

Happy New Year!

祝大家:

新的一年开开心心, 财源滚滚来



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2026, Beijing

# Introduction to Complex Geometry

- Reference.
1. Hörmander, Introduction to Several Complex variables  
local theory
  2. Kodaira, - Spencer, Complex manifold.  
Deformation of complex str.
  3. Daniel Huybrechts, Complex Geometry

## Introduction

What we learn in one Complex variable

$\Omega \subset \mathbb{C}$  local property of  $f \in \underline{O(\Omega)}$  holomorphic func.

$f: \Omega \rightarrow \mathbb{C}$  is holo if it satisfies Cauchy-Riemann Equ.

$$\underset{\substack{\uparrow \\ \mathbb{C}^1}}{f = u + vi}, \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy - Riemann formula for  $f \in O(\Omega)$

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta(z, r)} \frac{f(w)}{w - z} dw.$$

Properties of  $f \in O(\Omega)$ .



- Maximum Principle
- $f$  is  $C^\infty$
- $f$  has convergent Taylor Expansion.
- Higher order derivatives can be controlled by  $\|f\|_{L^\infty}$ .

Global Theory : Riemann Surface.  $S$

Poincaré : Topology : Study Topological Space locally homeomorphic to  $\Omega \subset \mathbb{R}^n$

For Riemann Surface  $S$ , we study holo-objects on  $S$ .

$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ .  $\exists f \in \mathcal{O}(\mathbb{P}^1)$ ,  $f$  must be constant.

$\leadsto$  Meromorphic func: Many!  $\leadsto$  Prescribed poles problem.

$\leadsto$  Riemann-Roth thm.

$$H^0(S, L) - H^1(S, L) = \deg L - g + 1$$

$L$  is formal sum:  $= \sum a_i p_i$   
 $a_i \in \mathbb{Z}$ ,  $p_i \in S$

$\swarrow$  genus of  $S$

$\deg L = \sum a_i$ ,  $H^0(S, L)$  is the dimension meromorphic func with prescribed poles at  $L$ .



$$L = P_1$$

$H^0(S, L)$  is the meromorphic func. with

at most a 1-order pole at  $P_1$

1-D to higher dimension.

- . local Theory of holomorphic func.  $f \in \mathcal{O}(\Omega)$ ,  $\Omega \subset \mathbb{C}^n$
- . global Theory of Complex Manifold.

$\leadsto$  Riemann-Roth-Hirzebruch.

Sheaf theory (1-dim  $\rightarrow$  higher dim).

Why? gluing holomorphic objects

Obstruction

?  $\uparrow$  some weak sense  
non-trivial Cohomology

# Lecture 1,

## 1.1 Review of 1-dim Complex analysis.

$$\Omega \subset \mathbb{C} \xrightarrow{f} \mathbb{C}, \quad f \in \mathcal{O}(\Omega) \Leftrightarrow \bar{\partial} f = 0.$$

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right)$$

$$dz = d(x + y\sqrt{-1}), \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}, \quad (0,1) \text{ form}$$

$$dz \left( \frac{\partial}{\partial z} \right) = 1.$$

Check:  $df = \underset{\substack{\uparrow \\ (1,0)}}{\partial} f + \underset{\substack{\uparrow \\ (0,1)}}{\bar{\partial}} f.$

Stokes formula:

$$(*) \quad f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(w)}{w-z} dw + \frac{1}{2\pi\sqrt{-1}} \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}}(w)}{w-z} dw \wedge d\bar{w}$$

$$\text{In particular, if } f \in \mathcal{O}(\Omega) \Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$$

$$\Rightarrow f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(w)}{w-z} dw.$$

Lemma: Given  $f \in C_c^\infty(\mathbb{C})$ , Def  $u(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f(w)}{w-z} dw \wedge d\bar{w}$

then  $\frac{\partial u}{\partial \bar{z}} = f$ , i.e.,  $\bar{\partial} u = f d\bar{z}$

+ good estimate

Rmk: Solving  $\bar{\partial}$  equation is one of the most important object.

Rmk: Solution to  $\bar{\partial} u = f$  is far from unique.

Proof:  $u(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{f(w+z)}{w} dw \wedge d\bar{w}$

$$\begin{aligned} \frac{\partial u}{\partial \bar{z}} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\frac{\partial f}{\partial \bar{z}}(w+z)}{w} dw \wedge d\bar{w} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\frac{\partial f}{\partial \bar{z}}(w)}{w-z} dw \wedge d\bar{w} \end{aligned}$$

By (\*),  $f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Omega} \frac{f(w) dw}{w-z} + \frac{1}{2\pi\sqrt{-1}} \int_{\Omega} \frac{\frac{\partial f}{\partial \bar{z}}(w) dw \wedge d\bar{w}}{w-z}$

choose suitable domain such that  $\int_{\partial\Omega} = 0$ .

□

## § 1.2. Holomorphic func of several Complex variables and Cauchy formula.

Fix notations:  $\Omega \subset \mathbb{C}^n$ ,  $(z_1, \dots, z_n)$ ,  $z_i = x_i + \sqrt{-1} y_i$

$$|z|^2 = \sum_i |z_i|^2, \quad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$$

$$|\alpha| = \sum \alpha_i, \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad \alpha! = \alpha_1! \dots \alpha_n!$$

$$B(a, r) \quad a \in \mathbb{C}^n$$

$$:= \{ z \in \mathbb{C}^n \mid |z - a| < r \}$$

polydisk.  $\Delta(a, \vec{r})$ ,  $a \in \mathbb{C}^n$ ,  $\vec{r} = (r_1, \dots, r_n)$

$$\Delta(a, \vec{r}) := \{ z \in \mathbb{C}^n \mid |z_i - a_i| < r_i \}$$

$$\text{Def} \quad f \in \mathcal{O}(\Omega) \Leftrightarrow \frac{\partial f}{\partial \bar{z}_i} = 0 \quad \forall i$$

$$\Leftrightarrow \bar{\partial} f_i = 0, \quad \bar{\partial} f = \sum \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

$$df = \partial f + \bar{\partial} f.$$

$$dz_i \sim (1, 0)$$

differential form

Def. Differential 1-form on  $\Omega$

1-form on  $\Omega$  is  $\omega = \sum a_i dx_i + \sum b_j dx_j$ ,  $a_i, b_j$  are func.



$(0,1)$ -form on  $\Omega$  is  $\alpha = \sum \alpha_i d\bar{z}_i$ ,  $\alpha_i$  is func.

$(1,0)$ -form on  $\Omega$  is  $\beta = \sum \beta_j dz_j$ ,  $\beta_j$  is func.

Ex. Any 1-form  $\gamma$  can be written as

$\gamma = \alpha + \beta$ ,  $\alpha$  is  $(0,1)$ -form,  $\beta$  is  $(1,0)$ -form.  
in a unique way

Cauchy - Formula:

$$f \in \mathcal{O}(\Omega). \quad f(z) := \int_{|z_1 - z_1| = r_1} \int_{|z_2 - z_2| = r_2} \dots \int_{|z_n - z_n| = r_n} \frac{f(z_1, \dots, z_n) dz_1 \dots dz_n}{\prod_{i=1}^n (z_i - z_i)}$$

proof. Iterate 1-d Cauchy Integral Formula.

~> • Maximum Principle

• Convergent Power series

• High-derivatives are controlled by  $\|f\|_{L^\infty}$

# § 1.3 Hartogs Thm

Let  $\Delta^*$  be the 1-dim punctured disk.

$\Delta^* = \Delta \setminus \{0\}$ ,  $\frac{1}{z} \in \mathcal{O}(\Delta^*)$  cannot extend holomorphically to  $\Delta$ .

Thm (Hartogs) Let  $\Omega \subset \mathbb{C}^n$ ,  $K \subset \Omega$  compact subset, such that  $\Omega \setminus K$  is connected. Then  $\forall f \in \mathcal{O}(\Omega \setminus K)$ ,  $\exists F \in \mathcal{O}(\Omega)$  and  $F|_{\Omega \setminus K} = f$ .

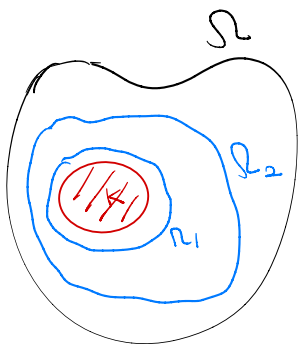
Proof:

$f$  is given on  $\Omega \setminus K$

idea: On  $\Omega$ ,  $\bar{\partial} F = 0$ .

Choose a cut-off function  $\eta$

$$\text{s.t. } \eta = \begin{cases} 0 & \text{on } \bar{\Omega}_1 \\ 1 & \text{on } \Omega - \Omega_2 \end{cases}$$



Hence,  $\eta f \in C^\infty(\Omega)$ .

$\bar{\partial}$   
"

$$\bar{\partial} G = \bar{\partial}(\eta f) \rightsquigarrow \bar{\partial}(-G + \eta f) = 0.$$

if  $\uparrow$  solvable

If  $U=0$  on  $\Omega \setminus \Omega_2$ , then  $F = 0 + 1 \cdot f = f$

almost holo  $\rightarrow$  holo, by solve  $\bar{\partial}$  equation.

Lemma Let  $\alpha$  be  $\bar{\partial}$  closed (i.e.) form with compact support in  $\Omega$ , then  $\exists G$ , s.t.  $\bar{\partial} G = \alpha$

$$\alpha = \sum \bar{\alpha}_i d\bar{z}_i \quad \bar{\alpha}_i, \bar{\alpha}_j \text{ are func.} \quad \frac{\partial G}{\partial \bar{z}_i} = \bar{\alpha}_i \quad \forall i$$

$\alpha$  is  $\bar{\partial}$ -closed  $\Leftrightarrow \bar{\partial} \alpha = 0$ .

$$\stackrel{\text{proof}}{\Leftrightarrow} \quad \frac{\partial \bar{\alpha}_i}{\partial \bar{z}_j} = \frac{\partial \bar{\alpha}_j}{\partial \bar{z}_i} \quad (\text{d}\alpha = 0, \text{ real case})$$

proof of Lemma:  $\alpha$  is closed is necessary

$$\bar{\partial} \bar{\partial} G = 0. \rightsquigarrow \text{If } \exists G \text{ s.t. } \bar{\partial} G = \alpha.$$

$$\rightsquigarrow \bar{\partial} \bar{\partial} G = \bar{\partial} \alpha = 0.$$

$$\text{Set } G(z) = \int_{\mathbb{C}} \frac{\bar{\alpha}_1 d\omega \wedge d\bar{\omega}}{\omega - z_1} \rightsquigarrow \frac{\partial G}{\partial \bar{z}_1} = \bar{\alpha}_1$$

$$\begin{aligned} \frac{\partial G}{\partial \bar{z}_2} &= \int_{\mathbb{C}} \frac{\frac{\partial \bar{\alpha}_1}{\partial \bar{z}_2} d\omega \wedge d\bar{\omega}}{\omega - z_1} = \int_{\mathbb{C}} \frac{\frac{\partial \bar{\alpha}_2}{\partial \bar{z}_1} d\omega \wedge d\bar{\omega}}{\omega - z_1} \\ &= \bar{\alpha}_2 \end{aligned}$$

$G(z)$  is compactly supported.

