & \xrightarrow[\widehat{F_j}]{\text{eq GH, smooth}} & (Y_\infty, O(n)) \\
 \downarrow & & \downarrow \\
 (M_j^n, g_j) & \xrightarrow[\widehat{F_j}]{\text{GH}} & (X_\infty^k, d_\infty) \\
 & & \text{(fibre is new?)}
 \end{array}$$

Con If $\pi_1(M^n) = 0$, then N -structure can be reduced to F -structure.

Pf:
$$\underbrace{\pi_2(Y_\infty)}_{\text{abelian}} \rightarrow \pi_1(\Gamma \backslash N) \rightarrow \underbrace{\pi_1(F(M^n))}_{\Gamma \text{ (零群)}} \rightarrow \pi_1(Y_\infty)$$

$\Rightarrow \Gamma$ is abelian. $\Rightarrow N$ is abelian $\Rightarrow \Gamma \backslash N \approx T^m$

□

Ehern - Gauss - Bonnet - (M^4, g) Einstein, compact unboundary

$$8\pi^2 \chi(M^4) = \int_{M^4} |Rm_g|^2 \geq 0$$

Hirzebruch:
$$12\pi^2 \sigma(M^4) = \int_{M^4} (|W_+|^2 - |W_-|^2)$$

$\downarrow$
 拓扑不变量
 $b^+(M^4) - b^-(M^4)$

$$Rm_g = W_g \oplus (A_g \otimes g), \quad A_g = \frac{1}{2} \left(Ric_g - \frac{Scal_g}{6} g \right)$$

$$2 \chi(M^4) \geq 3 \sigma(M^4)$$

Witchin: " $\equiv$ " iff (M^4, g) , up to a finite cover,
is isometric to (T^4, g) or the K_3 surface.

compact, complex surface
single connected.

first Chern class vanishes.

Thm (Witchin) (K, g) : TTFE

(1) g is Einstein

(2) $Ric_g \equiv 0$

(3) g is Kähler-Einstein (Ricci-flat Kähler)

(4) g is hyperkähler

$$b_2(K) = 22 \quad \begin{matrix} \text{harmonic 2-form} \\ \downarrow \\ b^+ = 3, \quad b^- = 19 \end{matrix}$$

$$\chi(K) = 24$$

$$\text{By Gauss-Bonnet} \quad \int_K |Rm_g|^2 = 192 \pi^2$$

$K_3 \not\cong \mathbb{R}^4$ collapsing $\rightarrow$ curvature uniform bdd.

if it's right. $8\pi^2 \chi(K_3) = \int_{K_3} |Rm|^2 \rightarrow 0, \quad \chi(K_3) = 0 \quad \times$

Bando - Kasue - Nakajima

$$(M_j^4, g_j) \xrightarrow{GH} (X_\infty^4, d_\infty) \quad \text{Einstein } |\text{Ric}_{g_j}| \leq 3$$

$$\int_{M_j^4} |\text{Rm}_{g_j}|^2 \leq \Lambda$$

Then (1) X_∞^4 has at most $C(\Lambda)$ orbifold singularities

(2) $GH \nearrow C^k$ outside the singular set.
 $\forall k \in \mathbb{Z}_+$

(3) for any singular point P_∞ , $\exists r_0 > 0$,
and a finite group Γ s.t.

$$B_{r_0}(P_\infty) = \widetilde{B_{r_0}(P_\infty)} / \Gamma \quad \text{and}$$

$\widetilde{g}_\infty$ is Einstein on $\widetilde{B_{r_0}(P_\infty)}$
(including $\widetilde{P_\infty}$)

Lemma $\forall \nu > 0$, $\exists \epsilon = \epsilon(\nu)$, s.t. if (M^4, g) Einstein, $|\text{Ric}_g| \leq 3$

$\text{Vol}(B_1(p)) \geq \nu > 0$, then

$$\int_{B_2(p)} |\text{Rm}_g|^2 < \epsilon \Rightarrow \sup_{B_1(p)} |\text{Rm}_g| \leq 1$$

可成比例. 下定理列为独立的命题

$$\Delta R_m = R_m * R_m + \nabla^2 Ric$$

$$\Delta |R_m| \geq -C |R_m|^2$$

(V-Noncollapse is necessary here)

high dim is not right.

