

Lecture 1.

Bishop Volume Comparison

$$\bullet \operatorname{Ric}(g) \geq 1 \cdot g \Rightarrow \operatorname{Vol}(M, g) \leq \operatorname{Vol}(\mathbb{S}^n, g_0)$$

$$(\operatorname{Ric}(\lambda \cdot g) = \operatorname{Ric}(g))$$

↑
round metric
 $\operatorname{Ric}(g_0) = g_0$

• Cheeger, Colding, Perelman

$$\operatorname{Ric}(g) \geq 1 \cdot g + \operatorname{Vol}(M, g) \geq (1 - \varepsilon(n)) \operatorname{Vol}(\mathbb{S}^n, g_0)$$

$$\Rightarrow M \subseteq \mathbb{S}^n$$

diffeomorphism

(Almost Maximal Vol)

⇓
rigidity

Thm (K. Fujita) (X, ω) Kähler-Einstein, metric with positive Ricci curvature.

$$\operatorname{Ric}(\omega) = \omega \Rightarrow \operatorname{Vol}(X, \omega) \leq \operatorname{Vol}(\mathbb{P}^n, \omega_{\text{FS}})$$

$(n+1)^n$
"

↑
complex projective space

still have ↗

Thm (K. Zhang) $\operatorname{Ric}(\omega) \geq \omega$

isometric
" = " $\Leftrightarrow (X, \omega) \cong (\mathbb{P}^n, \omega_{\text{FS}})$

Thm (Y. Liu) $\operatorname{Ric}(\omega) \geq \omega + \operatorname{Vol}(X, \omega) \geq (1 - \varepsilon(n)) \operatorname{Vol}(\mathbb{P}^n, \omega_{\text{FS}})$

$$\Rightarrow X \underset{\text{holomorphic}}{\cong} \mathbb{P}^n$$

Thm (L.-Miao) $(X, \omega) \text{ KE } (\text{Ric}(\omega) = \omega)$

$$X \not\cong \mathbb{P}^n \Rightarrow \text{Vol}(X, \omega) \leq 2n^n < (n+1)^n$$

$$\left(\begin{array}{l} \text{"="} \Leftrightarrow X \cong \mathbb{Q}, \mathbb{P}^1 \times \mathbb{P}^{n-1} \\ \{ z_0^2 + z_1^2 + \dots + z_{n+1}^2 = 0 \} \subset \mathbb{P}^{n+1} \end{array} \right)$$

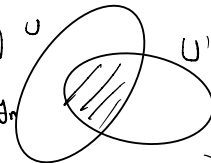
~~L-M-Zhang~~

$$\left. \begin{array}{l} \text{Ric}(\omega) \geq \omega \\ X \not\cong \mathbb{P}^n \end{array} \right\}$$

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1. Kähler - Einstein metric Futaki invariant
 2. Existence and uniqueness
Variational Approach
 3. K-stability
 4. Volume estimate (global)
 $\updownarrow$
 5. Local Volume

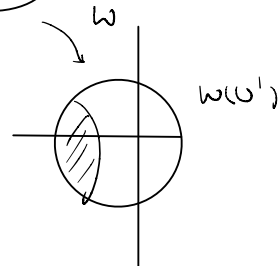
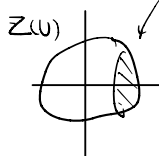
X^n complex manifold

$$Z = (Z_1, Z_2, \dots, Z_n)$$



$$U \xrightarrow[\text{diff. isom.}]{z} Z(U) \subset \mathbb{C}^n$$

$$U' \xrightarrow{w} W(U') \subset \mathbb{C}^n$$



$W = W(Z)$ holomorphic

$$T_{\mathbb{R}} X = \text{span} \left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\} \quad T_{\mathbb{R}} X \otimes \mathbb{C} = \left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\}$$

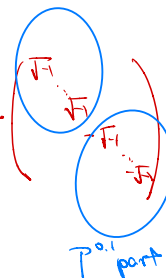
$$J \left(\frac{\partial}{\partial x_i} \right) = \frac{\partial}{\partial y_i}, \quad J \left(\frac{\partial}{\partial y_i} \right) = - \frac{\partial}{\partial x_i}$$

$$\frac{1}{2} \left(\frac{\partial}{\partial x_i} + i \frac{\partial}{\partial y_i} \right)$$

Hermitian metric

$$J: T_{\mathbb{C}} X \rightarrow T_{\mathbb{C}} X, \quad J^2 = -\text{id} \rightsquigarrow J \sim$$

$$T^{1,0} X \oplus T^{0,1} X$$



$$g = \sum_{i,j} g_{i,j} dz_i \otimes d\bar{z}_j \iff \text{Riemannian metric}$$

$$g: T_{\mathbb{C}} X \otimes T_{\mathbb{C}} X \rightarrow \mathbb{C}$$

$$g: T_{\mathbb{R}} X \otimes T_{\mathbb{R}} X \rightarrow \mathbb{R}$$

$$g(J \cdot, J \cdot) = g(\cdot, \cdot)$$

$$g \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) = 0 = g \left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j} \right)$$

$$g \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) = g_{i,\bar{j}}$$

$$(g_{i,\bar{j}})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

positive definite

$$g \longrightarrow \nabla^L \mathbb{C}, \quad \Gamma(TX) \longrightarrow \Gamma(T^*X \otimes TX)$$

$$v_1, v_2 \in \Gamma(TX), \quad \nabla_{v_1} v_2 \in \Gamma(TX)$$

$$T_{\mathbb{C}} X = T^{1,0} X \oplus T^{0,1} X$$

$$\text{span} \left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\} = \text{span} \left\{ \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\}$$

$$\Gamma(T_{\mathbb{C}}X) = \Gamma(T'^0X) \oplus \Gamma(T'^1X)$$

$$\Gamma(T_{\mathbb{C}}^*X \otimes T_{\mathbb{C}}X)$$

Kähler

$$\Leftrightarrow \nabla^{LC} (\Gamma(T'^0X)) \subseteq \Gamma(T^*X \otimes T'^0)$$

$$\Leftrightarrow \nabla^{LC} J = 0$$

$$\nabla_{\frac{\partial}{\partial \bar{z}_j}}^{LC} \frac{\partial}{\partial z_i} = T_{\bar{j}i}^k \frac{\partial}{\partial z_k} + \underbrace{T_{\bar{j}i}^{\bar{l}} \frac{\partial}{\partial \bar{z}_l}}_{=0}$$

$$T_{\bar{j}i}^{\bar{l}} = \frac{1}{2} g^{k\bar{l}} \left(\cancel{\frac{\partial g_{ki}}{\partial \bar{z}_j}} + \underbrace{\frac{\partial g_{k\bar{j}}}{\partial z_i} - \frac{\partial g_{i\bar{j}}}{\partial z_k}}_{=0} \right) = 0$$

$$\Leftrightarrow d\omega = 0, \quad \omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

$$d\omega = \sqrt{-1} \sum_{i,j} \frac{\partial g_{i\bar{j}}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j \quad ?$$

反对称化即有 0.

$$\text{Ric}(g) \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) = - \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \left(\log \left(\det(g_{k\bar{l}}(z)) \right) \right)$$

$$\begin{aligned} &\parallel \\ &R_{i\bar{l}k\bar{j}} g^{k\bar{l}} \\ &\parallel \end{aligned}$$

$$\det(g_{k\bar{l}}(z))$$

$$\langle R(\partial_i, \partial_{\bar{l}}) \partial_k, \partial_{\bar{j}} \rangle$$

$$\left| \det \left(\frac{\partial z}{\partial w} \right) \right|^2 \cdot \det(g_{p\bar{q}}(w))$$

$$\log \det(g_{k\bar{l}}(z)) = \log \left(\left| \det \left(\frac{\partial z}{\partial w} \right) \right|^2 \right)$$

$$\text{Fact. } \partial \bar{\partial} \log |f|^2 \stackrel{\leftarrow}{=} \text{pluri harmonic}$$

$$+ \log(\det(g_{p\bar{q}}(w)))$$

$$A^{p,q}(X) := \text{Span}_{\overbrace{A^0(X)}^{\text{smooth fun.}}} \{ dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{\bar{j}_1} \wedge \dots \wedge d\bar{z}_{\bar{j}_q} \}$$

$$\eta = \sum_{\substack{|I|=p \\ |J|=q}} a_{I\bar{J}}(z) (z) dz_I \wedge d\bar{z}_J$$

$$\parallel \begin{matrix} \{i_1, \dots, i_p\} \\ \{j_1, \dots, j_q\} \end{matrix}$$

$$\left[\begin{array}{c} d^2=0 \\ \Downarrow \\ \partial^2=\bar{\partial}^2=0, \partial\bar{\partial}=-\bar{\partial}\partial \end{array} \right]$$

$$\partial^2=0 \quad \partial: A^{p,q} \longrightarrow A^{p+1,q}, \quad \partial\eta = \sum_{I,J} \frac{\partial a_{I\bar{J}}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J$$

$$\bar{\partial}^2=0 \quad \bar{\partial}: A^{p,q} \longrightarrow A^{p,q+1} \quad \bar{\partial}\eta = \sum_{I,J} \frac{\partial a_{I\bar{J}}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

$$\partial + \bar{\partial} = d: A^{p,q} \longrightarrow A^{p+1,q} \oplus A^{p,q+1}$$

• Hodge Decomposition for compact Kähler mfd

$$\left\{ \begin{array}{l} H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \\ \overline{H^{p,q}} = H^{q,p} \end{array} \right.$$

$$H^k_d \simeq H^k(X, \mathbb{C}) \cong \frac{\{ \text{closed } k\text{-form} \}}{\{ \text{exact } k\text{-form} \}} = \{ \eta, \bar{\partial}\eta=0 \}$$

$$H^{p,q}_{\bar{\partial}} \simeq H^{p,q}(X) \cong \frac{\{ \bar{\partial}\text{-closed } (p,q)\text{-form} \}}{\{ \bar{\partial}\text{-exact } (p,q)\text{-form} \}}$$

//

$$\{ \bar{\partial}\partial \}$$

Hodge Theory

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

$\uparrow$ harmonic k -form $\nwarrow$ harmonic (p,q) -form

$$\Delta_d \eta = 0$$

$$\parallel$$

$$(dd^* + d^*d)$$

$$\Delta_d = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}$$

$$\parallel \qquad \parallel$$

$$\partial\partial^* + \partial^*\partial \qquad \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

• $U = \sum_i U^i \frac{\partial}{\partial z^i} \in T(T^{1,0}X)$ $U^i_{,\bar{j}}$ 协变导数

U holomorphic $\frac{\partial}{\partial \bar{z}^j} U^i = 0 \quad \forall i,j$

• Killing vector field $\xi = \xi^i \frac{\partial}{\partial z^i} + (\overline{\xi^i} \frac{\partial}{\partial \bar{z}^i}) \leftarrow$ real vector field.

$\mathcal{L}_\xi \eta = 0$

/ Lie derivative

$$\underbrace{\frac{1}{2}(\xi - \sqrt{-1}J\xi)}_U + \frac{1}{2}(\xi + \sqrt{-1}J\xi)$$

(1.0)-型 $J(\xi - \sqrt{-1}J\xi) = \sqrt{-1}(\xi - \sqrt{-1}J\xi)$

X-cpt.

$\Rightarrow U = \frac{1}{2}(\xi - \sqrt{-1}J\xi)$ is holomorphic.