□

## Lecture 2.

### Hartogs Thm

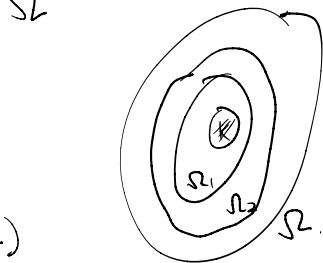
Let  $\Omega \subset \mathbb{C}^n$ , bdd,  $K \subset\subset \Omega$  cpt.  $\Omega \setminus K$  connected,  
( $n \geq 2$ )

then  $\forall f \in \mathcal{O}(\Omega \setminus K)$ , extends to  $\Omega$

proof: Choose cut-off  $\eta$  and then solve

$$\bar{\partial} G = \bar{\partial}(\eta f), \quad F = G - \eta f \in \mathcal{O}(\Omega)$$

verify  $F = f$  on  $\Omega$



$$K \subset \Omega_1 \subset \Omega_2 \subset \Omega,$$

( $n \geq 2$ )  
Lemma. Let  $\alpha$  be  $\bar{\partial}$  closed  $(0,1)$ -form and has compact Supp in  $\mathbb{C}^n$ , then  $\exists$  compact function  $G$  s.t.  $\bar{\partial} G = \alpha$

proof Def  $G := \int_{\mathbb{C}} \frac{\bar{\alpha}_1(w, z_2, z_3, \dots, z_n)}{w - z_1} dw \wedge d\bar{w}$

$$\Rightarrow \bar{\partial} G = \alpha$$

Claim:  $\exists R_2$  s.t. if  $|z_2| > R_2$ , then  $G(z) = 0$ .

Def  $\tilde{G} := \int_{\mathbb{C}} \frac{\bar{\alpha}_2(z_1, w, \dots, z_n)}{w - z_2} dw \wedge d\bar{w} \quad \therefore \bar{\partial} \tilde{G} = \alpha$

$$\bar{\partial}(G - \tilde{G}) = 0 \leadsto G - \tilde{G} \text{ is holo.}$$

$\exists R_1$  if  $|z_1| > R_1$ , then  $\tilde{G}(z) = 0$ .

If  $|z_1| > R_1$ ,  $|z_2| > R_2$ , both  $G$  and  $\tilde{G}$  are 0.

$$\leadsto G = \tilde{G}$$

□

Go back to  $F = \gamma f - G$

claim:  $G = 0$  on  $\Omega \setminus \Omega_2$ .

$$\bar{\partial} G = \bar{\partial} (\gamma f) = \bar{\partial} f = 0 \quad \text{on } \Omega \setminus \Omega_2$$

and  $G$  is compactly support.  $\leadsto G = 0$  on  $\Omega \setminus \Omega_2$

$$\leadsto F = f \quad \text{on } \Omega \setminus \Omega_2. \quad \begin{array}{c} \text{Both } F, f \\ \in \mathcal{O}(\Omega) \cap K \end{array} \leadsto F = f \quad \text{on } \Omega \setminus K$$

## § 1.4. Holomorphic and biholomorphic Maps

Def:  $\Omega \subset \mathbb{C}^n \xrightarrow{f} \mathbb{C}^n$ ,  $f = (f_1, f_2, \dots, f_n)$

We say  $f$  is holomorphic map if  $\forall i, f_i \in \mathcal{O}(\Omega)$ .

Def:  $f$  is biholomorphic to  $f(\Omega)$  if  $f$  is injective and  $f^{-1} \in \mathcal{O}(f(\Omega))$

### • Inverse function Thm

Let  $f: \Omega \rightarrow \mathbb{C}^n$  hol. map. if  $J_f(z_0)$  is not singular, then  $\exists$  a neighborhood of  $z_0 \in U \subset \Omega$  and

$f: U \rightarrow f(U)$  is biholomorphic.

Def  $J^{\mathbb{C}} f(z_0) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \dots & \frac{\partial f_n}{\partial z_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial z_n} & \dots & \frac{\partial f_n}{\partial z_n} \end{pmatrix}_{n \times n}$  no  $\frac{\partial f_i}{\partial \bar{z}_j}$

proof: It suffices to show that  $f^{-1}$  exists

Ex. If  $f^{-1}$  exist, then  $f^{-1}$  is holo.

$\left\{ \begin{array}{l} \text{reduce} \\ \leadsto J^{\mathbb{R}} f(z_0) \text{ is non-singular.} \end{array} \right.$

$J^{\mathbb{R}} f, f = (f_1, \dots, f_n) = (\operatorname{Re} f, \operatorname{Im} f)$

$J^{\mathbb{R}} f$  is the Jacobian of  $(\operatorname{Re} f, \operatorname{Im} f)$

w.r.t. variables  $(x_1, \dots, x_n, y_1, \dots, y_n)$

Claim:  $|\det(J^{\mathbb{C}} f(z_0))|^2 = \det(J^{\mathbb{R}} f(z_0)) > 0.$

$\nearrow$  need  $f$  to be holo.

proof of claim: Set  $U = (\operatorname{Re} f_1, \operatorname{Re} f_2, \dots, \operatorname{Re} f_n)$

$V = (\operatorname{Im} f_1, \dots, \operatorname{Im} f_n)$

$\frac{\partial U}{\partial x} := \left( \frac{\partial(\operatorname{Re} f_i)}{\partial x_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}, \quad \frac{\partial V}{\partial x}, \quad \frac{\partial U}{\partial y}, \quad \frac{\partial V}{\partial y}.$

$$\det J^{\mathbb{R}} f = \det_{J^{\mathbb{C}} f \text{ claim}} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \xrightarrow{\text{C-Regu}} \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}$$

$$\rightsquigarrow \det \begin{pmatrix} \frac{\partial u}{\partial x} - \sqrt{-1} \frac{\partial v}{\partial y} & \frac{\partial u}{\partial y} \\ -(\frac{\partial u}{\partial x} + \sqrt{-1} \frac{\partial v}{\partial y}) & \frac{\partial v}{\partial x} \end{pmatrix} \rightsquigarrow \det \begin{pmatrix} J^{\mathbb{C}} f & \frac{\partial u}{\partial y} \\ -\sqrt{-1} J^{\mathbb{C}} f & \frac{\partial v}{\partial x} \end{pmatrix}$$

$$\rightsquigarrow \det \begin{pmatrix} J^{\mathbb{C}} f & \frac{\partial u}{\partial y} \\ 0 & \frac{\partial v}{\partial x} + \sqrt{-1} \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\xrightarrow{\text{C-R}} \det \begin{pmatrix} J^{\mathbb{C}} f & * \\ 0 & J^{\mathbb{C}} f \end{pmatrix}$$

||

$$\det J^{\mathbb{C}} f \cdot \det J^{\bar{\mathbb{C}}} f = |\det J^{\mathbb{C}} f|^2$$

• Implicit func. theorem

$$f: \mathbb{C}^m \rightarrow \mathbb{C}^n, \quad m > n.$$

$$\text{If } \text{rk}(J^{\mathbb{C}} f(z_0)) = n, \quad z_0 = (w_0 \in \mathbb{C}^n, v_0 \in \mathbb{C}^{m-n})$$

$$\exists \text{ nbhd } U \text{ of } w_0 \text{ and } g: U \rightarrow \mathbb{C}^{m-n} \text{ s.t.}$$

$$g(w_0) = v_0 \quad \text{and} \quad f(w, g(w)) = f(z_0)$$

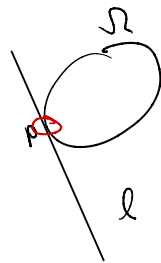
Rmk Poincare:  $B(1) \times B(1)$  is not biholomorphic to  $B(1,1) \subset \mathbb{C}^2$

dim=1

# § 1.5 Domain of holomorphy, holomorphic convex domain and pseudoconvex domain.

Def we say  $\Omega \subset \mathbb{C}^n$  is a domain of holomorphy if  $\forall \Omega' \not\subset \Omega, \exists f \in \mathcal{O}(\Omega)$  such that  $f$  cannot extend holomorphically to  $\Omega'$

e.g. Convex domain is a domain of holomorphy  
pick  $p \in \partial\Omega$ , pick supporting hyperplane at  $p$



$$l = \sum a_i z_i + \overline{\sum a_i z_i}, \quad \text{put } f = \frac{1}{\sum a_i z_i} \in \mathcal{O}(\Omega)$$

$f|_{p=\infty} \rightarrow$  cannot extend  $f$

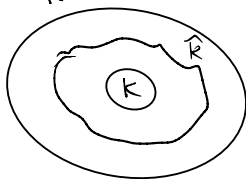
Def: Holomorphic Convexity

Let  $K \subset \subset \Omega$  cpt. if  $\hat{K}$  is also cpt in  $\Omega$

then we say  $\Omega$  is holomorphic convexity

$$\hat{K} := \{z \in \Omega \mid |f(z)| \leq \sup_K |f|, \forall f \in \mathcal{O}(\Omega)\}$$

$$K \subset \hat{K} \subset \Omega$$





e.g. Let  $P = \{z \in \Omega \mid \forall i, f_i(z) < 1, f_1, \dots, f_n \in \mathcal{O}(\Omega)\}$

$P$  is holo convex.

proof. Fix  $K \subset \Omega$  cpt. set  $r_i := \sup_K |f_i| < 1$ .

claim  $\hat{K} \subset \{z \mid |f_i(z)| < r_i, \forall i\} =: B$  is cpt. in  $\Omega$ .

pick  $w \notin B \rightarrow |f_j(w)| > r_j$  for some  $j$ . Goal:  $w \notin \hat{K}$   
 $= \sup_K |f_j| \leadsto w \notin \hat{K}$

Thm (Cartan-Tullen) Let  $\Omega \subset \mathbb{C}^n$  TFAE

(1) Domain of holomorphy

(2) holomorphic convex.

(3)  $\exists f \in \mathcal{O}(\Omega)$ , s.t.  $f$  cannot extend across any  $p \in \partial\Omega$  holomorphically.

Lemma: If  $\Omega$  holo convex, then  $\exists K_i \subset \subset \Omega$ , cpt

$K_i \subset \subset K_{i+1} \dots$  satisfies

1)  $\hat{K}_i = K_i$  2)  $\cup K_i = \Omega$  3)  $K_i \subset \subset K_{i+1}^\circ$

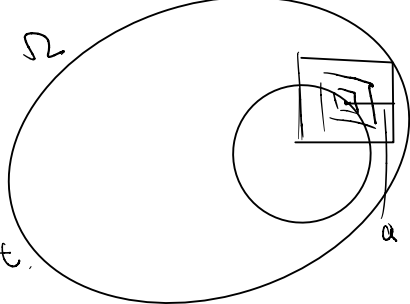
Lemma:  $\Omega$  holo. convex  $\Leftrightarrow$  for any  $z_i \rightarrow \partial\Omega$   
 $\exists f \in \mathcal{O}(\Omega)$  s.t.  $|f(z_i)|$  unbdd.

Ex

proof 3)  $\Rightarrow$  1) Trivial

1)  $\Rightarrow$  2) and 2)  $\Rightarrow$  3).

1)  $\Rightarrow$  2): Fix  $K$  Goal  $\hat{K}$  is cpt.



Def:  $\delta_K := \sup \{ a \in \mathbb{R} \mid \forall p \in K, \Delta(p, a) \subset \Omega \} > 0$ .

Similar def  $\delta_{\hat{K}}$

claim  $\delta_{\hat{K}} = \delta_K > 0 \leadsto \hat{K}$  is compact.

Argue By Contradiction: If  $\hat{K}$  is not compact,

$\forall f \in \mathcal{O}(\Omega)$

s.t.  $f$  extends across  $\partial\Omega$

*Idea is to estimate (This contradicts domain of holomorphy)  
convergence of radius.*

If  $z_i \in \hat{K} \xrightarrow{\text{run}} \partial\Omega$

$|\partial^\alpha f(z_i)| < \sup_K |\partial^\alpha f| \leadsto f$  has radius of  
convergence "almost"  $\delta_K$   
"almost"  $\text{dist}(K, \partial\Omega)$

□

2)  $\Rightarrow$  3) Step 1: pick  $z_i \rightarrow \partial\Omega$ ,  $\{z_i\} \subset \partial\Omega$

Step 2: pick  $K_i = \hat{K}_i \subset \subset K_{i+1} = \hat{K}_{i+1} \dots$

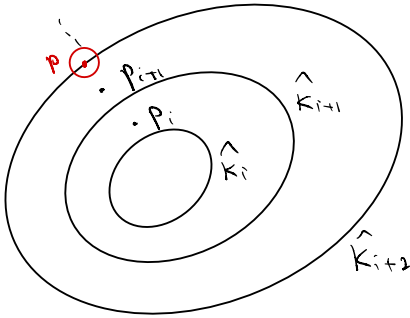
1)  $\cup K_i = \Omega$

$$2) K_i \subset \subset K_{i+1}$$

$$3) p \in K_{i+1} \setminus K_i$$

Step 3: For each  $i$ , pick  $f_i$  satisfies 1)  $f_i(p_i) = 1$

$$\in \mathcal{O}(\Omega) \quad 2) \sup_{K_i} |f_i| \leq 2^{-i}$$



(by log...)