Cheeger-Tian: $\exists$ absolute constants $\epsilon > 0, C > 0$ s.t. if

(M^4, g) Einstein $|Ric_g| \leq 3$, then

$$\int_{B_2(p)} |Rm_g|^2 < \epsilon \Rightarrow \sup_{B_1(p)} |Rm_g| \leq C.$$

scale
invariant.

(Volume collapse is allowed)

$2 \rightarrow 2R, 1 \rightarrow R$

$$\underline{\text{Cor 1}} \quad (M_j^4, g_j) \xrightarrow{GH} (X_\infty^K, d_\infty)$$

$$\int_{M_j^4} |Rm_{g_j}|^2 \leq \Lambda, \text{ Then, away from finitely}$$

many points. (M_j^4, g_j) is collapsing with bounded

curvature.

Example (Gross-Wilson) $(K, g_\epsilon) \xrightarrow{GH} (X_\infty^2, d_\infty)$

All tangent cones $\equiv \mathbb{R}^2$

Calabi-Yau structure

g_ϵ is collapsing with $|Rm_{g_\epsilon}| \leq 1$

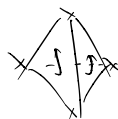
away from 24 points

Li-Tung

regular fiber

K_3 surface is π_1 -collapsing

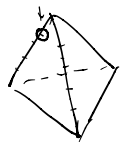
$$K \subset \mathbb{P}^3, \quad z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_1 z_2 z_3 z_4 = 0.$$



(程政已列明, 不能给出收敛的正列性)

bubble Toric-NOT

Sung Sun & Ruohong Zhang



complex structure

不定 - 在 $SO(3)$ 作用

置换 (w_1, w_2, w_3)

$$K_3 \xrightarrow{GH} x \rightarrow x$$

Hyperkähler structure

$$(w_1, w_2, w_3) \in H^2(M^4) \times H^2(M^4) \times H^2(M^4)$$

$$\frac{1}{2} w_i \wedge w_j = \frac{1}{6} \delta_{ij} (w_1^2 + w_2^2 + w_3^2)$$

$$d w_i = 0$$

Fix w_1 . Kähler form, $w_2 + \sqrt{-1} w_3 \rightarrow$ Hermitian structure.

4-dim Einstein metric / hyperkähler metric

Gibbons-Hawking Ansatz:

X^4 看作 $U(1)$ 的 S^1 bundle?

$$(X^4, S^1) \longrightarrow X^4/S^1 \text{ (flat)}$$

hyperkähler $\rightsquigarrow$ harmonic func. metric.

$$U \subseteq \mathbb{R}^3, \quad \Delta_{\mathbb{R}^3} V = 0 \text{ on } U.$$

Monopole equation Hodge *

$$d\theta = *_{\mathbb{R}^3} \circ dV, \quad V > 0$$

$$g = V g_U + V^{-1} \theta^2$$

↓
 $\mathbb{R}^3 \text{ 上的 平坦度量}$

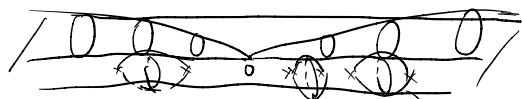
Taub-NUT:

$$U \cong \mathbb{R}^3 \setminus \{0\}, \quad V = \frac{1}{2r} + 1 \quad (\text{Green func.})$$

$$-\Delta_{\mathbb{R}^3} V = 2\pi \delta_0$$

$$g_{TN} = V \cdot g_{\mathbb{R}^3} + V^{-1} \theta^2$$

$$\mathbb{C}^2 \setminus \{0\} = (0, \infty) \times S^3$$



fiber length 1
at ∞ .

$$Vol(B_r) \sim r^3$$

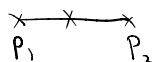
critical.

r large, far from 0.

if two points.

$$V = \frac{1}{2r_{p_1}} + \frac{1}{2r_{p_2}} \approx 1$$

critical point.



lift into rect.