$J^2 = -\text{id.}$

$$\left(\overset{\parallel}{\mathcal{L}_\xi J} = 0 \right) \Rightarrow \mathcal{L}_\xi \omega = 0$$

$$\omega(\xi_1, \xi_2) \stackrel{\text{def}}{=} g(J\xi_1, \xi_2)$$

$$\xi \text{ Killing} \Leftrightarrow \nabla \xi \text{ antisymmetric w.r.t. } g$$

Pf. $g_{i\bar{k}} U^i_{,\bar{\partial}\partial} = U_{\bar{k},\bar{\partial}\partial} \langle \nabla_{\xi_1}, \xi_2 \rangle = - \langle \nabla_{\xi_2}, \xi_1 \rangle$

$$= -U_{\bar{j},\bar{k}\partial}$$

$$g_{\bar{j}\partial} = \delta_{j\partial}$$

$$U_{\bar{k},\bar{\partial}\partial} = -U_{\bar{k},\bar{\partial}\partial}$$

$\Downarrow$

$$U^i_{,\bar{\partial}\partial} = 0$$

$$\begin{aligned} &= -U_{\bar{j},\bar{k}\partial} - U_{\bar{q}} R^{\bar{q}}_{\bar{\partial}\partial\bar{k}} \\ &= U_{\bar{j},\bar{k}\partial} - U_{\bar{q}} R_{\bar{q}\bar{k}} \quad \parallel \\ &= U_{\bar{j},\bar{k}\partial} - U_{\bar{q}} R_{\bar{q}\bar{k}} \\ &= -U_{\bar{k},\bar{\partial}\partial} - U_{\bar{q}} R^{\bar{q}}_{\bar{k}\bar{\partial}\partial} \\ &= -U_{\bar{k},\bar{\partial}\partial} - U_{\bar{q}} R_{\bar{q}\bar{k}} \quad \parallel \\ &= -U_{\bar{k},\bar{\partial}\partial} \end{aligned}$$

$$0 = \int U^i_{,\bar{\partial}\partial} U^{\bar{i}} = - \int U^i_{,\bar{\partial}} U^{\bar{i}}_{,\partial} = - \|U^i_{,\bar{\partial}}\|_{L^2}^2$$

$$\therefore \xi \text{ Killing} \Rightarrow U = \frac{1}{2}(\xi - \sqrt{-1}J\xi) \text{ is holomorphic} \quad \square$$

X cpt Kähler

X Kähler complete $\text{Ric}(\omega) \geq \delta \omega$, $\delta > 0$

$\Downarrow$

$b_1 = 0$ X cpt, $\text{Diam}(X, \omega) < +\infty$

$$\underset{\parallel}{H^{1,0}(X)} \oplus \underset{\parallel}{H^{0,1}(X)} = H^1(X, \mathbb{C}) = 0 \Leftarrow \underset{\parallel}{\pi_1(X)} \text{ finite}$$

Rmk: $\pi_1(X) = \{e\}$

Kähler-Einstein $\text{Ric}(\omega) = \lambda \omega$, $\lambda \in \mathbb{R}$

• ω : Kähler $\Leftrightarrow d\omega = 0 \Rightarrow [\omega] \in H^2(X, \mathbb{C})$

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det(g_{k\bar{l}}) \quad \underset{\cup}{H^{1,1}(X)}$$

$$= -\sqrt{-1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g) dz_i \wedge d\bar{z}_j$$

$$\Rightarrow [\text{Ric}(\omega)] \in H^{1,1}(X)$$

$C_1(X)$ first Chern class

don't depend on ω .

$$\text{Ric}(\omega_2) - \text{Ric}(\omega_1) = -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_2^n}{\omega_1^n}$$

$$[\text{Ric}(\omega)] = \lambda [\omega] \quad \text{recalling} \quad \text{Ric}(t \cdot \omega) = \text{Ric}(\omega)$$

$C_1(X)$

Kähler class

(Aubin-Yau)

• $\lambda = -1$ $C_1(X) < 0 \Rightarrow$ Always exist KE

• $\lambda = 0$ $C_1(X) = 0 \Rightarrow$ Always exist Yau

• $\lambda = -1$ $C_1(X) > 0 \Rightarrow \exists$ obstruction to existence.
 $\Leftarrow$ Fano

Ex. $\mathbb{CP}^n$ complex projective space 球面是 Kähler
 $\parallel$ $Gr(1, \mathbb{C}^{n+1})$
 $\mathbb{C}^{n+1} - \{0\} / \mathbb{C}^* = \{1\text{-dim subspaces of } \mathbb{C}^{n+1}\}$

$$0 \neq [z_0, z_1, \dots, z_n] \sim [tz_0, tz_1, \dots, tz_n]$$

$\uparrow$ satisfy $C_1(X) > 0$ condition $\mathbb{C}U \cong \mathbb{C}^n$
 $\bullet n=1 \quad X \cong \mathbb{CP}^1 \cong S^2, \quad \omega = \frac{\sqrt{-1} dz \wedge d\bar{z}}{(1+|z|^2)^2} = \sqrt{-1} \partial\bar{\partial} \log(1+|z|^2)$

$$\omega_{FS} = -\sqrt{-1} \partial\bar{\partial} \log(1+|z_1|^2 + \dots + |z_n|^2) \quad \mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}$$

$$\bullet n=2. \quad \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$$

$$\text{Blow up } \rightsquigarrow \begin{array}{ccc} \text{Bl}_1 \mathbb{P}^2, \text{Bl}_2 \mathbb{P}^2 & \dots & \text{Bl}_8 \mathbb{P}^2 \\ \parallel & & \parallel \\ \mathbb{P}^2 \# \mathbb{P}^2 & & \mathbb{P}^2 \# 8 \overline{\mathbb{P}^2} \end{array}$$

Fact: $\forall n, \exists$ finite (deformation) family of Fano manifolds

Thm (Matsushita) X Fano, $\exists KE \Rightarrow \overset{\circ}{\text{Aut}}(X)$ reductive
 $\parallel$
 $\{\text{biholomorphic automorphism}\}$
 $\text{Aut}^\circ(X) = (\text{Isom}(X, g_{KE}))^\mathbb{C}$

复李群并不总作为
 实李群的变化。

$$\text{auto} = \text{Lie}(\text{Aut}^\circ(X)) = \{\text{holomorphic v.f.}\}$$

$$\text{isom} = \text{Lie}(\text{Isom}(X)) = \{\text{Killing vector field}\}$$

$$\text{auto} = (\text{isom}) \oplus \sqrt{-1}(\text{isom}) \quad , \quad \psi = \psi_{\beta_1} + \sqrt{-1} \psi_{\beta_2}$$

$$\text{Aut}(\mathbb{P}^2) = \text{PGL}(3, \mathbb{C})$$

$$= \text{SL}(3, \mathbb{C}) / \mathbb{Z}_3$$

Lie-algebra

$$\hookrightarrow \text{aut}(\mathbb{P}^2) = \text{sl}(3, \mathbb{C}) = \text{SU}(3, \mathbb{C})^{\mathbb{C}}$$

$$A = S_1 + S_2$$

$$\text{SU}(3, \mathbb{C}) = \{ S \in M_{3 \times 3} \quad ,$$

$$S^* = -S \quad \}$$

$$\text{Tr} S = 0$$

$$\begin{array}{c} \nearrow \quad \nwarrow \\ S_1^* = -S_1 \quad S_2^* = S_2 \end{array}$$

$\text{stab}_{\mathbb{P}} \mathbb{P}^2$

$$\cdot \text{Aut}(\text{Bl}_{\mathbb{P}} \mathbb{P}^2) = \{ \sigma \in \text{PGL}(3, \mathbb{C}) \quad , \quad \sigma([1, 0, 0]) = [1, 0, 0] \quad \}$$

$$\text{Lie}(\downarrow) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \text{sl}(3, \mathbb{C}) \right\} \quad \text{Not reductive}$$

Lecture 2

Recall: (X, ω) Kähler mfd, compact

Kähler: $\omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is closed

Hodge Decomposition: $\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$

||

$\{ \eta, \Delta \eta = 0 \}$

ξ Killing v.f. $\Rightarrow \frac{1}{2}(\xi - \sqrt{-1} J \xi)$ is holomorphic.

$\partial\bar{\partial}$ -Lemma η is d-exact - (p,q) -form

$\downarrow$

$\eta = d\psi$

||

$\sum_{\substack{|I|=p \\ |J|=q}} a_{I\bar{J}} dz_I \wedge d\bar{z}_J$

$\Rightarrow \eta = \partial\bar{\partial} \beta$

$\uparrow$

$(p-1, q-1)$ form

Pf: For $(p,q) = (1,1)$ -form

$(1,1) \rightarrow \eta = d\psi$, $\psi = \psi^{1,0} + \psi^{0,1}$

$= d\psi^{1,0} + d\psi^{0,1}$ 1-form

$= (\underbrace{\partial\psi^{1,0}}_{(2,0)} + \underbrace{\bar{\partial}\psi^{1,0}}_{(1,1)}) + (\underbrace{\partial\psi^{0,1}}_{(1,1)} + \underbrace{\bar{\partial}\psi^{0,1}}_{(0,2)})$

$$\partial \psi^{1,0} = 0$$

$$A^{1,0} = \partial A^{0,0} + \partial^* A^{2,0} + \mathcal{H}^{1,0} \quad \leftarrow \text{Hodge theory}$$

$$\Phi: V \longrightarrow V, \quad V = \text{Im } \Phi \oplus \text{Im } \Phi^* \oplus (\text{Ker } \Phi \cap \text{Ker } \Phi^*)$$

finite-dim

$$\psi^{1,0} = \partial \beta^{0,0} + \partial^* \beta^{2,0} + \underbrace{\beta^{1,0}}_{\text{harmonic}}$$

$$\bar{\partial} \psi^{1,0} = \bar{\partial} \partial \beta^{0,0} + \bar{\partial} \partial^* \beta^{2,0} + 0$$

$$0 = \partial \psi^{1,0} = \cancel{\partial \partial \beta^{0,0}} + \partial \partial^* \beta^{2,0} + \cancel{\partial \beta^{1,0}}$$

$$0 = (\partial \partial^* \beta^{2,0}, \beta^{2,0})_{L^2} = (\partial^* \beta^{2,0}, \partial^* \beta^{2,0})_{L^2}$$

Kähler-Einstein with positive Ricci curv.

$$\text{Ric}(\omega) = \omega \Rightarrow \underline{C_1(X) > 0} \quad (\text{Fano mfd})$$

$$\begin{array}{c} \updownarrow \\ \exists \text{ Kähler form } \omega' \\ \text{s.t. } [\omega'] = C_1(X) \end{array}$$

Complete Kähler metric

$$\implies \bullet \text{Diam}(\omega) \leq C(n)$$

Myer's
thm

$$\bullet \pi_1(X) \text{ finite } H^1(X, \mathbb{C}) = 0$$

Thm (Matsushima) $\exists K \in \Rightarrow \text{Aut}^0(X)$ reductive
 $\downarrow$
 $\{\text{biholomorphic maps}\}$
 $\sigma: X \rightarrow X$