Step 4  $f = \prod_{i=0}^{\infty} (1 - f_i)^i$  • check  $f$  converge

•  $f(p_i) = 0$  with order  $i$

$\leadsto$  If  $f$  extends across  $p \in \partial\Omega$   
then  $f \equiv 0$ .

•  $f \not\equiv 0 \leadsto$  contradiction

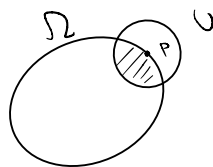
Fix a suitable compact set,  $\tilde{K}$ ,  $\sup_{\tilde{K}} |f_i| \leq 2^{-i}$

$\therefore f \not\equiv 0$ .

# Lecture 3.

Recall Domain of Holomorphy

Holomorphic Convex domain  $\Omega \subset \mathbb{C}^n$



Def:  $\Omega \subset \mathbb{C}^n$ ,  $\partial\Omega \in C^\infty$  we say  $\Omega$

$\forall p \in \partial\Omega$ , Fix a nbhd  $U \ni p$  and  $\varphi \in C^\infty(U, \mathbb{R})$

(Defining function) s.t.  $\Omega \cap U := \{ \varphi < 0 \}$ .

$\varphi = 0$  on  $\partial\Omega$

We say  $\Omega$  is pseudoconvex if  $\forall \xi \in \mathbb{C}^n$ ,  $(\xi) = (\xi_1, \dots, \xi_n)$

satisfies  $\xi \cdot \frac{\partial \varphi}{\partial \bar{z}} = 0 \quad \left( \frac{\partial \varphi}{\partial \bar{z}} \right) = \left( \frac{\partial \varphi}{\partial \bar{z}_1}, \dots, \frac{\partial \varphi}{\partial \bar{z}_n} \right)$

We have  $L_p(\varphi, \xi) := \sum_{i,j} \xi_i \bar{\xi}_j \varphi_{i\bar{j}} \geq 0$

Levi form

$$\varphi_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}$$

Remark: Pseudoconvexity is independent choice of  $\varphi$

Thm: (Levi) Domain of holomorphy  $\Rightarrow$  pseudoconvex

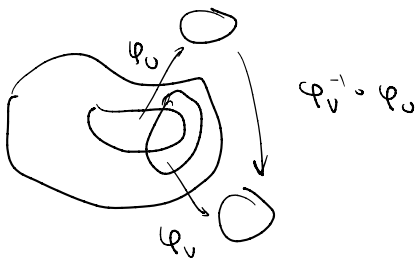
$\Leftarrow$  is the famous Levi Problem (OK).

## § 2.1 Complex mnd

Def  $X$  is complex mnd if  $X$  is a real manifold:  $\{U_i, \varphi_i\}_{i \in \Lambda}$

$\varphi_i: U_i \rightarrow \mathbb{C}^n$  homeomorphism

and  $\varphi_i \circ \varphi_j^{-1}$  is holomorphic



Def holomorphic function on  $X$

$$f \in \mathcal{O}(X) \Leftrightarrow f \in \mathcal{O}(U_i) \forall i \Leftrightarrow f \circ \varphi_i^{-1} \in \mathcal{O}(\varphi_i(U_i))$$

Def:  $(M, N)$  complex mnds  $f: M \rightarrow N$

when  $f$  is holo and  $(M, N)$  are holomorphic equivalent.  
 $f, f^{-1}$  both holo.

e.g. Torus:  $\mathbb{C}/\Lambda$ ,  $\Lambda$  is rank 2 lattice of  $\mathbb{C}$

e.g. Zeros of polynomials (generic)

$f(z_1, \dots, z_n)$  is polynomial if  $(\frac{\partial f}{\partial z} \neq 0)$ , then  $\{f=0\}$  is a complex mnd.

WLOG  $\frac{\partial f}{\partial z_1} \neq 0$ , By Implicit fun.

$$\exists U, g: U \rightarrow \mathbb{C}^{n-1} \text{ s.t. } f(z, g(z)) = 0.$$

$\{f=0\}$  is a graph

$$f(z_1, z_2) = 0 \text{ near } (a, b)$$

$$(\frac{\partial f}{\partial z_1} \neq 0) \Rightarrow z_2 = g(z_1)$$

$$f(z_1, g(z_1)) = 0$$

$\exists U \ni (a, b), \{f=0\}$  has coordinate  $z_1$

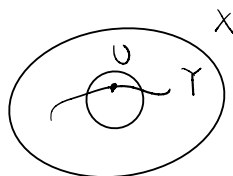
e.g.  $\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \sim$

$$(z_1, \dots, z_{n+1}) \sim (w_1, \dots, w_{n+1})$$

$$\text{iff } \vec{z} = \lambda \vec{w}, \lambda \in \mathbb{C}.$$

Cover  $\mathbb{P}$  by  $U_i \subset \mathbb{P} \quad 1 \leq i \leq n+1$ .

$$U_i = \{[Z] \in \mathbb{P} \mid z_i \neq 0\}$$



Define  $\varphi_i : U_i \rightarrow \mathbb{C}^n$  as

$$[Z] \mapsto \left( \frac{z_1}{z_i}, \frac{z_2}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i} \right)$$

Def: Complex submanifold  $Y^m \subset X^n$

1)  $Y$  is closed 2) locally  $U \cap Y := \{z_1 = z_2 = \dots = z_{n-m} = 0\}$ .

where  $z_1, \dots, z_n$  is coordinate on  $U$ .

(need holo. coordinate).

Def: Complex analytic subvariety  $A \subset X^n$

1)  $A$  closed 2) locally  $A$  is common zeros of holomorphic

$$\text{fun. } U \cap A = \{z \in U \mid f_i(z) = 0, \forall i, f_i \in \mathcal{O}(U)\}$$

Fact: If  $X$  is cpt Complex mfd,  $\neq$  nonconstant holomorphic fun

Ex

Fact:  $\neq$  cpt complex submanifold in  $\mathbb{C}^n$   
analytic subvariety (stronger).

Q1 Given  $X \in C^\infty$  mfd.  $\exists$ ? complex structure on  $X$ .  
 $(S_6)$  has complex str?

Def: Almost Complex str.  $J$  of a real mfd  $X$

1)  $J: T^R X \longrightarrow T^R X$  bundle map

$J_p: T_p^R X \longrightarrow T_p^R X$  Linear

2)  $J^2 = -1$ .

Q: Given  $X$ ,  $\exists J$ ?  $C^\infty$  continuous

$T^R X|_U = U \times \mathbb{R}^n$ ,  $J$  locally is family of matrix  
satisfies  $J^2 = -1$

$\exists J$  is a topological question has a "good" answer

e.g. If  $X$  is a complex mfd.

choose  $(z_1, z_2, \dots, z_n)$  on a chart  $U \subset X$

$$z_i = x_i + \sqrt{-1} y_i$$

Def  $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ ,  $J \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$

Rmk:  $J$  is independent of choice of holo coordinate.

Def A almost complex structure  $(X, J)$  is integrable if

$J$  is the complex str. of a complex mfd  $M$ ,  $M \cong X$ .

Q: When an almost complex str.  $J$  is integrable.

Def:  $T^{\mathbb{C}}X := T^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C}$

$$T_p^{\mathbb{R}}X = \mathbb{R}\left\{\frac{\partial}{\partial x}\right\}, \quad T_p^{\mathbb{C}}X = \mathbb{C}\left\{\frac{\partial}{\partial x}\right\}$$

We may extend  $J$  to  $T^{\mathbb{C}}X \rightarrow T^{\mathbb{C}}X$   $\mathbb{C}$ -linearly  
 $J^2 = -1$  still holds

Def:  $T^{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$ ,  $\dim T^{1,0}X = \dim^{0,1}X$   
 $\uparrow$   $\sqrt{-1}$  eigenvalue of  $J$   $\nwarrow$   $-\sqrt{-1}$  eigen. space of  $J$

Thm.  $J$  is integrable iff  $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$

(Newlander-Nirenberg)

Rmk. If  $J$  is the complex str. of complex mfd  $M$

$T^{1,0}X = \mathbb{C}\left\{\frac{\partial}{\partial z}\right\}$ , then  $\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right] = 0$ .

$\Leftarrow$  It suffices to construct  $n$  local func.  $f_1, \dots, f_n$  on  $X$

1)  $\bar{\partial}_J f_i = 0 \quad \forall i$     2)  $\text{Jacobi}(f) \neq 0$

construct hdo. coordinate

(Hormander).

$\wedge^{p,q} T^{\mathbb{C}}X$  is the bundle fiberspace spanned

by  $e_1^{1,0} \wedge e_2^{1,0} \wedge \dots \wedge e_p^{1,0} \otimes \beta_1^{0,1} \wedge \beta_2^{0,1} \wedge \dots \wedge \beta_q^{0,1}$



$\Lambda^{p,q} T^* X$  can be defined similarly

Lemma  $\Lambda^k T^{*\mathbb{C}} X = \bigoplus_{p+q=k} \Lambda^{p,q} T^* X$

$$\Lambda^k T^{\mathbb{C}} X = \bigoplus_{p+q=k} \Lambda^{p,q} T X$$

Def. For an almost complex  $J$

$$\bar{\partial}_J \alpha := \pi_{p,q+1} d\alpha^{p,q}, \quad \pi_{p,q+1}: \Lambda^{p,q+1} T^{*\mathbb{C}} X \rightarrow \Lambda^{p,q+1} T^* X$$

$$\partial \alpha^{p,q} = \pi_{p+1,q} d\alpha^{p,q}$$

Rmk. If  $(X, J)$  is integrable,  $\bar{\partial}_J$  is the standard  $\bar{\partial}$

$$\bar{\partial} \left( \sum_{p,q} f_{p\bar{q}} dz_p \wedge d\bar{z}_{\bar{q}} \right) = \sum_{j,p,q} \frac{\bar{\partial} f_{p\bar{q}}}{\bar{\partial} z_j} d\bar{z}_j \wedge dz_p \wedge d\bar{z}_{\bar{q}}$$

Ex.  $[T^{1,0} X, T^{1,0} X] \subset T^{1,0} X \iff \bar{\partial}_J \circ \bar{\partial}_J = 0$

almost complex str case.

$$\bar{\partial}_J \bar{\partial}_J = \pi \underset{\substack{\curvearrowright \\ \text{cannot}}}{d} \pi d$$

## § 2.2. Vector Bundle

- 一族 holomorphic 的 linear space  
in some way.

We say  $E \xrightarrow{\pi} X$  is hol. VB of rank  $r$  if

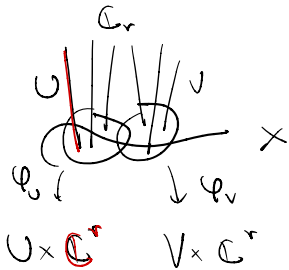
1)  $E, X$  are complex mfd,  $\pi$  is submersion

$E_p := \pi^{-1}(p)$  has  $\mathbb{C}^r$ -linear str.

2) locally trivial:  $\exists \{U_i\}_{i \in \mathbb{N}}$  cover of  $X$

$$\varphi_i: (\pi^{-1}(U_i)) \xrightarrow{\text{holo}} U_i \times \mathbb{C}^r$$

$p \in U, \varphi_i: E_p \rightarrow \{p\} \times \mathbb{C}^r$  is linear isom.



we call  $\{\varphi_i\}$  is a trivialization of  $E$

Def:  $s$  is a section of  $E$  on  $U \subset X$

if 1)  $s: U \rightarrow E$  is hol.

2)  $\pi \circ s = \text{id}$  on  $U$

$s \in \Gamma(U, E)$  If choose  $U = X$ ,  $\Gamma(X, E)$  is called global section.

Def: Let  $U \subset X$ ,  $E \xrightarrow{\pi} X$  hol. VB,  $\text{rank } E = r$ .

$\{e_1, e_2, \dots, e_r\}$  is a local frame of  $E$  on  $U$  if

$$1) e_i \in \Gamma(U, E) \quad \forall i$$

2)  $e_1(p), e_2(p), \dots, e_r(p)$  are linearly independent at any  $p \in U$ .

Lemma Trivialization of  $E$  on  $U \iff$  Find frame on  $U$ .

$\Rightarrow \varphi_U: E|_U \rightarrow U \times \mathbb{C}^r$ , pick standard basis

$B_i = (0, 0, \dots, 1, \dots, 0) \quad 1 \leq i \leq r$  and pull it back to  $E|_U$  gives frame. <sup>ith</sup>

$\Leftarrow$  Given basis  $\leadsto$  find the coefficient

Given  $v \in E_p$ ,  $v = \sum_{i=1}^r a_i(p) e_i(p)$  ( $e_i$  is frame)

$\varphi_i$  is trivialization of  $E$

Def.  $\varphi_i: E_{U_i} := \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}^r$

$\varphi_j: E_{U_j} \rightarrow U_j \times \mathbb{C}^r$

$\psi_{ij}: U_i \cap U_j \rightarrow GL(r, \mathbb{C})$  is defined as

the unique map s.t.  $(\varphi_i \circ \varphi_j^{-1})(p, v) = (p, \psi_{ij}(p) \cdot v)$

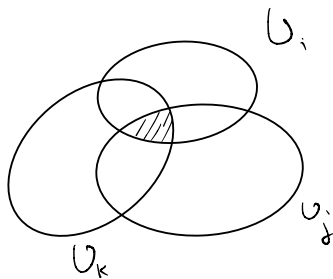
Ex.  $\psi_{ij}$  is holo. map.

$\nwarrow$   $\psi_{ij}$  is called transition map of  $E$

Properties of  $\psi_{ij}$

$$1) \psi_{ij} \circ \psi_{ji} = \text{Id.}$$

$$2) \psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \text{Id.}$$



Conversely: If given  $\psi_{ij} \in U_i \cap U_j \rightarrow GL(r, \mathbb{C})$

satisfies 1)  $\psi_{ij} \psi_{ji} = \text{Id}$  2)  $\psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \text{Id}$ .

then may define  $E := \bigsqcup U_i \times \mathbb{C}^r / \sim$  as holo. VB

$p \in U_i \cap U_j$ ,  $(p, v) \sim (p, w)$  iff  $\psi_{ij}(p)w = v$ .

Correction.  $f(x, y)$ ,  $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$ .

Implicit func. locally  $x = g(y)$ ,  $x_0 = g(y_0)$

and  $\{f = f(x, y) = 0\} = \{g(y), y\}$ .

Let  $f = (z_1, z_2, \dots, z_n)$   $\frac{\partial f}{\partial z} \neq 0$ , then  $f=0$  defines

a complex mfd.

wlog.  $\frac{\partial f}{\partial z_1} \neq 0$ .  $z_1 = g(z_2, \dots, z_n)$  locally

and  $\{f=0\} = \{g(z_2, \dots, z_n), z_2, \dots, z_n\}$ .

# Lecture 4.

Recall Almost Complex Str.  $J: T^R X \rightarrow T^R X$ ,  $J^2 = -1$ .

$$T^{\mathbb{C}} X = T^{1,0} X \oplus T^{0,1} X, \quad T^{*\mathbb{C}} X = T^{*,1,0} X \oplus T^{*,0,1} X.$$

$$\wedge^{p,q} T^{\mathbb{C}} X \text{ and } \wedge^{p,q} T^{*\mathbb{C}} X \xrightarrow{\sim} \bar{\partial}_J \alpha^{p,q} := \pi_{p,q+1} d\alpha^{p,q}$$

Thm (Nirenberg-Newman)  $J$  is integrable  $\Leftrightarrow \bar{\partial}_J \circ \bar{\partial}_J = 0$ .

$$\Leftrightarrow [T^{1,0} X, T^{1,0} X] \subset T^{1,0} X$$

Vector Bundle: Transition map  $\psi_{ij}: U_i \cap U_j \rightarrow GL(r, \mathbb{C})$

over complex manifold

satisfies: 1)  $\psi_{ij} \circ \psi_{ji} = \text{Id}^{\text{holo.}}$   $\forall i, j$

$\text{Fix } \{U_i\}_{i \in \Lambda}$  over  $X$ .

2)  $\psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \text{Id}$   $\forall i, j, k$

$$E = \coprod_{U_i, i \in \Lambda} U_i \times \mathbb{C}^r / \sim$$

$$(p, v) \in U_i \times \mathbb{C}^r \sim (p, w) \in U_j \times \mathbb{C}^r \iff \psi_{ij}(p) \cdot w = v$$

e.g.  $TX$ .  $X$  is complex manifold

Holomorphic tangent bundle.

$p \in X$ ,  $T_p X := \mathbb{C} \left\{ \frac{\partial}{\partial z_1}(p), \dots, \frac{\partial}{\partial z_n}(p) \right\}$  where  $z_1, \dots, z_n$  is holomorphic

coordinate near  $p$ .

check  $TX$  is holo. VB.

Let  $p \in (U, Z)$ .  $p \in (V, W)$

$$v \in T_p X$$

$$v = \sum a_i \frac{\partial}{\partial z_i} \Big|_p = \sum b_j \frac{\partial}{\partial w_j} \Big|_p$$

$$\begin{array}{ccc} \varphi_U: T \times|_U \longrightarrow U \times \mathbb{C}^n & | & \varphi_V: T \times|_V \longrightarrow V \times \mathbb{C}^n \\ \forall p \in U & v \mapsto (p, \vec{a}(p)) & v \in T_p X \mapsto (p, \vec{b}(p)), \forall p \in V \end{array}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial w} \cdot \boxed{\frac{\partial w}{\partial z}} \quad \begin{array}{l} \swarrow \text{Holomorphic?} \\ \text{Yes. By def of complex mfd.} \end{array}$$

Similarly,  $T_p^* X = \mathbb{C} \{ dz_1(p), \dots, dz_n(p) \}$ .

Defines holo. cotangent bundle  $T^*X$

e.g. Topological line Bundle  $\mathcal{O}(-1) \rightarrow \mathbb{P}^n$

$$\mathbb{P}^n = \{ \mathbb{C}^{n+1} \setminus \{0\} \} / \sim \quad [Z]$$

$$\mathcal{O}(-1) := \{ ([Z], v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in [Z] \text{ or } v=0 \}$$

$\mathcal{O}(-1)|_{[Z]}$  is a linear space. (1-dim?)

Recall  $\mathbb{P}^n$  has a cover  $\{U_i\}_{1 \leq i \leq n+1}$ ,  $U_i = \{z_i \neq 0\}$

on  $U_i$  we have coordinate  $\left\{ \frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_{n+1}}{z_i} \right\}$ .

Suppose  $v \in [Z]$   $v = (v_1, \dots, v_{n+1})$

$$\varphi_i: \mathcal{O}(-1)|_{U_i} \longrightarrow U_i \times \mathbb{C}$$

$$v \longmapsto ([z], v_z)$$

$$\varphi_j: \mathcal{O}(-1)|_{U_j} \longrightarrow U_j \times \mathbb{C}$$

$$v \longmapsto ([z], v_j)$$

$$\frac{v_i}{v_j} = \frac{z_i}{z_j} \quad \text{since } v = \lambda z, \lambda \in \mathbb{C}.$$

of c holo.  $\nearrow$  since  $(\frac{z_i}{z_j}) \psi_{ij}$

Bundle Operation

$$E_1 \oplus E_2, E_1 \otimes_{\mathbb{C}} E_2, E_1^*, \text{Sym}^m E, \wedge^m E, \text{Hom}(E_1, E_2)$$

Ex: If  $E$  is holo VB on  $X$ . then all Bundles above are holomorphic.

Pull back Vector Bundle  $f: X \xrightarrow{\text{holo.}} Y$

$$\begin{array}{ccc} f^{-1}E|_p (p \in X) = E_{f(p)} & \uparrow & \uparrow \\ f^{-1}E & \longrightarrow & E \end{array}$$

Ex. Check  $f^{-1}(E)$  is holo.

e.g.  $\mathcal{O}(-1), \mathcal{O}(1) \longrightarrow \mathbb{P}^n$  hyperplane bundle. Tensor product

•  $\mathcal{O}(1)$  is the dual of  $\mathcal{O}(-1)$

•  $\gamma_{ij} = \frac{z_i}{z_j}$  is transition of  $\mathcal{O}(-1)$

$$\mathcal{O}(m) := \begin{cases} \mathcal{O}(1)^{\otimes m} & \text{if } m \geq 1 \\ \mathcal{O}(-1)^{\otimes (-m)} & \text{if } m \leq -1 \end{cases}$$

$\uparrow$   
line bundle

$\psi_{ij} = \frac{z_j}{z_i}$  is transition of  $\mathcal{O}(1)$ .

Key Q Given a holomorphic Bundle  $E$ , do we have  $\Gamma(X, E)$ ?

Thm  $\dim_{\mathbb{C}} \Gamma(\mathbb{P}^n, \mathcal{O}(m)) = \begin{cases} 0 & \text{if } m < 0, \\ \binom{n+m}{m} & \text{if } m \geq 0 \end{cases}$

General discussion Fix  $\{U_i\}_{i \in I}$  of  $X$ . Suppose  $L$  is line bundle,  $e_i$  is a frame of  $L$  on each  $U_i$

Suppose  $S \in \Gamma(X, L)$ ,  $S|_{U_i} = f_i e_i$ ,  $f_i \in \mathcal{O}(U_i)$ ,  $\forall i$

$$\begin{aligned} S|_{U_j} &= f_j e_j & f_i e_i &= f_j e_j \\ & & \Rightarrow f_i &= \psi_{ij} f_j \end{aligned}$$

To give  $S \in \Gamma(X, L)$

$\Leftrightarrow$  give a sequence  $f_i \in \mathcal{O}(U_i)$ , s.t.  $f_i = \psi_{ij} f_j$

Let  $m > 0$ .  $\psi_{ij} = \left(\frac{z_j}{z_i}\right)^m$

To give  $S \in \Gamma(\mathbb{P}^n, \mathcal{O}(m)) \Leftrightarrow$  give a sequence  $f_i \in \mathcal{O}(U_i)$

s.t.  $\frac{f_i}{f_j} = \left(\frac{z_i}{z_j}\right)^m$

$\frac{f_i z_i^m}{f_j z_j^m} = 1 \rightsquigarrow$  Such  $f_i$  gives a holomorphic function  
 $\in \mathbb{C}^{n+1} / \{z_i \neq 0\}$  on  $\mathbb{C}^{n+1} / 0$



By Hartogs Extension

$$F = f_i z_i^m = f_j z_j^m \rightsquigarrow \text{holo. function on } \mathbb{C}^{n+1}$$

Actually  $f_i$  define on  $\mathbb{P}^n$  of homogeneous degree  $m$ .

Conversely, if  $F$  is a homogeneous polynomial of degree  $m$

$$\text{on } \mathbb{C}^{n+1} \rightsquigarrow f_i \in \mathcal{O}(U_i) \quad \forall i \quad \text{and} \quad \frac{f_i}{f_j} = \left( \frac{z_i}{z_j} \right)^m$$

$$\text{precisely define } f_i := \frac{F}{z_i^m}$$

Q:  $n+1$  variables  $m$  degree, how many homogeneous polynomial.

$$A: \binom{n+m}{n}.$$

□.