More precisely

$$\{ \text{hol. v. } f \} = \{ \text{kill } g, \text{ v. } f \}^{\mathbb{C}}$$

Prop $\text{Ric}(w) \geq tw \quad (t > 0)$

$$\Rightarrow \lambda_1(-\Delta) \geq t, \quad \Delta f = g^{\bar{i}\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}$$

real operator

(" = $\frac{1}{2} \Delta_{\mathbb{R}}$ ")

pf: $(-\Delta) f = \lambda_1 f$

$$-f_{,i\bar{i}} = -f_{,i\bar{j}} g^{\bar{j}\bar{i}}$$

$$-f_{,i\bar{i}j} = \lambda_{,j} f_{,i} - R^k_{\bar{i}j} f_{,k}$$

$$-\left(f_{,ij\bar{i}} + f_R R_{ij\bar{i}}\right) = -f_{,ij\bar{i}} + f_R R_{ij\bar{i}}$$

i.e. $-f_{,i\bar{j}} + f_{,k} R_{\bar{k}j} = \lambda, f_{,j}$

$$- f_{,ij} f_{\bar{j}} + \frac{f_k R_{\bar{k}j} f_{\bar{j}}}{\nu} = \frac{\lambda_1 f_{,j} f_{\bar{j}}}{\nu} \quad \parallel \lambda_1 |f_{,j}|^2$$

Integral.

$$0 \leq \int |\varphi_{i,\bar{j}}|^2 = (\lambda_i - t) \int |\varphi_{i,\bar{j}}|^2 \Rightarrow \lambda_i \geq t.$$

$$\parallel$$

$$|\varphi_{i,\bar{j}}|^2$$

Cor, $\text{Ric}(\omega) = \omega \Rightarrow \lambda_i(\omega) \geq 1$. $\ker(-\Delta - 1) = \left\{ \varphi : \begin{array}{l} \varphi_{i,\bar{j}} = 0 \\ \int \varphi \omega^n = 0 \end{array} \right\}$

特征函数 give us a hol. v.f.
(全纯向量场)

$$f^i \partial_i \text{ hol. v.f.}$$

inj.

$$F: V_1 \longrightarrow \{ \text{hol. v.f.} \}$$

$$\parallel$$

$$\frac{\partial \varphi}{\partial \bar{z}_i} g^{i\bar{j}} \frac{\partial}{\partial \bar{z}_j}$$

Claim: $V_1 \cong \{ \text{hol. v.f.} \}$

ψ holomorphic v.f.

$\swarrow (0,1)$

$$\frac{\iota_{\psi} \omega}{\uparrow (1,0) \text{ v.f.}} = \sqrt{-1} \bar{\partial} f \text{ for some } f \in C^\infty(X, \mathbb{C})$$

$\nwarrow (0,1) \text{ - form}$

$$\bar{\partial} \iota_{\psi} \omega = \bar{\partial} (\psi^i g_{i\bar{j}} d\bar{z}_j)$$

$$\parallel$$

$$\iota_{\psi} \bar{\partial} \omega = \psi^i \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_p} d\bar{z}_p \wedge d\bar{z}_j$$

$$\parallel$$

$$0$$

$$\psi^i g_{i\bar{j}} = f_{\bar{j}}$$

$$\updownarrow$$

$$\psi^i = f_{\bar{j}} g^{i\bar{j}}$$

$$0 = H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

$$\parallel$$

$$\frac{\ker(\bar{\partial})}{\text{Im}(\bar{\partial})}$$

Lemma $F(\text{Ker}(\Delta+1)_{\mathbb{R}} = \text{Ker}(\Delta+1) \cap C^\infty(X, \mathbb{R}))$

$$\parallel$$

$$\{ \text{Holomorphic v.f } U \text{ s.t. } \text{Im}(U) \text{ is Killing} \}$$

$$\text{Ker}(\Delta+1) = \text{Ker}(\Delta+1)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

$$= \text{Ker}(\Delta+1)_{\mathbb{R}} \oplus \sqrt{-1}(\text{Ker}(\Delta+1)_{\mathbb{R}})$$

$$\begin{array}{ccc} & \text{IS} & \\ \downarrow & & \\ \{ \text{holo. v.f} \} & \cong & \underbrace{\text{Ker}(\Delta+1)_{\mathbb{R}}}_{\text{IS}} \oplus \sqrt{-1} \{ \text{Killing v.f} \} \end{array}$$

$$U = \frac{1}{2}(\xi_1 - \sqrt{-1} J \xi_1) + \sqrt{-1} \left(\frac{1}{2}(\xi_2 - \sqrt{-1} J \xi_2) \right)$$

Ex. For $\mathbb{P}^2$

$$\{ \text{hol. v.f} \} \cong \text{sl}(3, \mathbb{C}) \cong \underbrace{\text{SU}(3, \mathbb{C})}_{\uparrow \{ \text{Killing v.f} \}} \oplus \sqrt{-1} \text{SU}(3, \mathbb{C})$$

Ex. For $\text{Bl}_{\mathbb{P}} \mathbb{P}^2$ Aut^0 not reductive $\Rightarrow \nexists \text{KE}$

$$\parallel$$

$$\text{For } \text{Bl}_{\mathbb{P}_1, \mathbb{P}_2} \mathbb{P}^2 \quad \text{Stab}_{\mathbb{P}} \mathbb{P}^2$$

$$\nexists \text{KE}$$

• Futaki invariant $g = \{ \text{hol. v.f.} \}$

Fut: $g \rightarrow \mathbb{C}$, $X: \text{Fano} \Leftrightarrow C_1(X) > 0$

$\nu \mapsto \int \nu(h_\omega) \omega^n \exists$ Kähler metric ω , s.t. $[\omega] \in C_1(X)$

$$\text{Ric}(\omega) \in C_1(X)''$$

$\downarrow$

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega$$

Thm (Futaki) • Fut is independent of $\omega \in C_1(X)$

• $KE \Rightarrow \text{Fut} = 0$

volume form

Pf: $\Omega = e^{h_\omega} \omega^n$

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \omega^n$$

$\det(g_{k\bar{l}})$

local 坐标
和 global-坐标
作变换-差的
 $\partial \bar{\partial}$ 后为 0.

$$C_1(-K_X, \Omega) = \text{Ric}(\Omega) = -\sqrt{-1} \partial \bar{\partial} \log(e^{h_\omega} \omega^n)$$

$$\downarrow$$

$$e^{h_\omega} \det(g_{k\bar{l}})$$

real positively oriented

$\{ \text{volume form} \} \longleftrightarrow \{ \text{Hermitian metric on } \underline{(-K_X)} \}$

ω
 Ω

$$\left| \frac{\partial}{\partial \bar{z}} \right|_\Omega^2 = \left\langle \Omega, \frac{\partial}{\partial \bar{z}} \wedge \frac{\partial}{\partial \bar{z}} \right\rangle \wedge^n (T_{\text{real}} X)$$

$$\frac{\partial}{\partial \bar{z}} = \det\left(\frac{\partial \omega}{\partial \bar{z}}\right) \frac{\partial}{\partial \omega}$$

$$\alpha(z)(\sqrt{-1})^* dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$$

$$\text{Ric}(e^{h_\omega} \omega^n) = -\sqrt{-1} \partial \bar{\partial} (h_\omega + \log \omega^n)$$

$$\cong \text{Ric}(\omega) - \sqrt{-1} \partial \bar{\partial} h_\omega = \omega$$

$$F_{\text{Ric}}(\Omega) = \int_X \left(\int_V \log \Omega \right) \cdot \frac{\text{Ric}(\Omega)^n}{\omega^n} \quad \text{diff. form}$$

$$= \int \frac{\int_V \Omega}{\Omega} \omega^n = \int \left(\nu(h_\omega) + \frac{\int_V \omega^n}{\omega^n} \right) \cdot \omega^n$$

$$\int_X \int_V \omega^n = 0.$$

$$\Omega_1 = e^\phi \Omega_0$$

$$F_t = \int_X \frac{\int_V (\log \Omega_t) \cdot (\text{Ric}(\Omega_t))^n}{\int_V (t\phi + \log \Omega_0)} \quad \Omega_t = e^{t\phi} \Omega_0$$

$$\dot{F}_t = \int_X \frac{\nu(\phi) \frac{\text{Ric}(\Omega_t)^n}{\omega_t^n} + \theta_V \cdot n \omega_t^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \phi}{\int (\theta_V \Delta_{\omega_t} \phi) \omega_t^n}$$

Here, ω is a metric

$$= 0$$

but it's not necessary!

$$\int \theta_V \phi_i^i \omega_t^n$$

$$= \int \nu^i \phi_i \omega_t^n$$

$$F_{\text{Ric}} = \int \theta_V \cdot \omega^n$$

$$= \frac{1}{n+1} \int (\omega + \theta_V)^{n+1}$$

$$\theta_V = \int_V \log \Omega = \frac{\int_V \Omega}{\Omega}$$

$$\omega = \text{Ric}(\Omega)$$

$$\int_V \omega = \sqrt{-1} \partial \bar{\partial} \theta_V$$

$$\Downarrow$$

(equivariant
cohomology)

$$\begin{array}{ccc} (\sqrt{-1} \bar{\partial} - l_v) (w + \theta_v) & = & 0 \\ \downarrow & & \downarrow \\ \bar{\partial}_v & & \gamma \end{array}$$

$\rightsquigarrow$ localization formula

(Bott residue formula)

$$\begin{aligned} (S')^G & \\ \cong & \text{group action} \\ \mathbb{C}^* & \hookrightarrow X \end{aligned}$$

$$(t, x) \mapsto t \cdot x$$

Ex $\mathbb{C}^* \hookrightarrow \mathbb{P}^2$

$$t \circ (z_1, z_2) = (t z_1, t^k z_2)$$

$$t \circ ([z_0, z_1, z_2]) = ([z_0, t z_1, t^k z_2])$$

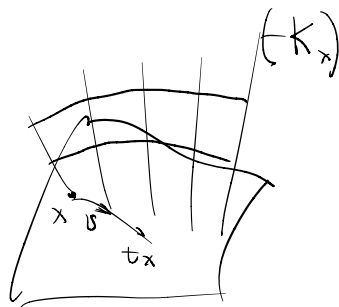
$$\rightsquigarrow v = z_1 \frac{\partial}{\partial z_1} + k z_2 \frac{\partial}{\partial z_2} \quad \hookrightarrow \text{Im}(v) \text{ Killing}$$

$$l_v w = \sqrt{-1} \bar{\partial} \theta_v$$

$$\begin{aligned} & \downarrow \\ \theta_v &= \frac{|z_1|^2 + k|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} + \text{const.} \end{aligned}$$

X Fano, $-K_X = \Lambda^n T_{\text{hol}} X$

$$\mathbb{C}^* \hookrightarrow (X, -K_X)$$



$$t \circ (v_1 \wedge \dots \wedge v_n) = t_*(v_1) \wedge \dots \wedge t_*(v_n)$$

$$H^0(X, m(-K_X)) = \{ \text{holomorphic sections} \}$$

$$\parallel$$

$$(-K_X)^{\otimes m}$$

$$\begin{array}{ccc} E_X & H^0(P^n, -mK_X) \cong & \left\{ \begin{array}{l} \text{homogeneous polynomials} \\ \text{in } [Z, Z_1, \dots, Z_n] \\ \text{of degree } (n+1)m \\ \parallel \\ \text{dim} \end{array} \right\} \\ \swarrow & \uparrow & \\ & \mathbb{C}^* & \\ & \parallel & \\ & \mathbb{C}^{\binom{n+1}{m}} & \end{array}$$

$$\parallel$$

$$\bigoplus_{\lambda_i \in \mathbb{Z}} V_m(\lambda_i^{(m)})$$

$$\parallel$$

$$\{ F \in V_m : t_0 F = t^{\lambda_i} F \}$$

$$\lambda_1^{(m)} \leq \lambda_2^{(m)} \leq \dots \leq \lambda_{N_m}^{(m)}$$

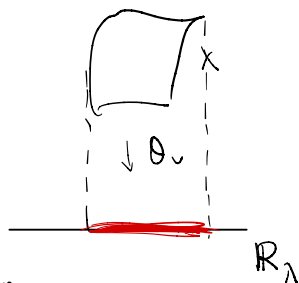
$$N_m = \dim H^0(P^n, -mK_X)$$

$$= \binom{(n+1)m+n}{n}$$

Another view. (几何)

$$Fut(v) = \int_X \theta_v \omega^n = \int_{R_\lambda} \lambda(\theta_v)_* \omega^n$$

↓
measure



Disternact - Heckmann
不依赖于平形式选取

$$S' \hookrightarrow (X, \omega)$$

$$L_{S'} \omega = d\theta$$

$$[w]$$

↓

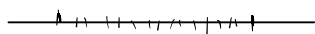
Fact:

$$\rightarrow \theta_* \omega^n \text{ independent of } w$$

Fact

$$(\theta_v)_* \omega^n = \lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \int \frac{\lambda_i^{(m)}}{m} \quad \text{下标}$$

$m \uparrow$, δ -measure 稠密.