Ex. if  $m < 0$ , then no nontrivial section of  $T(\mathbb{P}^n, \mathcal{O}(m))$

## § 2.4 DeRham Cohomology and Dolbeault Cohomology

In general Given a complex of (group-ring)  $(C^\bullet, \delta)$

$$\dots C_1 \xrightarrow{\delta_1} C_2 \xrightarrow{\delta_2} C_3 \xrightarrow{\delta_3} C_4 \dots \text{ satisfies } \delta_{i+1} \delta_i = 0 \quad \forall i$$

then we may define cohomology as  $H^i(C^\bullet, \delta) = \frac{\ker(\delta_i)}{\operatorname{Im}(\delta_{i-1})}$

Let  $X$  be a complex mfd

Def: DeRham Complex  $(A^\bullet(X), d)$

$$A^p(X) = \begin{cases} 0 & p < 0 \\ \text{smooth } p\text{-form } p \geq 0 \\ \text{with complex coefficients} \end{cases}$$

$\leadsto (A^\bullet(X), d)$  define DeRham Cohomology  $H^p(X, \mathbb{C})$

Dolbeault Cohomology  $(A^{p,\bullet}(X), \bar{\partial})$   $p$  is fixed  
 $0 \leq p \leq n$

$$A^{p,q}(X) := \begin{cases} 0 & \text{if } q < 0 \\ C^\infty(p,q) \text{ form on } X & \text{if } q \geq 0 \end{cases} \quad \left. \begin{array}{l} \text{Dolbeault} \\ \text{complex} \end{array} \right\}$$

$$\bar{\partial} : A^{p,q}(X) \rightarrow A^{p,q+1}(X)$$

$\left\{ \begin{array}{l} \text{Dolbeault} \\ H_{\bar{\partial}}^{p,q}(X) \end{array} \right\} \leftarrow \text{coho.}$

(Poincare,  $\bar{\partial}$ -poincare) If  $X$  is complex Ball

$$\text{then } H^p(X, \mathbb{C}) = 0 \quad H_{\bar{\partial}}^{p,q}(X) = 0$$

$p \geq 1 \qquad q \geq 1$

### § 3.1 Basic Concept of Sheaf Theory

Motivation Let  $X$  Riem. surface  $p_1, \dots, p_m \in X$

At Each  $p_j$  give Laurent series  $\sum_{k=1}^n \frac{C_k^{(j)}}{z^k} \in M_p / \mathcal{O}_p$

meromorphic

Mittag-Leffler:  $\exists f$  meromorphic on  $X$  with prescribed  $\sum_{k=1}^n \frac{C_k^{(j)}}{z^k}$

locally. Yes on open set  $U_i$

If not very lucky,  $f_i \neq f_j$  on  $U_i \cap U_j$

$f_{ij} := f_i - f_j$  on  $U_i \cap U_j$  satisfies

$$1) f_{ij} + f_{ji} = 0, \quad 2) f_{ij} + f_{jk} + f_{ki} = 0.$$

Some hope find  $h_i \in \mathcal{O}(U_i)$  s.t.  $h_i + f_i = h_j + f_j$  on  $U_i \cap U_j$

$$h_i - h_j = f_{ji}$$

In Smm: Given  $f_{ij}$  satisfies 1) and 2)

can we find  $h_i$  s.t.  $h_i - h_j = f_{ji}$

$\leadsto$  Čech Cohomology of  $X$ .

Let  $X$  topological space.

Def: A pre-sheaf  $\mathcal{F}$  is a rule assigning  
an abelian group  $\mathcal{F}(U)$  to any open  $U \subset X$ ,  
and for each pair  $V \subset U$ , a group homo.

$r_V^U: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$  (Restriction) satisfying

$$1) r_U^U = \text{Id} \quad 2) r_K^U \circ r_V^U = r_K^U \quad \text{for any } K \subset V \subset U.$$

• an element  $s \in \mathcal{F}(U)$ , we say  $s$  is a section of  $\mathcal{F}$  on  $U$

•  $\mathcal{F}_p$  stalk at  $p \in X$

$$\mathcal{F}_p := \varinjlim_U \mathcal{F}(U)$$

$$= \coprod_{p \in U} \mathcal{F}(U) / \sim$$

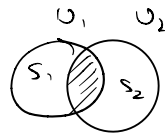
$$a \in \mathcal{F}(U) \sim b \in \mathcal{F}(V)$$

$$\text{iff } \exists K \subset U, K \subset V.$$

$$\text{s.t. } r_K^U(a) = r_K^V(b).$$

Def. Morphism of presheaf  $\mathcal{F} \rightarrow \mathcal{G}$

$$\begin{array}{ccc} \mathcal{F}(U) & \rightarrow & \mathcal{G}(U) \\ r_V^U \downarrow & & \downarrow r_V^U \\ \mathcal{F}(V) & \rightarrow & \mathcal{G}(V) \end{array} \quad V \subset U.$$



Def: Sheaf: we say a pre-sheaf  $\mathcal{F}$  is a sheaf if.

1) Given an open cover  $\{U_i\}$  of  $U \subset X$  and

$$s_i \in \mathcal{F}(U_i) \text{ satisfies } r_{U_i \cap U_j}^{U_i}(s_i) = r_{U_i \cap U_j}^{U_j}(s_j)$$

$$\text{then } \exists s \in \mathcal{F}(U) \text{ s.t. } r_{U_i}^U(s) = s_i$$

2) Given open cover  $\{U_i\}$  of  $U \subset X$

If  $s \in \mathcal{F}(U)$  satisfies  $r_{U_i}^U(s) = 0$ , then  $s = 0$ .

e.g. let  $X \subset \mathbb{C}^n$ ,  $\mathcal{F}_b(U) := \{ \text{bounded hol. func.} \}$ .

$r_V^U$  is restrict. func. on  $U$  to  $V$ .

( $\mathbb{C}^n$  noncpt. locally bdd  $\nRightarrow$  global bdd,  $\perp$  is false.)

$\mathcal{F}_b$  is not sheaf.

e.g. Given any group  $G$

Presheaf:  $\underline{G}(U) := G$ ,  $r_V^U := \text{Id. } G \rightarrow G$

$\underline{G}(\emptyset) = G$ ,  $r_\emptyset^U = \text{Id.}$ ,  $r_\emptyset^V = \text{Id.}$

If  $U \cap V = \emptyset$ , take  $a \in \underline{G}(U) = G$   $a \neq b$   
 $b \in \underline{G}(V) = G$

$r_\emptyset^U(a) = r_\emptyset^V(b)$   $\leadsto$   $\exists s \in \underline{G}(U \cup V = G)$   
s.t.  $s = a$ ,  $s = b$ .

Impossible.

# Lecture 5

Correction: Last time Define presheaf  $\leadsto$  sheaf  
 $(\mathbb{C}, \mathcal{O}_b)$  is not sheaf.

Constant sheaf  $\underline{G}$  ( $G$  is a group)

Def:  $\underline{G}(U) := \{ U \rightarrow G, \text{ locally constant} \}$

If  $U$  is connect,  $\underline{G}(U) = G$

$$\underline{G}(\emptyset) = \{e\}.$$

$\gamma_U^U$  is identity map if  $\emptyset \neq V \subset U$

Ex: Verify  $\underline{G}$  is a sheaf.

e.g.:  $E \rightarrow X$  holo VB

Sheaf of holomorphic sections of

$$\sigma(U) := \{ \Gamma(U, E) \}$$

e.g. locally constant sheaf in group  $G$

$X \xrightarrow{f} Y$ ,  $X, Y \subset^\infty$  mfd,  $f$  is submersion

Define a presheaf  $R^i f_* \underline{\mathbb{C}}(U) := H^i(f^{-1}(U), \underline{\mathbb{C}}_X)$

$U \subset Y$

$$H_{\text{de}}^i(f^{-1}(U), \mathbb{C})$$

" $y$  changes stalk doesn't change"

If  $f: X \rightarrow Y$  is good fibration, then  $(R^i f_* \underline{\mathbb{C}}_X)_y = H^i(X_y, \mathbb{C})$

on  $Y$   
 "isom. differ to each other."

Def Morphism of sheaf  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .

$$\begin{array}{ccc} \mathcal{F}_1(U) & \longrightarrow & \mathcal{F}_2(U) \\ r_U^\vee \downarrow & & \downarrow r_U^\vee \\ \mathcal{F}_1(V) & \longrightarrow & \mathcal{F}_2(V) \end{array}$$

Def.  $(\text{Ker } \varphi)(U) := \text{Ker}[\mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U)]$

Ex:  $\text{Ker } \varphi$  is a sheaf.

Def  $(\text{Im } \varphi)(U) := \text{Im}[\mathcal{F}_1(U) \xrightarrow{\varphi_U} \mathcal{F}_2(U)]$

In general:  $\text{Im}(\varphi)$  defined above is NOT sheaf.

e.g.  $X = \mathbb{C}^*$   $\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$ ,  $X = U_1 \cup U_2$   
 $\downarrow \quad \quad \downarrow$   $U_1 = X \setminus (-\infty, 0)$   
 $\quad \quad \quad \downarrow$   $U_2 = X \setminus (0, +\infty)$

$$Z \in \text{Im}(\exp)(U_1)$$

$$Z = e^{\log_1 Z} \text{ on } U_1$$

$$Z \in \text{Im}(\exp)(U_2)$$

$$r_{U_1 \cap U_2}^{U_1}(Z) = r_{U_1 \cap U_2}^{U_2}(Z).$$

we cannot log

If  $\text{Im}(\exp)$  is sheaf,  $\exists F \in \mathcal{O}_*(\underbrace{U_1 \cup U_2}_{\mathbb{C}^*})$  s.t.  $e^F = Z$  on  $U_1 \cup U_2$   
 (By gluing property of sheaf). ↑ Contradiction!

## Thm (Sheafification)

Let  $\mathcal{F}$  be a presheaf, then  $\exists!$  sheaf  $\mathcal{F}^+$  and  $i: \mathcal{F} \rightarrow \mathcal{F}^+$ , s.t. for any sheaf  $G$  and  $\mathcal{F} \xrightarrow{p} G$ ,

$\exists f: \mathcal{F}^+ \rightarrow G$  satisfying

$$\mathcal{F} \xrightarrow{i} \mathcal{F}^+$$

$\mathcal{F} \xrightarrow{p} G$  is a presheaf morphism  
 $f \circ i$  is a sheaf morphism

## Sketch of proof:

$$\mathcal{F}^+(U) := \left\{ s: U \rightarrow \coprod_{p \in U} \mathcal{F}_p \text{ satisfies } \forall q \in U \right.$$

$$\left. \exists q \subset V \subset U \text{ and } f \in \mathcal{F}(V) \text{ s.t. } \tilde{f} = s \right\}$$

where  $\tilde{f}: V \rightarrow \coprod_{p \in V} \mathcal{F}_p$  defined as  $\tilde{f}_p :=$  the stalk of  $[f]$  at  $p$

$$\underline{\text{Ex:}} \quad \mathcal{F} \xrightarrow{i} \mathcal{F}^+, \quad \mathcal{F}(U) \xrightarrow{i_U} \mathcal{F}^+(U)$$
$$s \longmapsto \tilde{s}$$

Ex: verify  $\mathcal{F}^+$  is a sheaf.

$$s_1 \in \mathcal{F}^+(U), \quad s_2 \in \mathcal{F}^+(V)$$

$$s_1 = s_2 \text{ on } \mathcal{F}^+(U \cap V) \Rightarrow \exists f \in \mathcal{F}^+(U \cup V)$$



$$F(p) = \begin{cases} S_1(p) & \text{if } p \in U \\ S_2(p) & \text{if } p \in V \end{cases} \quad \text{well-defined.}$$

$$U \cup V \longrightarrow \coprod_{p \in U \cup V} \mathcal{F}_p \quad q \in U \cup V. \quad \square$$

Ex: Verify  $\mathcal{F}^\#$  satisfies the universal property.

Im( $\varphi$ ) denotes the sheafification sheaf.

$$\text{Hence, } \text{Im}(\varphi)(U) \cong \mathcal{F}_1(U) \xrightarrow{\varphi_U} \mathcal{F}_2(U).$$

Ex.  $(*) \quad 0 \rightarrow \mathcal{F}_1 \xrightarrow{f} \mathcal{F}_2 \xrightarrow{g} \mathcal{F}_3 \rightarrow 0$

where  $f$  and  $g$  are sheaf morphism

We say  $(*)$  is exact.

$$\frac{\text{Ker } g}{\text{Im } f} \quad \text{Ker } f = 0 \quad \mathcal{F}_3 = \text{Im}(g)$$

In general,  $0 \rightarrow C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} C_3 \xrightarrow{d_3} \dots C_n \rightarrow \dots$  exact

means  $\text{Ker } d_i = \text{Im } d_{i-1}$ .

Ex:  $(*)$  is exact  $\Leftrightarrow$  Induced sequence stalk's complex?

$$0 \rightarrow \mathcal{F}_{1,p} \rightarrow \mathcal{F}_{2,p} \rightarrow \mathcal{F}_{3,p} \rightarrow 0$$

is exact  $\forall p \in X$

### § 3.2 Čech Cohomology

Fix  $(X, \mathcal{F})$  open cover  $\underline{U} = \{U_i\}_{i \in I}$  of  $X$

Define  $H^i(\underline{U}, \mathcal{F})$  is defined as the Cohomology of the following complex:  $0 \rightarrow C^0(\underline{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\underline{U}, \mathcal{F}) \rightarrow \dots$

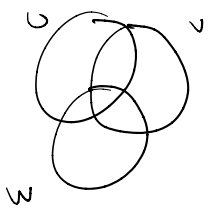
where  $C^p(\underline{U}, \mathcal{F}) = \bigoplus_{i_0, i_1, \dots, i_p \in I^{p+1}} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_p})$

satisfies 1)  $\sigma_{i_0, \dots, i_p} = 0$  if  $i_m = i_n$  for  $m \neq n$

2)  $\sigma_{i_0, \dots, i_p} = \sigma_{\tau(i_0, \dots, i_p)} (-1)^\tau$  where  $\tau \in S^{p+1}$

$$\sigma = (\sigma_{i_0, \dots, i_p})$$

$$\text{If } \sigma \in C^p, (\delta \sigma)_{i_0, i_1, \dots, i_{p+1}} := \sum_{n=0}^{p+1} (-1)^n \sigma_{i_0, \dots, i_n, i_{p+1}}$$

Eg 

$$\sigma \in C^1(\underline{U}, \mathcal{F}),$$

$$\sigma_{uv}, \sigma_{uw}, \sigma_{vw}$$

$$\delta \sigma = \sigma_{uv} - \sigma_{uw} + \sigma_{vw}$$

Ex.  $\delta \circ \delta = 0$ . (check!)

$$E_x \quad H^0(\underline{U}, \mathcal{F}) = H^0(\underline{V}, \mathcal{F}) = \mathcal{F}(X)$$

$\underline{U}, \underline{V}$  are different open covers.

i.e.  $H^0$  is independent of choice of cover

for  $i \geq 1$ .  $H^i(\underline{U}, \mathcal{F})$  depends on the choice of cover.

If  $\underline{V}$  is refinement of  $\underline{U}$ , then

$$\begin{array}{ccc}
 C^p(\underline{U}, \mathcal{F}) & \xrightarrow{?} & C^p(\underline{V}, \mathcal{F}) \\
 \downarrow \delta & \text{restriction which smaller} & \downarrow \delta \\
 C^{p+1}(\underline{U}, \mathcal{F}) & \longrightarrow & C^{p+1}(\underline{V}, \mathcal{F}) \\
 \vdots & & \vdots
 \end{array}$$

$U_i \supseteq U_{j(i)}$   
 $\downarrow$   
 $U_i \supseteq U_{j(i)}$   
 $\downarrow$   
 $U_i \supseteq U_{j(i)}$

General theory:

$$H^p(\underline{U}, \mathcal{F}) \longrightarrow H^p(\underline{V}, \mathcal{F}) \text{ for any } p$$

choose an element  $\rightarrow H^p(\underline{V}, \mathcal{F})$ . explain 不依赖于代表元选取.

General theory

$$H^p(X, \mathcal{F}) = \varinjlim_{\underline{U}} H^p(\underline{U}, \mathcal{F}) = \frac{\coprod_{\underline{U}} H^p(\underline{U}, \mathcal{F})}{\sim}$$

### § 3.3 The most Important Theorem in Sheaf Theory.

Let  $0 \rightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \rightarrow 0$  be exact sequence of sheaf

then we have the follow exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{G}) \rightarrow H^0(X, \mathcal{H}) \rightarrow H^1(X, \mathcal{F}) \\ \rightarrow H^1(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{H}) \rightarrow \dots \rightarrow H^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{G}) \\ \rightarrow H^p(X, \mathcal{H}) \rightarrow \dots \end{aligned}$$

prop: Let  $0 \rightarrow \mathcal{F} \xrightarrow{K_0} \mathcal{G}_0 \xrightarrow{K_1} \mathcal{G}_1 \rightarrow \dots \rightarrow \mathcal{G}_n \rightarrow 0$   
be exact sequence of sheaf.

If  $H^i(X, \mathcal{G}_j) = 0$ ,  $\forall i \geq 1, 0 \leq j \leq n$  = RHS

then  $H^i(X, \mathcal{F}) \cong H^i(H^0(X, \mathcal{G}_0) \rightarrow H^1(X, \mathcal{G}_0) \rightarrow \dots \rightarrow H^1(X, \mathcal{G}_p) \rightarrow \dots)$

sheaf cohomology

Global section's complex's cohomology

proof: Let  $K_p := \text{Ker}(\mathcal{G}_p \rightarrow \mathcal{G}_{p+1}) \sim K_0 = \mathcal{F}$

Step 1.  $0 \rightarrow K_p \rightarrow \mathcal{G}_p \rightarrow K_{p+1} \rightarrow 0$ .

$H^1(\mathcal{G}_p)$

$$H^0(K_p) \rightarrow H^0(\mathcal{G}_p) \rightarrow H^0(K_{p+1}) \rightarrow H^1(K_p) \rightarrow 0$$

$$H^1(K_p) = \frac{H^0(K_{p+1})}{\text{Im } H^0(G_p)} = \frac{\text{Ker } [G_{p+1}(X) \rightarrow G_{p+2}(X)]}{\text{Im } [G_p(X) \rightarrow G_{p+1}(X)]} = H^0(X, G_{p+1})$$

$$\subseteq H^{p+1}(\text{RHS})$$

Step 2.  $H^1(K_p) \stackrel{?}{=} H^2(K_{p-1}) \stackrel{?}{=} H^3(K_{p-2}) \dots H^{p+1}(K_0) = H^{p+1}(\mathcal{F})$   
 RHS  $\checkmark$

Claim  $H^1(K_p) \subseteq H^2(K_{p-1})$

$$0 \rightarrow K_{p-1} \rightarrow G_{p-1} \rightarrow K_p \rightarrow 0$$

$$\begin{array}{ccccccc} \sim & H^1(G_{p-1}) & \rightarrow & H^1(K_p) & \rightarrow & H^2(K_{p-1}) & \rightarrow H^2(G_{p-1}) \rightarrow H^2(K_p) \rightarrow \dots \\ & \parallel & & \text{---} & & \text{---} & \parallel \\ & 0 & & & & & 0 \end{array}$$

iso?

Def: We say  $\mathcal{F}$  is acyclic if  $H^i(X, \mathcal{F}) = 0 \quad \forall i \geq 1$ .

Def: Fine sheaf:  $\mathcal{F}$

We say  $\mathcal{F}$  is fine if for any open cover  $\underline{U} = \{U_i\}_{i \in \Lambda}$

We can find  $\eta_i: \mathcal{F} \rightarrow \mathcal{F}$  s.t.

(i) For each  $i$ ,  $\eta_i|_p: \mathcal{F}_p \rightarrow \mathcal{F}_p$  is 0 for  $p$  outside

a compact set  $W \subset U_i$



(ii)  $\sum_i \eta_i = \text{Id}$ .

Ex. fine sheaf is acyclic

e.g. sheaf of  $C^\infty$   $p$ -form. on  $X$ .

e.g.  $\mathcal{O}_X$  sheaf of holo. func is not fine.

Def: Flabby Sheaf:  $\mathcal{F}$ .

We say a sheaf  $\mathcal{F}$  is flabby if for any  $U \subset V$ ,

$$\text{then } r_U^V : \mathcal{F}(V) \rightarrow \mathcal{F}(U) \rightarrow 0$$

Ex Flabby Sheaf is acyclic.

DeRham Thm

$$H^i(X, \mathbb{C}) \cong H_{\text{dR}}^i(X, \mathbb{C}) = [ \underset{H^0(X, \mathcal{A}^0)}{A^0(X)} \xrightarrow{d} \underset{H^1(X, \mathcal{A}^1)}{A^1(X)} \xrightarrow{d} \dots ]$$

Dolbeault Thm

$$H^i(X, \Omega^p) \cong H_{\bar{\partial}}^{p,q}(X) = [ A^{p,0}(X) \xrightarrow{\bar{\partial}} A^{p,1}(X) \xrightarrow{\bar{\partial}} \dots ]$$

proof:  $0 \rightarrow \mathbb{C}_X \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \mathcal{A}^2 \rightarrow \dots$  (Why exact?  
By Poincaré Lem.)

$\mathcal{A}^p$  denotes the sheaf of smooth  $p$ -form

$$\underline{\mathcal{A}^p \text{ is acyclic!}} \Leftrightarrow H^i(X, \mathcal{A}^p) = 0 \quad \forall i \geq 1$$

$$0 \rightarrow \Omega^p \rightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \dots$$

• The exactness is Poincaré Lemma

•  $\mathcal{A}^{p,q}$  is acyclic

$$H^i(X, \Omega^p) \cong H^i[ A^{p,0} \rightarrow A^{p,1} \rightarrow \dots ]$$

# Lecture 6.

## § 3.4. Applications of sheaf cohomology

$$0 \rightarrow \mathcal{F}_0 \rightarrow \underbrace{\mathcal{G}_0 \rightarrow \mathcal{G}_1 \rightarrow \dots}_{\text{acyclic}} \text{ Exact Sequence}$$

$$H^i(X, \mathcal{G}_j) = 0 \quad \forall i \geq 1.$$

$$\text{Then } H^i(X, \mathcal{F}_0) \cong H^i(\Gamma(X, \mathcal{G}_j)) \quad \Gamma(X, \mathcal{G}_0) \rightarrow \Gamma(X, \mathcal{G}_1) \rightarrow \dots$$

$$H^i(X, \mathbb{C}_X) \cong H_{dR}^i(X)$$

$$H^q(X, \Omega^p) \cong H^q[\mathcal{A}^{p,0}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}(X) \rightarrow \dots]$$

$$0 \rightarrow \Omega^p \rightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}$$

$$\text{Cohomology of } \mathbb{Z}_X, \text{ then } H^i(X, \mathbb{Z}_X) \cong H_{\text{sing}}^i(X, \mathbb{Z})$$

Picard Group:  $H^1(X, \mathcal{O}_X^*)$ ,  $\mathcal{O}_X^*$  is the sheaf of nonzero holomorphic fnc. algebraic topology

$$[L_1] \cdot [L_2] = [L_1 \otimes L_2] \text{ isomorphic class of Line Bundle } L.$$

- Given Line Bundle  $L$  +  $\underline{U}$  open cover

$$\rightarrow f_{ij}: U_i \cap U_j \xrightarrow{\text{holo}} \mathbb{C}^* \leadsto H^1(X, \mathcal{O}_X^*)$$

check: Independent of open cover

Independent of trivialization of  $L$ .