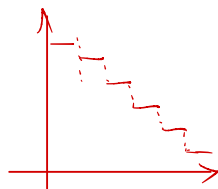
(DH. Brion)

$$Fut(v) = \lim_{m \rightarrow \infty} \frac{1}{N_m} \sum_{i=1}^{N_m} \frac{\lambda_i^{(m)}}{m}$$

↑
某种 weight.

$$= \int_{-A(v)}^T \lambda \cdot \frac{DH(v)}{e^*} = -dvol$$

measure.



↓ $m \rightarrow \infty$

$$v \hookrightarrow \mathbb{C}^* \hookrightarrow \mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1$$

$$t \cdot (z_1, z_2) = (t^{\underline{1}} z_1, t^{\underline{k}} z_2)$$

$$A_x(v) = \underline{1 + k} \quad \frac{\lambda_i^{(m)}}{m} \rightarrow -A_x(v)$$

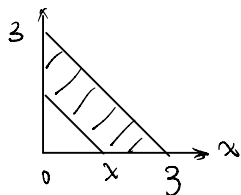
$$\underline{Fut} = \frac{1}{v} \int_{-A(v)}^T \lambda \cdot (-dvol(\lambda))$$

$$= \left[-\lambda vol(\lambda) \Big|_{-A(v)}^T - \int_{-A(v)}^T -vol(\lambda) d\lambda \right] \frac{1}{v}$$

transport?

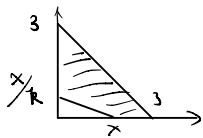
$$= A(v) - \frac{1}{v} \int_0^\infty Vol((-K_x) - xE) dx$$

$$k=1, \quad A(v_1) = 2$$



$$\frac{3^2}{2} = vol(\mathbb{P}^2)$$

$$A(v_k) = 1 + k$$



$$Fut = 0. \quad \checkmark$$

Lecture 3

$$\text{vol}(\omega) = \int \omega^n = \langle [\omega]^n, [X] \rangle$$

$$F_{\text{ut}}(v) = \frac{1}{\text{vol}} \int_X \frac{L_X \Omega}{\Omega} \left(\text{Ric}(\Omega) \right)^n \quad \Omega \text{ sm. volume form}$$

Assume
Im v Killing

$\Downarrow$

θ_v real

$$L_{v-\bar{v}} \omega = 0$$

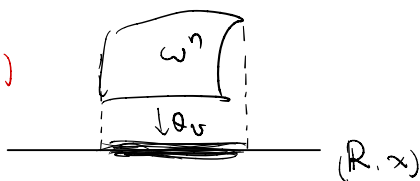
$$\sqrt{-1} \partial \bar{\partial} (\theta - \bar{\theta})$$

$$= \frac{1}{\text{vol}} \int_X \theta_v \cdot \omega^n \quad (L_v \omega = \sqrt{-1} \partial \bar{\partial} \theta_v)$$

$$= \frac{1}{\text{vol}} \int_{\mathbb{R}} \chi(\theta_v)_* \omega^n$$

$$\theta_v : X \rightarrow \mathbb{R}$$

$$L_{\text{Im}(v)} \omega = d\theta_v$$



Assume

Im v generates S^1 -action

$\Leftrightarrow v$ generates a $\mathbb{G}^*$ action

$$\hookrightarrow (X, -K_X)$$

$$\rightsquigarrow \mathbb{G}^* \hookrightarrow H^0(X, -mK_X) = V_m$$

$$\overset{\parallel}{\wedge^n T_{\text{hol}} X}$$

$$\parallel (-K_X)^{\otimes m}$$

$$N_m = \dim V_m$$

Riem. Roach.

$$\bigoplus_{\lambda_i^{(m)} \in \mathbb{Z}} V_m(\lambda_i^{(m)}) = \{ s \in V_m : t \cdot s = t^{\lambda_i^{(m)}} s \}$$

$$\lambda_1^{(m)} < \lambda_2^{(m)} \leq \dots \leq \lambda_{N_m}^{(m)}$$

$$\frac{m^n}{n!} \text{Vol}(\omega) \approx O(m^{n-1})$$

$$\mu_m = \frac{1}{N_m} \sum_{i=1}^{N_m} \int \chi_{\lambda_i^{(m)}} \omega^n \xrightarrow{m \rightarrow +\infty} ((\theta_v)_* \omega^n) \frac{1}{\text{Vol}(\omega)} = \mu_\infty$$

$$\frac{1}{N_m} \sum \frac{\lambda_i^m}{m} = \int_{\mathbb{R}} \lambda \mu_m \xrightarrow{m \rightarrow +\infty} \int_{\mathbb{R}} \lambda \cdot \mu_\infty$$

$$\frac{1}{N_m} \left(-d \left(\dim \left\{ s: \underbrace{\frac{wt(s)}{m} \geq \lambda}_{\substack{\text{weight} \\ \frac{m^n}{n!} a(s)}} \right\} \right) \right) \xrightarrow{m \rightarrow +\infty} \frac{1}{Vol} \int_{\mathbb{R}} \lambda (-da(s))$$

$\parallel$
 $F_{wt}(U)$

Ex. $V = \mathbb{C} z_1 \frac{\partial}{\partial z_1} + k z_2 \frac{\partial}{\partial z_2} \rightsquigarrow t(z_1, z_2) = (t z_1, t^k z_2)$
 $k \in \mathbb{Z}_{>0}$

$$S \in V_m = H^0(X, -mK_X)$$

$$f_{(s_1, s_2)} \left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \right)^{\otimes m}$$

$$wt(s) = wt(f) - \underbrace{(\frac{1}{2}k)}_{A(s)} m$$

$$\left\{ s: \frac{wt(s)}{m} \geq \lambda \right\} \Leftrightarrow \left\{ s: \frac{wt(f)}{m} \geq A(s) + \lambda \right\}$$

$\cap$
 $H^0(X - mK_{P^2}) \quad wt(z^i z^j) = i + jk$

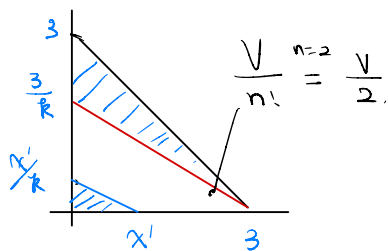
homogeneous polynomial of degree

$$\{z_1^i z_2^j, i+j \leq 3m\} = \text{inhomogeneous poly. in } z_1, z_2 \stackrel{(3m)}{\deg \leq 3m}$$

$\parallel$
 $\binom{3m+2}{2} \sim \frac{3^2}{2} m^2 + O(m)$

$$\# \left\{ i+jk \geq m x', \hat{i}+\hat{j} \leq 3m \right\} \sim \frac{m^2}{2!} \text{vol}((-kx) - \underbrace{x'v}_{x \mapsto A(v)})$$

$$\begin{aligned} F_{\text{up}}(v) &= \frac{1}{\text{vol}} \int_{-A_x}^{+\infty} (-d(\text{vol}(-kx) - (x \mapsto A(v))v)) \\ &= A(v) - \frac{1}{\text{vol}} \int_0^{+\infty} \text{vol}(-kx - x'v) dx' \end{aligned}$$



$$0 = (1+k) - \frac{1}{9} \int_0^{3/k} (9 - \frac{x'^2}{k}) dx' - \int_{3/k}^3 (3 - \frac{3}{k}) 3 \frac{(3 - \frac{x'}{k})^2}{3 - \frac{3}{k}} dx$$

$$\underline{\text{KE}} \quad \text{Ric}(w) = w, \quad C_1(x) > 0$$

ψ
 w_0

$$\begin{aligned} H^2(X, \mathbb{R}) \ni [w] &= [w_0] \Rightarrow w - w_0 = \sqrt{-1} \partial \bar{\partial} \varphi \\ &\Rightarrow w_\varphi = w_0 + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \end{aligned}$$

$$\text{WTFind } \varphi, \text{ s.t. } \text{Ric}(w_\varphi) = w_\varphi.$$

$$\mathcal{H}(w_0) = \{ \varphi \in C^\infty(X, \mathbb{R}), w_\varphi > 0 \}$$

Space of Kähler potential $\left((g_{i\bar{j}})_{i,j} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right)_{i,j} > 0$

$$\mathcal{H}(\omega_0) = \{ \varphi \in L^1(X), \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0 \}$$

$\uparrow$
 ω_0 -plurisubharmonic.

$$\begin{aligned} 0 &= \text{Ric}(\omega_\varphi) - \omega_\varphi \\ &= \underbrace{\text{Ric}(\omega_\varphi) - \text{Ric}(\omega_0)}_{\parallel -\sqrt{-1} \partial \bar{\partial} \log \frac{\omega_\varphi^n}{\omega_0^n}} + \underbrace{\text{Ric}(\omega_0) - \omega_0}_{\parallel \sqrt{-1} \partial \bar{\partial} h_{\omega_0}} + \underbrace{\omega_0 - \omega_\varphi}_{\parallel -\sqrt{-1} \partial \bar{\partial} \varphi} \end{aligned}$$

$$0 = -\sqrt{-1} \partial \bar{\partial} \left(\log \frac{\omega_\varphi^n}{\omega_0^n} - h_{\omega_0} + \varphi \right)$$

$$0 = \sqrt{-1} \partial \bar{\partial} f = \sqrt{-1} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$$

$$0 = g^{i\bar{j}} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} = \Delta f = \text{tr}_{\omega_0}(\sqrt{-1} \partial \bar{\partial} f)$$

compact

$$K \in \mathcal{H} \Leftrightarrow \log \frac{\omega_\varphi^n}{\omega_0^n} - h_{\omega_0} + \varphi = C$$

$$\Leftrightarrow \omega_\varphi^n = e^{h_{\omega_0} - \varphi} \omega_0^n$$

$$\parallel$$

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi)^n$$

complex Monge-Ampère
 复号 会出问题

$$\omega_\varphi^n = e^{h + \varphi} \omega_0^n \quad \text{一定有解.}$$

$$\sqrt{-1} \partial \bar{\partial} \varphi(x_0) \leq 0$$

Max. Principle. $\log \frac{\omega_\varphi^n}{\omega_0^n} - h = \varphi(x) \quad \varphi(x_0) = \max \varphi$

$$\log \frac{\omega_n}{\omega_0} - h$$

$$\Rightarrow \varphi(x_0) \leq -h(x_0)$$

Yau: $\omega_\varphi^n = e^h \omega_0^n$ Max. argument doesn't work.

Tian $\text{Ric}(\underbrace{e^{h_\omega}}_{\Omega} \omega^n) = \omega$

$$F_\omega(\varphi) = -\underbrace{E(\varphi)}_{\substack{\uparrow \\ \text{Monge-Ampère energy} \\ (\text{Aubin-Yau})}} - \log \left(\int e^\varphi \Omega \right) \quad \forall \varphi \in \mathcal{H}(\omega)$$

$F_\omega''(\varphi + c)$ $\uparrow$ Monge-Ampère energy (Aubin-Yau)

$$\begin{matrix} \varphi \\ \downarrow \\ (\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n \end{matrix}$$

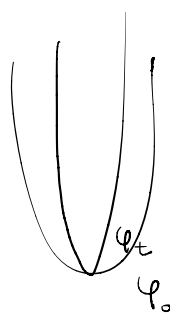
$$\underbrace{(\delta E)(\delta \varphi)}_{\{}} = \frac{1}{V} \int (\delta \varphi) \omega_\varphi^n = \det'' \left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right) dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$$

$$\begin{aligned} \delta \int |\nabla u|^2 \Omega &= \int (\nabla u) (\nabla \delta u) \Omega \\ &= - \int (\delta u) \Delta u \Omega \end{aligned}$$

Ex $E(\varphi) = \frac{1}{n+1} \sum_{i=0}^n \int \varphi \omega^i \wedge \omega_\varphi^{n-2}$

$$\frac{d}{dt} E(\varphi_t) = \int \left(\frac{\partial \varphi}{\partial t} \right) \omega_{\varphi_t}^n$$

$$0 = \frac{d}{dt} F(\varphi_t) = -\frac{1}{v} \int \dot{\varphi} \omega_{\varphi}^n - \frac{\int (\bar{\partial}^* \Omega)(\dot{\varphi})}{\int e^{-\varphi} \Omega}$$



$$= -\frac{1}{v} \int_X \dot{\varphi} \left(\omega_{\varphi_0}^n - \frac{e^{-\varphi_0} \Omega}{\int_X e^{-\varphi_0} \Omega} \right)$$

$$\underline{\omega_{\varphi_0}^n = C e^{-\varphi_0} \Omega} \quad \text{KE-equ. (*)}$$

(Tian, Dag-Tian)

Thm : $\exists$ KE (i.e. solution to (*)) $\omega_{KE} = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_{KE}$
 $\Rightarrow \omega_{KE}$ obtains the minimum of $F_{\omega}(\varphi) (= F_{\omega}(\omega_{\varphi}))$

Thm (Uniqueness) KE is unique up to $\text{Aut}^0(X)$

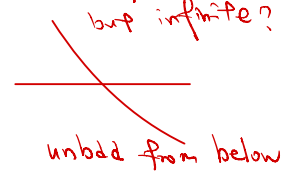
(Bando-Mabuchi) $\omega_{KE} = \sigma^* \omega'_{KE}$

$$(\sigma^* \text{Ric}(\omega) = \text{Ric}(\sigma^* \omega))$$

finite dim
easy
but infinite?

Thm (Bordignon) $F_{\omega}(\varphi)$ is convex.

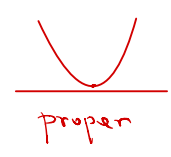
$$\left[\varphi_t = (1-t) \varphi_1 + t \varphi_2, \quad \underline{\omega_{\varphi_t} > 0} \right]$$



Thm (Tian) Assume $\text{Aut}(X)$ discrete

$\Updownarrow$
 $(\neq \text{holomorphic vector field})$

Then $\exists$ KE $\Leftrightarrow F_{\omega}(\varphi)$ is proper



Tian-Zhu $\Leftrightarrow F_W(\varphi)$ is coercive
 Phong - Sturm - Song - Meinke (modulo $\text{Aut}^0(X)$)

Dairas - Rubinstein
 $\exists \delta > 0$

$$F_W(\varphi) \geq \delta \|w_\varphi\|^2 - C \quad \underline{\text{coercive}}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ F_W(w_\varphi) & & J_W(\varphi) \end{array}$$

$$F_W(\varphi) \geq \delta \|w_\varphi\|^2 - C \quad (\text{Tian})$$

$$J_W(\varphi) = \frac{1}{V} \int \varphi w^n - E(\varphi)$$

$$= \sum_{i=1}^n \underbrace{\left(\frac{n-i}{n+1} \right)}_{\downarrow 0} \int \underbrace{\sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi}_{\int |\nabla \varphi|^2} \wedge w^{n-1-i} \wedge w_\varphi^i$$

up to a constant
 they are almost equivalent.

$$I_W(\varphi) = \frac{1}{V} \int \varphi (w^n - w_\varphi^n)$$

$$= \frac{1}{n+1} \sum_{i=0}^{n-1} \frac{\sqrt{-1}}{V} \int \partial \varphi \wedge \bar{\partial} \varphi \wedge w^{n+1-i} \wedge w_\varphi^i$$

pf

" $\Leftarrow$ " Continuity Method

$$(*)_t \quad \text{Ric}(w_\varphi) = t w_\varphi + (1-t) w_0, \quad 0 \leq t \leq 1$$

$$t=0: \text{Ric}(w_\varphi) = w_0 \Leftrightarrow \underline{w_\varphi^n = e^{h_w} w^n} \quad (*) \quad \checkmark$$

Tian's Fields medal.

$$S = \{ t \in [0, 1], (*) \text{ is solvable} \}$$

$$\bullet 0 \in S$$

$$\bullet S \text{ is open set } (\Leftarrow \lambda_1(-\Delta w_t) > t)$$

$$\bullet S \text{ is closed?} \quad \begin{array}{c} \uparrow \\ \text{Ric}(w_t) \geq t w_t \end{array}$$

If $(*)_t$ is solvable in $[0, \beta)$, $0 \leq \beta < 1$,

then $\|\varphi_t\|_{C^{k,\alpha}} \leq C < +\infty, \forall t \in [0, \beta)$ How to prove?

$$\Rightarrow (*)_\beta \text{ is solvable}$$

$$\Rightarrow S \text{ is open and closed} \Rightarrow S \in [0, 1]$$

Step 1 $F(w_{\varphi_t}) \leq C < +\infty$

$$\forall$$

$$\delta \cdot J(w_{\varphi_t}) - C_1$$

coercive

$$\mid \Rightarrow J(w_{\varphi_t}) \leq \frac{C + C_1}{\delta}$$

bdd in a compact

set.

$$I(w_{\varphi_t})$$

the unbounded case

is impossible



$$\sup \varphi_t - \inf \varphi_t$$

$$\parallel$$

Step 2

$$\text{Osc}(\varphi_t) \leq C_1 I(w_{\varphi_t}) + C_2$$

Step 3

$$(C^2\text{-estimate})$$

Calabi Aubin-Yau

$$\|\Delta \varphi\|_{L^2} \leq C_1 e^{C_2 \text{Osc}(\varphi)}$$

Step 4 ($C^{k,\alpha}$ -estimate) Evens-Krylov

$$\text{Step 1: } \sup \varphi \leq \frac{1}{V} \int \varphi \omega^n + C$$

uniformly bdd.

$$\varphi(x) = \frac{1}{V} \int \varphi(y) dy + \int \underbrace{\frac{G(x,y)}{V}}_0 \underbrace{(-\Delta \varphi(y))}_{< n} dy$$

$$\omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0$$

$$\Delta_y G(x,y) = \delta(x-y)$$

$$n + \Delta \varphi > 0$$

$$\frac{1}{|x-y|^{n-2}}$$

$$-\Delta \varphi < n$$

$$\bullet -\inf \varphi_t \leq C_1 \int (-\varphi_t) \omega_{\varphi_t}^n + C_2 \quad t \geq \epsilon > 0$$

$$\omega = \omega_{\varphi_t} - \sqrt{-1} \partial \bar{\partial} \varphi_t > 0$$

$$(\varphi - \sup \varphi) - 1 \leq -1$$

$$n - \underbrace{\Delta_t \varphi_t}_{\Delta \omega_t} > 0$$

$$\psi = -\varphi \geq 1 > 0$$

$$\int (-\varphi)^k \Delta_t \varphi_t \omega_t^n < \int n (-\varphi)^k \omega_t^n \leq n \int \underbrace{\varphi^{k+1}}_{\|\varphi\|_{k+1}^{k+1}} \omega^n$$

$$-\int \varphi^k \Delta \varphi \omega_t^n = \int \underbrace{k \varphi^{k-1} |\nabla \varphi|^2}_{\|\varphi\|^2} \omega_t^n$$

$$\int C_p |\nabla \varphi^p|^2 = \frac{4k}{(k+1)^2} \int |\nabla \varphi^{\frac{k+1}{2}}|^2 \omega_t^n$$

$$p^2 \cdot p^{2p-2} |\nabla \varphi|^2, \quad 2p-2 = k-1$$

$$p = \frac{k+1}{2} \quad C_p \cdot p^2 = k \Rightarrow C_p = \frac{4k}{(k+1)^2}$$

$$W^{k,p} \hookrightarrow L^{\frac{2np}{2n-kp}} \quad (k=1, p=2)$$

$$W^{1,2} \hookrightarrow L^{\frac{2n}{n-1}}$$

$$\left(\int |u|^{\frac{2n}{n-1}} \right)^{\frac{n-1}{n}} \leq \int |\nabla u|^2$$

$$\frac{4p}{(p+1)^2} \left[\int \left(\varphi^{\frac{p+1}{2}} \right)^{\frac{2n}{n-1}} \right]^{\frac{n-1}{n}}$$

$$\parallel \\ C_p \parallel \varphi \parallel_{\left(\frac{p+1}{2} \right)^{\frac{n-1}{n}}}^{p+1}$$

$$\Rightarrow \frac{1}{C_p} \parallel \varphi \parallel_{\left(\frac{p+1}{2} \right)^{\frac{n-1}{n}}} \leq \parallel \varphi \parallel_{p+1}$$

"right ~"

$$\rightsquigarrow \parallel \varphi \parallel_{L^\infty} \leq C \parallel \varphi \parallel_{L^2}$$

$$\parallel \varphi \parallel_{L^2} \leq \frac{1}{c\sqrt{v}} \int |\varphi|^2 \omega^n \leq \int \varphi \omega^n$$

$$\lambda_1(-\Delta_\varphi) \geq t > c > 0$$

$$\parallel \\ \int -\varphi \omega_\varphi^n$$

$$\therefore \sup \varphi' \leq C_1 \int -\varphi' \omega_\varphi^n + C_2$$

$$\begin{aligned} \text{osc } \varphi &= \varphi - \inf \varphi \\ &\leq \varphi - \sup \varphi - 1 \\ &\leq C_1 I_\omega(\varphi) + C_2 \end{aligned}$$