- Given an element,

$$H^1(X, \mathcal{O}_X^*) := \coprod_{\underline{U}} H^1(\underline{U}, \mathcal{O}_X^*) / \sim$$

$\leadsto$  A rep  $[f_{ij}] \in U_i \cap U_j \xrightarrow{\text{holo}} \mathbb{C}^* \leadsto$  glue to line bundle.

check: Different  $[f_{ij}]$  gives isomorphic Line Bundle.

Divisor Class on a Complex mfd  $X$ .

Def: Prime Divisor  $D$ : Codim-1 Subvariety of  $X$ , locally defined by reduced and irreducible holomorphic functions.



$$\underline{U} = \{U_i\}$$

$$\underline{D} \cap U_i = (f_i = 0) \quad f_i \in \mathcal{O}(U_i)$$

fact.  $\downarrow g(1) \cdot (f_i = 0)$  since line bundle's rank = 1

Def (Divisor) =  $\mathbb{R}$ -linear combination of Prime divisor

$$\sum a_i D_i, \quad a_i \in \mathbb{R}, \quad D_i \text{ is prime.}$$

Def  $D_1 \sim D_2$  linear equivalent iff  $\exists g \in M^*(X)$  s.t.

$$D_1 = \sum a_i D_i$$

$$(g) = D_1 - D_2$$

$$D_2 = \sum b_j D_j$$

e.g.  $g = \frac{z_1^2}{z_2 z_3}$  on  $\mathbb{CP}^2$   $(g) = 2(z_1=0) - 1(z_2=0) - 1(z_3=0)$

$\mathbb{Z}$  coefficient Divisor class  $\xleftrightarrow{|\cdot|} \text{iso class of line bundle.}$

• Given  $D$  is a divisor  $\leadsto D \cap U_i = (f_i = 0)$  选取时不能过多要 prime  
 $\underline{U}$  is open cover + some poles

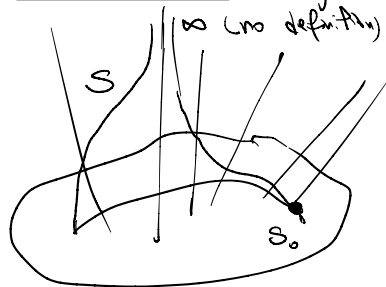
$$f_{ij} = \frac{f_i}{f_j} = U_i \cap U_j \xrightarrow{\text{holo}} \mathbb{C}^*$$

why it's well-defined?

$\leadsto$  glue to Line Bundle.



- Given a Line Bundle  $\rightsquigarrow$  pick a rational section  $S$  of  $L$   
 $\rightsquigarrow D = (S_0) - (S_\infty)$



First Chern Class of a Line Bundle

$$0 \rightarrow \mathbb{Z}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 0$$

$$f \mapsto \exp(2\pi i f)$$

$$\dots H^1(X, \mathbb{Z}_X) \rightarrow H^1(\mathcal{O}_X) \xrightarrow{i} \underbrace{H^1(\mathcal{O}_X^*)}_{(L)} \xrightarrow{c_1} H^2(X, \mathbb{Z}_X) \dots$$

$c_1(L) \in H^2(X, \mathbb{Z}_X)$  is called the first Chern Class of  $L$ .

$\text{Pic}^0(X) := \text{Ker}(c_1) \subset \text{Im}(i) \rightarrow$  degree zero line Bundle.

$$\frac{H^1(X, \mathcal{O}_X)}{\text{Im}(H^1(X, \mathbb{Z}_X))} \xrightarrow{i} H^1(X, \mathcal{O}_X^*)$$

In fact,  $\text{Pic}^0(X)$  is a compact Torus

## § 4. Differential Geometry of Vector Bundle

Def. Hermitian metric  $h$  on  $E \rightarrow X$  holo. of rank of  $r$ .  
a smooth family of Hermitian metric  $h_p (p \in X)$  on  $E_p \cong \mathbb{C}^r$ .

$h_p(u, v)$  for  $u, v \in E_p$

If choose a local frame  $(e_\alpha, 1 \leq \alpha \leq r)$  of  $E$ .

then  $h(e_i, e_j) = h_{ij}$

Ex: Prove  $h$  exists.

Def: (connection) connection  $D$  of  $E \rightarrow X$

A connection  $D: C^\infty(X, E) \rightarrow A^1(X, E)$  satisfies

1)  $D$  is  $\mathbb{C}$ -linear

2) (Leibniz rule)  $D(fe) = df \cdot e + fDe$

$f$  is  $C^\infty$ -func and  $e$  is  $C^\infty(X, E)$ .

local form of a connection: pick a local frame of  $E$

$De_\alpha = \overset{\text{row}}{\theta_\alpha^\beta} \overset{\text{column}}{e_\beta}$ ,  $\theta_\alpha^\beta$  is 1-form  $\rightsquigarrow \theta_\alpha^\beta$  1-form

$$(De_1, De_2, \dots) = (e_1, e_2, \dots, e_n) \begin{pmatrix} \theta_1^1 & \theta_1^2 & \dots \\ \theta_2^1 & \theta_2^2 & \dots \\ \vdots & \vdots & \ddots \\ \theta_n^1 & \theta_n^2 & \dots \end{pmatrix}$$

$\xi \in C^\infty(X, E)$

$$D\xi = D(\xi^\alpha e_\alpha) = d\xi^\alpha e_\alpha + \xi^\alpha De_\alpha$$

$$= d\xi^\alpha e_\alpha + \xi^\alpha \theta_\alpha^\beta e_\beta$$

$$= d\xi^\alpha e_\alpha + \xi_\alpha \theta_\beta^\alpha e_\alpha$$

$$= (d\xi^\alpha + \xi_\alpha \theta_\beta^\alpha) e_\alpha.$$

Curvature  $\Theta$  of  $D$   $\Theta: D \rightarrow D$

Def: Extend  $D: A^k(x, E) \rightarrow A^{k+1}(x, E)$

$$D(\varphi \cdot e) := d\varphi \cdot e + \overset{\deg(\varphi)}{(\varphi) \wedge D e}$$

$\nearrow$   $k$ -form  $\nearrow$  local section of  $E \rightarrow \Theta$  is well-defined.

$$\Theta(f \cdot g) := D \cdot D(f \cdot g) = D(df \cdot g + f \cdot Dg)$$

$$f \in C^\infty(x, \mathbb{C})$$

$$g \in C^\infty(x, E)$$

$$= (\cancel{d \cdot df})g - \cancel{df \wedge Dg} + \cancel{df \wedge Dg}$$

$$+ f D \cdot Dg$$

$$= f(\Theta g)$$

$$\text{Hence } \Theta \in A^2(x, \text{End}(E))$$

consider as bundle.

local form of  $\Theta$  In terms of  $\theta_\alpha^\beta$

Ex:

$$\Theta_\beta^\alpha = d\theta_\beta^\alpha + \theta_\gamma^\alpha \wedge \theta_\beta^\gamma$$

$$\begin{bmatrix} \downarrow & \text{1-form} \\ 0 & 0 & \dots \end{bmatrix} \begin{bmatrix} \downarrow & \text{1-form} \\ 0 & \dots \end{bmatrix} = \begin{bmatrix} \downarrow & \downarrow & \text{2-form} \\ 0 & 0 & \dots \end{bmatrix}$$

$$\text{Ex: If } \tilde{e}_\alpha = \underbrace{a_\alpha^\beta}_{\substack{\downarrow \\ n \times n \text{ matrix}}} e_\beta, \text{ then } \tilde{\Theta} = a^{-1} \Theta a.$$

(Gauge transformation)

$$\text{Ex. } \tilde{\Theta} = [\tilde{\Theta}] \Theta$$

$$\text{From } \tilde{\Theta} = a^{-1} \Theta a$$

S.S. Chern Some invariants of  $D$

$$\text{Def: } \det \left( I_r + \frac{\sqrt{-1}}{2\pi} \Theta \right) = I + C_1(E, D) + C_2(E, D) + \dots$$

where  $C_i(E, D)$  is  $2i$ -Chern form  $+ \dots + C_k(E, D)$

(the class of  $C_i(E, D)$  is called  $i$ -Chern class) depend on connection !!!

## § 4.2 Connection Chern.

Thm  $(E, h) \rightarrow X$ ,  $\exists!$  connection  $D$  satisfies

1) Metric Compatible

$$d(s, t)_h = (ds, t)_h + (s, dt)_h \quad \forall s, t \in C^\infty(X, E)$$

2) Compatible with  $J$  i.e.

if  $s$  is holo section of  $E$  ( $\bar{\partial}s = 0$ )

then  $Ds$  is  $E$ -valued  $(1, 0)$  form

i.e.  $\{e_\alpha\}$  is holo frame

$D e_\alpha \sim \theta_\alpha^P$  is  $(1, 0)$ -form, no  $(0, 1)$  component.

$$D = D^{1,0} + \underbrace{D^{0,1}}_{= \bar{\partial}}$$

Ex.  $\theta_i^P = h^{P\bar{j}} \partial h_{i\bar{j}}$ , Here  $h_{i\bar{j}} = h(e_i, e_{\bar{j}})$ ,  $\{e_\alpha\}$  is holo. frame.

$$\theta = h^{-1} \cdot \partial h$$

Ex.  $\text{photo} \rightsquigarrow \textcircled{H} = \bar{\partial}(h^{-1} \cdot \partial h)$

$$\textcircled{H} \nearrow = d\theta + \theta \wedge \theta.$$

In particular, if  $E$  is a line Bundle  $L$ .

$$\begin{aligned} \text{then } \textcircled{H} &= \bar{\partial} \partial \log f & f &= h(e, e) \\ &= -\partial \bar{\partial} \log f \end{aligned}$$

# Chern Class of Line Bundle.

$$\det(I + \frac{\sqrt{-1}}{2\pi} \textcircled{F}) = 1 + \underbrace{\frac{\sqrt{-1}}{2\pi} \textcircled{F}}_{\text{First Chern form}} + 0$$

e.g.  $\mathcal{O}(-1) \rightarrow \mathbb{P}^n$   
 $\parallel$   
 $\mathbb{P}^n \times \mathbb{C}^{n+1}$

Define  $h(v, w) = v \cdot \bar{w}$

$v, w \in \mathcal{O}(-1) \downarrow_{[Z]}$

$([Z], v) \downarrow_{v \in [Z]}$

$\textcircled{F}_h$  on  $U_0 := (Z_0 \neq 0)$

on  $U_0$ , we have coordinate  $(z_i := \frac{z_i}{z_0}, 1 \leq i \leq n)$

We have local frame of  $\mathcal{O}(-1)$ ,  $e = \left(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)$

$$h(e, e) = 1 + \sum_{i=1}^n \left| \frac{z_i}{z_0} \right|^2$$

$\downarrow$   
[Z]

$$= 1 + \sum_{i=1}^n |z_i|^2$$

$$\textcircled{F} = -\partial \bar{\partial} \log \left( 1 + \sum_{i=1}^n |z_i|^2 \right)$$

## § Chern-Weil Theory

Thm. Chern Class is independent of  $h$

i.e.  $C_i(E, h_1) - C_i(E, h_2)$  is d exact.

# § 4.5 Some properties of Chern class (BTL)

e.g. 
$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ X & \longrightarrow & Y \end{array} \quad c_1(f^*E) = f^*c_1(E)$$

e.g.  $c_1(E \otimes F) \stackrel{?}{=} c_1(E) \text{ and } c_1(F)$

$$c_1(E \oplus F) \stackrel{?}{=}$$

$$c_1(L, h) \in H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$$

$$c_1(L) \in H^2(X, \mathbb{Z}) \quad \nearrow \text{embed}$$

are they same.?

# Lecture 7

Correspondence: Divisor class  $\xleftrightarrow{1-1}$  class of Line bundle

On a compact complex mfd  $X$ .

- If  $X$  is projective  $X \subset \mathbb{CP}^N$  submfd  $\checkmark$
- If  $X$  is not projective, not necessarily true  $\hookrightarrow \checkmark$ .

Reason: If  $X$  is projective, rational section of  $L$  always exists.