Lecture 4

Recall Thm $\exists KE \Leftrightarrow F_\omega(\varphi)$ coercive

(modulo $\text{Aut}(X)$)

Thm $F_\omega(\varphi)$ is a convex functional

$\leadsto \omega_{KE}$ minimizer $F_\omega(\varphi)$

unique (modulo $\text{Aut}(X)$)

$$\mathcal{H}(\omega) = \left\{ \varphi \in C^\infty(X, \mathbb{R}) \mid \omega_\varphi > 0, \int \varphi \omega^n = 0 \right\}$$

(Mabuchi)

$$\langle \varphi_1, \varphi_2 \rangle_{\omega_\varphi} = \int_X \varphi_1 \varphi_2 \omega_\varphi^n$$

$\varphi \in C^\infty(X, \mathbb{R})$



$$2 \langle \nabla \varphi_1 \varphi_2, \varphi_3 \rangle = \varphi_1 \langle \varphi_2, \varphi_3 \rangle + \varphi_2 \langle \varphi_1, \varphi_3 \rangle$$

$$\left\{ \begin{array}{l} - \varphi_3 \langle \varphi_1, \varphi_2 \rangle \pm \langle [\varphi_1, \varphi_2], \varphi_3 \rangle \end{array} \right.$$

$$\varphi_0 = \varphi$$

$$\varphi_t = \varphi + t \varphi_1$$

$$\left. \frac{d}{dt} \right|_{t=0} \int_X \varphi_2 \cdot \varphi_3 \omega_{\varphi_t}^n$$

$$V(f) = \frac{d}{dt} f(X_t)$$

$$\text{i.e.} = \int_X \varphi_2 \cdot \varphi_3 \Delta \varphi_1 \omega_\varphi^n$$

$$+ \int_X \varphi_1 \cdot \varphi_3 \Delta \varphi_2 \omega_\varphi^n$$

$$- \int_X \varphi_1 \cdot \varphi_2 \Delta \varphi_3 \omega_\varphi^n$$

$$\Delta \varphi_3 (\varphi_1, \varphi_2)$$

$$= -2 \int_X \langle \nabla \psi_1, \nabla \psi_2 \rangle \psi_3 \omega_\varphi^n$$

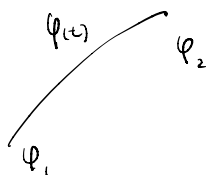
$$\Rightarrow \nabla_{\varphi_1} \psi_2 = - \nabla \psi_1 \cdot \nabla \psi_2$$

$$\Phi = (\varphi(t))_{t \in I} \quad \text{geodesic}$$

$$\ddot{\varphi} + \nabla_{\dot{\varphi}} \dot{\varphi} = 0 \Leftrightarrow \ddot{\varphi} - |\nabla \dot{\varphi}|_{\omega_\varphi}^2 = 0$$

$$\text{local} \quad \ddot{\varphi} - \dot{\varphi}_i \dot{\varphi}^i = 0$$

$$\ddot{\varphi}_i \dot{\varphi}_{\bar{j}} g^{i\bar{j}}$$



$$\text{Thm (X.-X. Chen)} \quad \varphi_1, \varphi_2 \in \mathcal{H}(\omega), \exists \Phi = (\varphi(t))_{0 \leq t \leq 1}$$

$$\varphi(t) \text{ is } C^\infty$$

同样的 Kähler potential
可以连光滑测地线

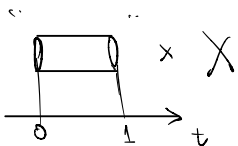
Semmes, Donaldson

$$\Phi(t, z) = \varphi_t(z)$$

Geodesic equation

$$S^1 \times I \times X$$

$$(\pi^* \omega + \sqrt{-1} \partial \bar{\partial} \Phi)^{n+1} = 0$$



$$\Phi(0, z) = \varphi$$

$$\Phi(1, z) = \varphi$$

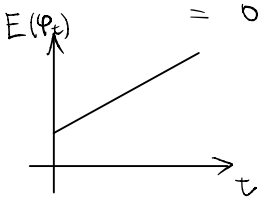
$$F = -E + L, \quad L(\varphi) = -\log\left(\frac{1}{V}\right) e^{-\varphi} \Omega$$

prove $\hookrightarrow$ convex

$$\Phi = (\varphi_t)$$

$\{e^{\varphi} \omega^n\}$

$$\begin{aligned} \frac{d^2}{dt^2} E &= \frac{d}{dt} \int \dot{\varphi} \omega_{\varphi}^n = \int \ddot{\varphi} \omega_{\varphi}^n + \int \dot{\varphi} \frac{n \omega_{\varphi}^{n-1} \wedge \sqrt{-1} \partial \bar{\partial} \varphi}{\Delta \dot{\varphi} \omega_{\varphi}^n} \\ &= \int (\ddot{\varphi} - |\nabla \dot{\varphi}|^2) \omega_{\varphi}^n \\ &\quad \underbrace{\quad}_{\text{Ric}_{\omega_{\varphi}}(\sqrt{-1} \partial \bar{\partial} \varphi)} \end{aligned}$$



φ is regularity
may be not good.

$$\frac{d^2}{dt^2} L(\varphi) = \frac{d}{dt} \frac{\int \dot{\varphi} e^{-\varphi} \Omega}{\int e^{-\varphi} \Omega}$$

You can use approx argument
or something technique to deal
with it!

$$= \frac{\int \ddot{\varphi} e^{-\varphi} \Omega - \int \dot{\varphi}^2 e^{-\varphi} \Omega}{\int e^{-\varphi} \Omega} + \frac{\left(\int \dot{\varphi} e^{-\varphi} \Omega\right)^2}{\left(\int e^{-\varphi} \Omega\right)^2}$$

$$\underbrace{d\mu = \frac{e^{-\varphi} \Omega}{\int e^{-\varphi} \Omega}}_{\text{measure}} = \int \ddot{\varphi} d\mu - \int \dot{\varphi}^2 d\mu + \left(\int \dot{\varphi} d\mu\right)^2$$

$$\int d\mu = 1$$

$$\dot{\varphi}_1 = \dot{\varphi} - \int \dot{\varphi} d\mu$$

$$= \int (\ddot{\varphi} - \dot{\varphi}^2) d\mu + \int |\nabla \dot{\varphi}_1|^2 d\mu - \int |\dot{\varphi}_1|^2 d\mu \geq 0$$

$$\text{Ric}(d\mu) = -\sqrt{-1} \partial \bar{\partial} \log(e^{-\varphi} \Omega)$$

$$= \sqrt{-1} \partial \bar{\partial} \varphi + \underbrace{\text{Ric}(\Omega)}_{\omega}$$

$$= \omega_{\varphi}$$

Lemma $d\mu = e^{-\varphi} \omega^n$, $\text{Ric}(d\mu) \geq \omega$

$$\Rightarrow \lambda_1(-\Delta_{\varphi}) = \inf_{\substack{u \in C^{\infty} \\ \int u d\mu = 1}} \frac{\int u_i \bar{u}_i d\mu}{\int u^2 d\mu} \geq 1$$

$$\text{Ker}(-\Delta_{\varphi} - 1) = \{ u \in C^{\infty}(X), u|_{\partial} = 0 \}$$

凸泛函边界即最小点

Ex. $U = \text{Re}(v) + \sqrt{-1} \text{Im}(v)$

← Killing

$$\rightsquigarrow \sigma: X \rightarrow X$$

$$\sigma(t)^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$$

全纯向量场

↓
测地线

• φ_t is a geodesic

$$\bullet \frac{d}{dt} F_{\omega}(\varphi_t) = -F_{\omega}(\nu)$$

$$\frac{d}{dt} F_{\omega}(\varphi_t) = -\frac{1}{V} \int \dot{\varphi} \omega_{\varphi}^n + \frac{\int \dot{\varphi} e^{-\varphi} \Omega}{\int e^{-\varphi} \Omega}$$

$$= -\frac{1}{V} \int (\dot{\varphi} - \overset{\text{mean}}{\dot{\varphi}}) \omega_{\varphi}^n$$

$$\dot{\varphi}_t - \dot{\varphi}_t = 0$$

$$= -\frac{1}{V} \int \varphi_{\nu} \omega_{\varphi}^n = -F_{\omega}(\nu)$$

$J_\omega \rightarrow +\infty$. $F_\omega \rightarrow +\infty$ linear

$J_\omega \rightarrow +\infty$. $F_\omega \rightarrow +\infty$ linear

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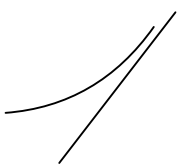
$J_\omega \rightarrow +\infty$. $F_\omega \rightarrow +\infty$ linear

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$J_\omega \rightarrow +\infty$. $F_\omega \rightarrow +\infty$ linear



$$J(\varphi_s) \sim s$$

$$\lim_{s \rightarrow \infty} \frac{F(s)}{s} = \lim_{s \rightarrow \infty} F'(s)$$

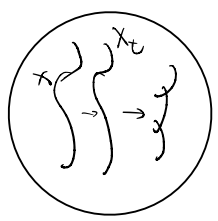
$$\begin{aligned} & \wedge T_{\text{hol}} X \\ & \parallel \\ L &= -K \end{aligned}$$

$$\begin{aligned} & L \\ & \downarrow \\ X & \hookrightarrow \mathbb{P}^{N_m} = \mathbb{P}(H^0(X, -mK_X)) \end{aligned}$$

$$\begin{aligned} G_1(L) & \ni \omega \\ \dim H^0(X, -mK_X) \end{aligned}$$

• Test configuration

$$\begin{aligned} & \mathbb{P}^{N_m} \\ & \nwarrow \text{Aut}(\text{PGL}(N_m+1)) \end{aligned}$$



$$\begin{aligned} & \uparrow \\ \mathbb{C}^* & \ni t \end{aligned}$$

$$(\sigma(t) : \mathbb{P}^N \rightarrow \mathbb{P}^N)$$

$$X_t = \sigma(t)(X)$$

$$\downarrow t \rightarrow 0$$

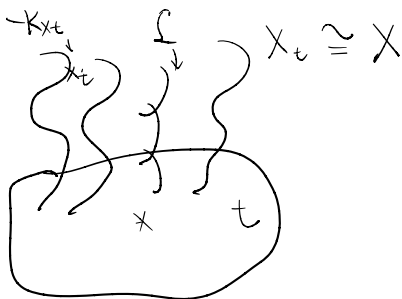
$$\begin{aligned} & \mathbb{P}^3 \\ & \cup \end{aligned}$$

$$X = (z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0)$$

$$\begin{aligned} \sigma(t)X &= \left(t^{3a_0} z_0^3 + t^{3a_1} z_1^3 + t^{3a_2} z_2^3 + t^{3a_3} z_3^3 = 0 \right) \\ & \quad a_0 \leq a_1 \leq a_2 \leq a_3 \\ &= \left(z_0^3 + t^{3(a_1-a_0)} z_1^3 + \dots = 0 \right) \end{aligned}$$

$$X \xrightarrow{\sigma_t} (\mathbb{P}^N, \omega_{FS}) \quad \sigma_t(X) = X_t = \sigma(t)(X)$$

$$\sigma_t^* \left(\frac{1}{m} \omega_{FS} \right) = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t. \quad \varphi_t = \mathcal{H}(\omega)$$



$$\lambda \rightarrow \mathbb{P}^N \times \mathbb{C}$$

$$(X_t) \mapsto (\sigma(t)(X), t)$$

$$\frac{F(\varphi_t)}{-\log|t|^2} \geq \frac{\delta J(\varphi_t) - C}{-\log|t|^2}$$

$$\downarrow$$

$$F^\infty(\Phi) \geq \delta > 0.$$

$$\omega + \sqrt{-1} \partial \bar{\partial} \Phi \geq 0 \quad \text{geodesic subray}$$

$$(\omega + \sqrt{-1} \partial \bar{\partial} \Phi)^{n-1} = 0 \quad \text{geodesic ray}$$

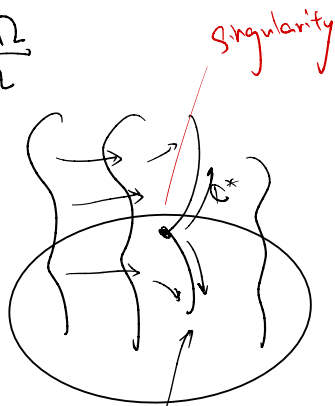
$$\leftarrow \theta_v = \frac{L_v \Omega}{\Omega}$$

$$F = -E + L$$

$$\begin{cases} \int_X \theta_v \Omega = 0 \\ \sqrt{-1} \partial \bar{\partial} \theta_v = L_v \text{Ric}(\Omega) \end{cases}$$

For special TC
(X, -K_X)

$$F^\infty(\Phi) = \int_{X_0} \theta_v \text{Ric}(\Omega)^n$$



X_0 - $\mathbb{Q}$ -Fano variety
special test
configuration
(Tian, Ding-Tian)

↑ Ding-Tian prove it.

Def (Tian) X is K -stable $\iff \forall$ special test configuration
 for special tc

$$F^\infty(X, -K_X) > 0$$

$\longleftrightarrow$ F is coercive along the ray associate

$$\parallel$$

$$F^{\text{NA}}(X, -K_X)$$

$$(X, -K_X)$$

$$\parallel$$

$$F^{\text{ut}}(X, -K_X) = \text{CM}(X, -K_X)$$

Def X is K -stable $\iff \forall \text{ TC}(X, \mathbb{L})$

$$F^\infty(X, \mathbb{L}) = \lim_{t \rightarrow \infty} \frac{F(\varphi_t)}{-\log |t|^2} > 0$$

X is uniform K -stable $\iff \exists \delta > 0$ s.t.

$$\forall \text{ TC}(X, \mathbb{L}) \quad F^\infty(X, \mathbb{L}) \geq \delta > 0$$

Thm (Li-Xu, Liu-Xu-Zhang)
 Fujita

All the above conditions are equivalent.

(Aut(X) discrete)
Thm $\exists KE \iff (X, -K_X)$ is K -stable

Tian $\nearrow$ (Tian 97) $\Downarrow$

$$F^\infty(X, \mathbb{L}) > 0, \forall (X, \mathbb{L}) \text{ TC}$$

F is coercive

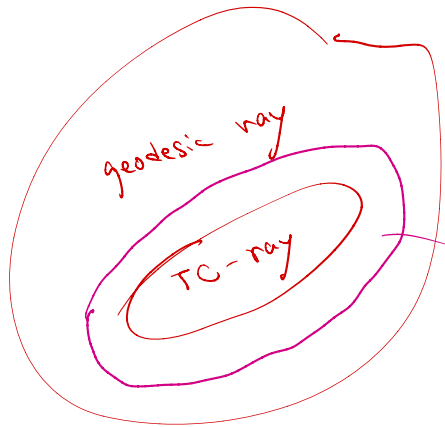
$\Updownarrow$

$$F^\infty(\mathbb{Q}) \geq \delta > 0$$

$\nwarrow$ geodesic ray

$$\Updownarrow$$

$$F^\infty(X, \mathbb{L}) \geq \delta > 0, \forall (X, \mathbb{L}) \text{ TC}$$



$$\{(\mathcal{X}_m, \mathcal{L}_m)\} = \mathcal{F}$$

— ~~is~~ test configuration

we need.

$$\Phi_{\mathcal{F}}$$

$$F^{\infty}(\Phi_{\mathcal{F}}) = \lim_{n \rightarrow \infty} \mathbb{P}^{\infty}(\mathcal{X}_n, \mathcal{L}_n)$$

$\parallel$

$$\beta(v)$$

$\mathbb{C}^*$ action

valuation $v: \mathbb{C}(X) \rightarrow \{\mathbb{C}(L_m, L_m)\}$

$\tilde{\text{Fut}}(v) = \beta(v)$

↓ In this case $\tilde{\text{Fut}}(v) = \beta(v)$

Thm X is K-stable $\Leftrightarrow \beta(v) > 0$, $\forall$ valuation

Lecture 5.

Assume $\text{Aut}(X)$ discrete

Thm X has KE $\iff X$ is K-stable



F_W is coercive



$$+TC(X, L)$$

$$F^\infty(X, L) > 0$$

Thm (Valuation criterion) X is K-stable iff

for any nontrivial valuation V .

$$\beta(V) \triangleq A_x(V) - \frac{1}{\text{Vol}(X)} \int_0^\infty \text{Vol}(-K_X - xD) dx$$

Def A (real) valuation

$$\exists W: \mathbb{C}^3, \text{Ric} = 0$$



$W \neq \text{flat}$.

$$v: \mathbb{C}(X) \rightarrow \mathbb{R} \cup \{+\infty\}$$



$$\mathbb{C} \subset \{ \text{meromorphic func. on } X \}$$



$$f \mapsto v(f)$$

Satisfying

$$(1) \quad v(f_1 f_2) = v(f_1) + v(f_2)$$

$$(2) \quad v(f_1 + f_2) \geq \min \{ v(f_1), v(f_2) \}$$

$$(3) \quad v(1) = 0, \quad v(0) = +\infty$$

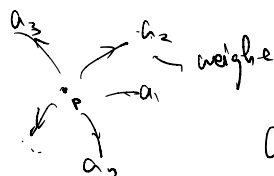
E_x 1. $X = \Sigma$ Riemann surface, $n = \dim_{\mathbb{C}} X = 1$

$$\{z=0\} = p \in \Sigma, \quad \text{ord}_p(f) = \text{ord}_{z=0} f(z) = \min \{k, a_k \neq 0\}$$

$$\sum_{k \geq n} a_k z^k$$

2. $\mathbb{C}^* \hookrightarrow X^n$

$X^n \ni p$ extracting point



$$\exists \{z_1, \dots, z_n\} \text{ s.t. } t(z_1, \dots, z_n) = (t^{a_1} z_1, \dots, t^{a_n} z_n) \quad \forall t \in \mathbb{C}^*$$

$$(\Leftrightarrow v = a_1 z_1 \frac{\partial}{\partial z_1} + \dots + a_n z_n \frac{\partial}{\partial z_n})$$

$$\frac{f_1}{f_2} = f \in \mathbb{C}(X)$$

$f = \text{holomorphic}$

$$v(f) = v(f_1) - v(f_2)$$

f_1, f_2 holomorphic

$$\sum_{i=1}^n C_i z^i$$

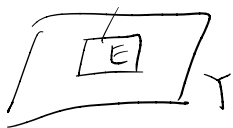
$$(i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$$

$$v(f) = \min \left\{ \sum_i (i_1 a_1 + \dots + i_n a_n) \mid C_i \neq 0 \right\}$$

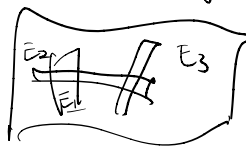
$$\text{codim} = 1 \quad \langle (a_1, \dots, a_n), (i_1, \dots, i_n) \rangle$$

irreducible
(codim $\mathbb{P}^n = 1$)

3



$$\mu^* f$$



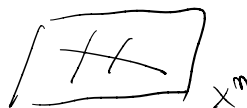
divisor

$\downarrow \mu$ blow up

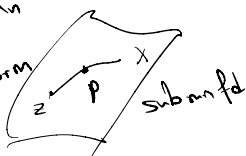
$$\downarrow$$

$$\downarrow \mu$$

$$f$$



$\dim n \dots n \dim$
hol- n -form



$$\text{ord}_E(f) = \min \{ m : b_m(z_1, \dots, z_n) \neq 0 \}$$

$$E = \{ z_1 = 0 \} \int \text{define}$$

$$f = \sum_{m \geq 0} b_m(z_1, \dots, z_n) z_1^m$$

$A_X(V) : \log \text{ discrepancy}$

$$A_X(V) = a_1 + \dots + a_n \quad \text{wt}(dz_1 \wedge \dots \wedge dz_n) = a_1 + \dots + a_n$$

$$\mathbb{C}^* \hookrightarrow X - K_X = \Lambda^n T_{\text{hol}} X \hookleftarrow \text{wt}\left(\frac{\partial}{\partial z_1} \wedge \dots \wedge \frac{\partial}{\partial z_n}\right) = -(a_1 + \dots + a_n)$$

$$\mu^*(du_1 \wedge \dots \wedge du_n) = \int h(z) dz_1 \wedge \dots \wedge dz_n$$

$$A_X(\text{ord}_E) = \overset{\text{hol}}{\text{ord}_E(h(z))} + 1$$

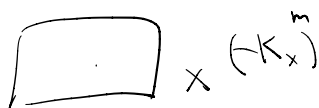
$$\Downarrow$$

$$\text{ord}_E(K_{Y/X}) + 1$$

$$v \in \text{Vol}(X), \quad R_m = H^0(X, -mK_X)$$



$\downarrow \mu$



$$E \longleftrightarrow L_E$$

$$\parallel$$

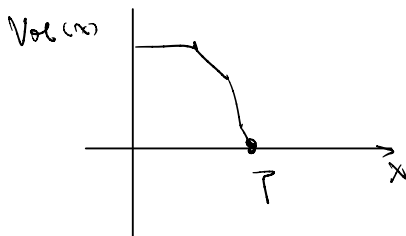
$$(f_\alpha = 0) \quad \left(\frac{f_\alpha}{f_\beta} \right) = (a_{\alpha\beta})$$

$$a_{\alpha\beta} a_{\beta\gamma} a_{\gamma\alpha} = 1$$

$$\frac{H^0(X, m(\mu^*(-K_X) \otimes L_E^{-X}))}{\parallel} \sim \frac{m^n}{n!} \text{Vol}(-K_X - X \cdot v) + O(m^{n-1})$$

$$\{s \in H^0(m(-K_X)), \text{ord}_E(s) \geq m \cdot X\}$$

↑
section



$$E_X \quad \mathbb{P}^2 = \underbrace{\mathbb{C}^2 \cup \mathbb{P}^1}_{(z_1, z_2)}$$

$$-K_{\mathbb{P}^2} = \mathcal{O}(3)$$

$$U = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \rightsquigarrow U, \text{ valuation}$$

$$\cdot \quad H^0(m(-K_X)) = \frac{(3m)^2}{2!} + \mathcal{O}(m^1) \Rightarrow \text{Vol}(-K_X) = 9$$

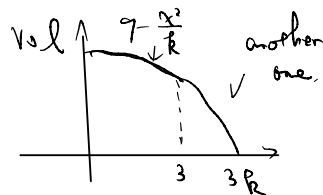
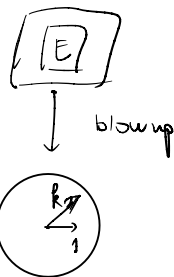
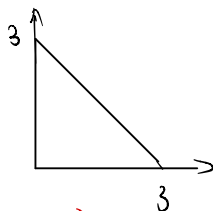
line bundle

$$\cdot \quad H^0(\underbrace{m(-K_X)}_{\text{line bundle}} - mXE)$$

$$= \text{Vol}(-K_X - XE) \frac{m^2}{2!} + \mathcal{O}(m)$$

$$\# \{ (i, j) \mid i+j \leq 3m, i+j \geq m \cdot X \}$$

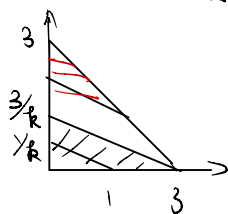
$\begin{matrix} \uparrow \\ m \\ \geq 0 \end{matrix}$



$$\underbrace{(-K_X - XE)^2}_{\mathbb{P}^2} = \int_{\mathbb{P}^2} C_1(-K_X - XE)$$

$$(3H - XE)^2 = 9H^2 - 0 - X^2 \frac{1}{R} = 9 - \frac{X^2}{R}$$

$\alpha > 3$ 不能用上述相交数做。



$$(3 - \frac{3}{k})^3 \frac{(3 - \frac{x}{k})^2}{(3 - \frac{3}{k})^2} = \frac{(3k - x)^2}{k(k-1)} = \text{vol}(-K_X - xE)$$

$$-K_X - xE = \frac{x-3}{k-1} (H - kE) + \frac{3k-x}{k-1} (H - E)$$

$\underbrace{\hspace{10em}}_{\substack{\text{N 不贡献 vol.} \\ p > 0}} \quad \text{把正的部分找出}$

$$h^0(X, mL) = \text{Vol}(L) \cdot \frac{m^n}{n!} + O(m^{n-1})$$

Thm, X Fano KE, $Z \subset X$, holomorphic submfld of codim r . Assume $N_{Z/X}$ is trivial.

Then $\text{Vol}(X) \leq \text{Vol}(Z \times \mathbb{P}^r) = \binom{n}{r} d(r+1)^r$

$$\begin{aligned} \text{"="} &\Leftrightarrow \int_X \omega_{KE}^n \stackrel{\parallel}{=} \int_X \left(c_1(Z) + c_1(\mathbb{P}^r) \right)^n \\ &\stackrel{\parallel}{=} \int_X c_1(X)^n \stackrel{\parallel}{=} \binom{n}{r} \underbrace{\int_Z c_1(Z)^{n-r}}_d \underbrace{\int_{\mathbb{P}^r} c_1(\mathbb{P}^r)^r}_{(r+1)^r} \end{aligned}$$

Thm (Fujita) Take $Z = \{pt\} \in X$

$$\Rightarrow \text{Vol}(X) \leq \text{Vol}(\mathbb{P}^n) = (n+1)^n$$

$$\text{"="} \Leftrightarrow X \cong \mathbb{P}^n$$

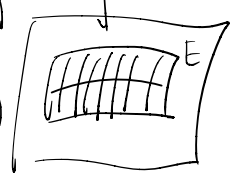
$$\text{Thm (K. Zhang)} \quad \text{Ric}(\omega) \geq \omega \Rightarrow \int_X \omega^n \leq (n+1)^n$$

$$(X, \omega) \text{ Kähler} \iff (X, \omega) \cong (P^n, \omega_{KE})$$

$$\beta_{\text{ord}_E}^{\text{ord}_E}(v) = A_x(v) - \frac{1}{v} \int \text{Vol}(-K_X - xE) dx \geq 0$$

$$A_x(v = \text{ord}_E) = r.$$

$$P(N_{Z/X}) = (Z \times P^{r-1})$$

$$P(\mathcal{O}^{\oplus r})$$


$$\text{Vol}(-K_X - xE) = \lim_{m \rightarrow \infty} \frac{1}{m^n} h^0\left(\frac{m(-K_X) + mxE}{15}\right)$$

$$h^0(m(-K_X) \otimes I_Z^{mx}) \left| \frac{dz_1 \wedge \dots \wedge dz_r}{z} \right|$$

$$\{f \in \mathcal{O}(X) : \text{ord}_Z(f) \geq mx\}$$

$$\{z_1 = \dots = z_r = 0\}$$

$$0 \rightarrow I_Z^{k+1} \rightarrow I_Z^k \rightarrow I_Z^k / I_Z^{k+1} \rightarrow 0$$

$$\text{sym}^k(N_{Z/X}^v)$$

$$Z \leftarrow \text{sym}^k(\mathcal{O}^{\oplus r})$$

$$\mathcal{O} \oplus \binom{k+r-1}{r-1}$$

$$dz_1 \wedge \dots \wedge dz_r \wedge (dz_{r+1} \wedge \dots \wedge dz_n)$$

$$dz_1 \wedge d(w_1 z_1) \wedge \dots \wedge d(w_r z_r)$$

$$z_1^{r-1} dz_1 \wedge dw_1 \wedge \dots \wedge dw_r$$

$$\frac{1}{m^n} h^0(m(-K_X) \otimes I_Z^{xm}) \geq h^0(m(-K_X)) - h^0(\mathcal{O}_X / I_Z^{xm})$$

$$\geq h^0(m(-K_X)) - \sum_{k=0}^{x-1} h^0\left(\frac{I_Z^k}{I_Z^{k+1}}\right)$$

$$\equiv \text{sym}^k(\mathcal{O}^{\oplus r})$$

$$\left[\begin{array}{l} 0 \rightarrow I_2^k \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/I_2^k \rightarrow 0 \\ 0 \rightarrow I_2^k / I_2^{k+1} \rightarrow \mathcal{O}_X / I_2^{k+1} \rightarrow \mathcal{O}_X / I_2^k \rightarrow 0 \end{array} \right]$$

$$= \frac{1}{m^n n!} \left(\frac{h^0(m(-K_X))}{\text{Vol } \frac{m^n}{n!}} - \sum_{k=0}^{xm-1} \binom{k+r-1}{r-1} \cdot \frac{h^0(Z, m(-K_X)/Z)}{\frac{m^{n-r} \text{Vol}(Z)}{(n-r)!}} \right)$$

$\underbrace{(r+xm-1)(r+xm-2)\dots(xm)}_{r!} \quad \binom{r+xm-1}{r}$

$$\begin{aligned} \text{Vol}(-K_X - xE) &\geq V - d \binom{n}{r} x^r, \quad d = (-K_X)^{n-r} Z \\ &= (-K_X)^{n-r} Z \\ &= \int_Z c_1(Z)^{n-r} \end{aligned}$$

$$A_{X(V-\text{ord}_E)} = r \geq \frac{1}{V} \int_0^\infty (V - d \binom{n}{r} x^r) dx$$

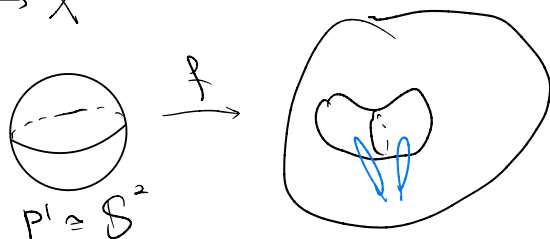
$$r \geq \frac{1}{V} \int_0^{\left(\frac{V}{d \binom{n}{r}}\right)^{\frac{1}{r}}} (V - d \binom{n}{r} x^r) dx$$

$$\Rightarrow V \leq \binom{n}{r} d (r+1)^r$$

$$" = " \Leftrightarrow \text{Vol}(-K_X - xE) = \frac{V - d \binom{n}{r} x^r}{(-K_X - xE)^n} \quad \forall 0 \leq x \leq r+1.$$

Thm (Mori) $\forall$ Fano mfd $\exists$ nontrivial rational curve

holomorphic map $f: \mathbb{P}^1 \rightarrow X$



Thm (Kollar-Mori-Miyaoka)

$$\exists f: \mathbb{P}^1 \rightarrow X \text{ s.t. } f^*TX / T\mathbb{P}^1 = \underbrace{\mathcal{O}(1)}_{\substack{\uparrow \\ \text{line bundle}}}^{\oplus (d-2)} \oplus \underbrace{\mathcal{O}}_{\substack{\uparrow \\ \text{minimal} \\ \text{rational} \\ \text{curve.}}}^{n-d+1}$$

$\mathbb{P}^1 \xrightarrow{f} X$
 $\parallel$
 $N_{\mathbb{P}^1/X}$

$$2 \leq \deg f = \int_{\mathbb{P}^1} f^* c_1(X) \leq n+1.$$

$$f^*TX = \underbrace{\mathcal{O}(2)}_{T\mathbb{P}^1} \oplus \mathcal{O}(1)^{\oplus d-2} \oplus \mathcal{O}^{n-d+1}$$

$r = \text{codim} = n-1$

$$d=2, N_{\mathbb{P}^1/X} = \text{trivial} \Rightarrow \text{Vol}(X) \leq \binom{n}{n-1} 2(n-1+1)^{n-1}$$

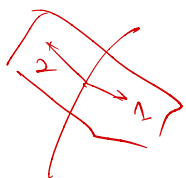
$$3 \leq d \leq n-1$$

不直接对 Z 做 blow up

会做 neighbor - blow up

$$n 2^{n-1} = 2^n$$

$$\text{Vol}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$$



$$Z = \mathbb{P}^1, N_{Z/X} \cong \frac{\mathcal{O}(1)^{\oplus (d-2)}}{(1^{\oplus d-2}, 2^{\oplus (n-d+1)})} \oplus \mathcal{O}^{n-d+1}$$

non-trivial

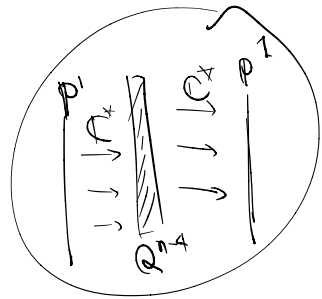
$$A_x(v) = (d-2) \cdot 1 + (n-d+1) \cdot 2 = 2n - d$$

$$\text{Vol}(-K_X - xE) \geq V - \phi_d(x)$$

$$\Rightarrow \text{Vol} < 2n^n$$

$$\underline{d=n} : Q^n = \{z_0^2 + z_1^2 + \dots + z_{n+1}^2 = 0\} \subset \mathbb{P}^{n+1}$$

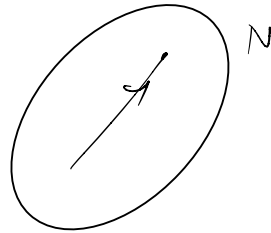
$$N_{\mathbb{P}^1} = \mathcal{O}(1) \oplus \mathcal{O}(n-2) \oplus \mathcal{O}$$



Miyooka if $\forall$ minimal rel. cur.
 $\deg(C) \geq n$

Then $X \cong Q^n, P^n$

$$\begin{array}{ccc} \text{Vol} = 2n^n & \nearrow & \nwarrow \\ & d=n & d=n+1 \\ & \text{Vol} = (n+1)^n \end{array}$$



" = " $X \cong Q^n, P^n$ blow up some codim 2 subvariety
 $P^n,$