Last time: Chern connection of  $(E, h) \rightarrow X$

- Metric compatible.
- $J$  compatible

## § 4.6 Hermitian metric and Kähler metric

$\underbrace{TX}_{\substack{\uparrow \\ \text{holomorphic tangent bundle} \\ \{\partial_i\}}}$   $\longrightarrow X$

$T^{\mathbb{R}}X$  real tangent bundle

$T^{\mathbb{C}}X$  complexified tangent bundle

$\parallel$

$T^{1,0}X \oplus T^{0,1}X$   
 $\uparrow \quad \quad \uparrow$   
 $\partial_i \quad \quad \bar{\partial}_i$

$(X, J, g)$  be complex mfd  
 with  $g$  be a Riem. metric

$g: T^{\mathbb{R}}X \otimes T^{\mathbb{R}}X \rightarrow \mathbb{R}$

We say  $g$  is  $J$  invariant if  $g(JX, JY) = g(X, Y), X, Y \in T\mathbb{R}X$

Extend  $g$   $\mathbb{C}$ -linear to  $T^{\mathbb{C}}X$

Then we define a Hermitian metric  $h$  on  $TX$ .

$$\langle u, v \rangle_h := g(u, \bar{v}), u, v \in TX$$

Conversely, if we have a Hermitian metric  $h$  on  $TX$ .

$\leadsto J$  invariant. Rie. metric on  $T\mathbb{R}X$ .

Ex:  $g(T^{1,0}, T^{1,0}) = 0 = g(T^{0,1}, T^{0,1})$ .

Def: Kähler Metric. If  $g$  is  $J$  invariant.

$$\omega(\cdot, \cdot) := g(J\cdot, \cdot)$$

$\omega$  is a 2-form.  $\downarrow$   
symplectic.

We say  $(J, g)$  is a Kähler metric if  $d\omega = 0$ .

Ex: Set  $g_{i\bar{j}} = g(\partial_i, \partial_{\bar{j}})$ , then  $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_{\bar{j}}$

Ex:  $d\omega = 0 \Leftrightarrow \frac{\partial g_{i\bar{j}}}{\partial \bar{k}} = \frac{\partial g_{i\bar{k}}}{\partial \bar{j}} \quad \forall i, j, k$

$\bar{\partial}\omega = 0 + \partial\omega = 0$

$g_{i\bar{j}} = \delta_{ij}$   
at  $p$

Ex:  $d\omega = 0 \Rightarrow \exists$  normal holomorphic coordinate at  $p$ .

i.e.  $\exists$  a coordinate  $(z_1, z_2, \dots, z_n)$  at  $p$  s.t.  $dg_{i\bar{j}} = 0$ .



Cor.  $(X, J, g)$  is a Kähler mfd  $\Rightarrow \nabla \overset{\text{tensor}}{J} = 0$ .

where  $\nabla$  is Levi-Civita Con.,  $J$  is complex structure.

i.e.  $\nabla[J(x)] = J(\nabla x) = 0$

proof: check using normal holomorphic coordinate (Ex).

e.g. Fubini-Study metric on  $\mathbb{CP}^n$

pick  $[Z] \in \mathbb{CP}^n$ .  $T_{[Z]}\mathbb{CP}^n = \{ u \in \mathbb{C}^{n+1} \mid \langle u, Z \rangle = 0 \}$   
 $u \in [Z]^\perp$  is also right.  
 $(u, v)_h = u \cdot \bar{v}$

$$W_{FS} = g(J \cdot, \cdot)$$

$$(*) := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |Z|^2, \quad |Z|^2 = \sum_{i=0}^n |Z_i|^2 \text{ not well-defined.}$$

Fact:  $\partial \bar{\partial} \log a = \partial \bar{\partial} \log(a|f|^2)$ , where  $f$  is holomorphic.

i.e.  $\partial \bar{\partial} \log |f|^2 = 0$  if  $f$  is holomorphic.

$$\begin{aligned} \text{On } U_0 := (Z_0 \neq 0), \quad W_{FS} &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( 1 + \sum_{i=1}^n \left| \frac{Z_i}{Z_0} \right|^2 \right) \\ &= \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left( 1 + \sum_{i=1}^n |\xi_i|^2 \right) \end{aligned}$$

where  $\xi_i = \frac{Z_i}{Z_0}$ .

Ex. Verify  $\omega_{FS} = g(J\cdot, \cdot)$  is indeed of form (\*)

$$\boxed{d\omega_{FS} = 0}$$

eg. If  $(X, g, J)$  is Kähler, then any complex submanifold is Kähler with induced metric

i.e.  $(Y, g_Y, J_Y)$  is Kähler: If  $u \in T^R Y$ , then  $Ju \in T^R Y$

Ex:  $\omega_Y := g_Y(J_Y \cdot, \cdot)$ , check  $\omega_Y = i^* \omega_X$ ,  $i: Y \rightarrow X$

Let  $(X, J, g)$  is Kähler: Levi-civita Connection is the same as Chern Connection.

• Let  $\nabla$  be Levi-civita connection do  $\mathbb{C}$ -linear Extension to  $T\mathbb{C}X$ . And then we can restrict  $\nabla$  to  $T^{1,0}X = TX$ .

Proof: •  $\nabla$  is metric compatible. ✓

•  $\nabla$  is  $J$  compatible.

$$\text{i.e. } \nabla_{\bar{\partial}} \partial_i = 0 \quad \forall i, \bar{j}$$

$$J(\nabla_{\bar{\partial}} \partial_i) = \nabla_{\bar{\partial}}(J\partial_i) = \sqrt{-1} \nabla_{\bar{\partial}} \partial_{\bar{j}} = -\sqrt{-1} \nabla_{\bar{\partial}} \partial_{\bar{j}}$$

$$J(\nabla_{\bar{\partial}} \partial_i) = J(\nabla_{\bar{i}} \partial_{\bar{j}}) = \nabla_{\bar{i}}(J\partial_{\bar{j}}) = -\sqrt{-1} \nabla_{\bar{i}} \partial_{\bar{j}}$$

$$\leadsto \nabla_{\bar{j}} \partial_i = 0, \quad \nabla_i \partial_{\bar{j}} = 0.$$

( $\nabla_j \partial_i, \nabla_{\bar{j}} \partial_{\bar{i}}$  may not be zero.)

Set  $\nabla_i \partial_j = \Gamma_{ij}^k \partial_k \leadsto$  Define  $\Gamma_{ij}^k$

Ex.  $\Gamma_{ij}^k = g^{k\bar{p}} \cdot \frac{\partial g_{j\bar{p}}}{\partial \bar{j}} = g^{k\bar{p}} \frac{\partial g_{j\bar{p}}}{\partial \bar{j}}$

$\Gamma_{ij}^k$

$$(\nabla_i \partial_j, \partial_p) = \Gamma_{ij}^k g_{k\bar{p}}$$

$$\partial_i (h_{j\bar{p}}) = \partial_i (\partial_j, \partial_{\bar{p}}) - (\partial_j, \nabla_i \partial_{\bar{p}})$$

Curvature of Kähler Metric

In general, for a general bundle  $(E, D)$ .

$$\Theta = D \circ D, \quad \Theta: T(X, \wedge^2 T^*M \otimes \text{End}(E)).$$

Ex.  $R(JX, JY)Z = R(X, Y)Z \leadsto R(\partial_i, \partial_{\bar{j}})Z = 0.$   
 $R(\partial_{\bar{i}}, \partial_j)Z = 0.$

$$R_{ij\bar{k}\bar{l}} := \langle R(\partial_i, \partial_{\bar{j}}) \partial_k, \partial_{\bar{l}} \rangle \stackrel{?}{=}$$

$\mathbb{C}$ -linear  $\langle \cdot, \cdot \rangle$  anti- $\mathbb{C}$ -linear  $(\cdot, \cdot)$

$$(\nabla_i (\nabla_{\bar{j}} \partial_k), \partial_{\bar{l}}) - (\nabla_{\bar{j}} \nabla_i \partial_k, \partial_{\bar{l}}) = - \langle \nabla_{\bar{j}} (\Gamma_{ik}^p \partial_p), \partial_{\bar{l}} \rangle$$

$$= (-\partial_{\bar{j}} \Gamma_{ik}^p) g_{p\bar{e}} \rightsquigarrow (E_x)$$

Ex.  $\text{Ric}(X, Y) = \text{Ric}(JX, JY)$

Similar to  $g(X, Y) = g(JX, JY) \rightsquigarrow \omega(\cdot, \cdot) = g(J\cdot, \cdot)$

Def. Ric form  $\text{Ric}(\omega) := \text{Ric}(J\cdot, \cdot)$  is a 2-form

$$\text{Ric}(\omega_g) = \sqrt{-1} R_{i\bar{j}} dz_i \wedge d\bar{z}_{\bar{j}} \quad , \quad R_{i\bar{j}} = \text{Ric}(\partial_i, \partial_{\bar{j}})$$

Ex.  $R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} \log(\det g_{i\bar{j}})$

$\rightsquigarrow \text{Ric}(\omega_g)$  is a rep. of First Chern class  $(\Lambda^n T_x)$

Ex  $C_1(TX) \stackrel{?}{=} C_1(\Lambda^n TX)$

$(L, h)$ ,  $\text{Ric}(h) = -\partial\bar{\partial} \log h(e, e)$ ,  $e$  local frame of  $L$ .

$$-\partial\bar{\partial} \log h(e, e)$$

$$= -\partial\bar{\partial} \log [h(e, e) |f|^2]$$

$$= -\partial\bar{\partial} \log h(e, e) - \cancel{\partial\bar{\partial} \log |f|^2} \rightarrow 0$$

independent of choice of frame.

## § 5. Hodge Theorem

$\dim_{\mathbb{R}}$   
↓

Let  $(X^n, g)$  Rie. mfd.

Sheaf Theory  $H^i(X, \mathbb{R}) = H_{dR}^i(X, \mathbb{R})$ .

Find a canonical Rep of  $H_{dR}^i(X, \mathbb{R})$

$\alpha$   $p$ -form in  $[\alpha] \rightsquigarrow \alpha + \underbrace{d\eta}_{\in [\alpha]} \quad \eta: (p-1) \text{ form.}$

Minimizing  $L^2$  energy of rep. form in  $[\alpha]$ .

$$\|\alpha + d\eta^t\|_L^2 := \int \langle \alpha + d\eta^t, \alpha + d\eta^t \rangle_g dV_g = F(t)$$

More generally. for any 2-form  $\beta, \gamma$ .

$\langle \beta, \gamma \rangle_g$  can be pointwisely defined

Suppose  $\|\alpha\|_L^2$  is critial. (or minimal in  $[\alpha]$ )

$$F'(t)|_{t=0} = 2 \int \langle \alpha, d\eta \rangle dV_g = 0 \rightsquigarrow d^* \alpha = 0.$$

$d^*$  is defined as  $(d^* \beta, \gamma) = (\beta, d\gamma)$ .

Def:  $\alpha \in A^p(X)$  is called harmonic iff  $d\alpha = 0$  and  $d^* \alpha = 0$ .

Def:  $\Delta = dd^* + d^*d \quad \downarrow A^p \text{ form.} \quad d^*: A^p \rightarrow A^{p-1}$   
Hodge Laplacian.  $A^p(X) \rightarrow A^p(X)$ .

Fact  $\Delta \alpha = 0$  iff  $d\alpha = 0 = d^* \alpha$

$$\begin{aligned} (\Delta \alpha, \alpha) &= ((dd^* + d^*d) \alpha, \alpha) \\ &= (dd^* \alpha, \alpha) + (d^*d \alpha, \alpha) \\ &= (d^* \alpha, d^* \alpha) + (d \alpha, d \alpha). \end{aligned}$$

What  $d^*$ ?

Def: Hodge  $*$  operator  $A^p \rightarrow A^{n-p}$   
 $(i_1, \dots, i_p, j_1, \dots, j_{n-p})$   
 $* (dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p})_p = \delta_{(i_1, \dots, i_p)(j_1, \dots, j_{n-p})} dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_{n-p}}(p).$

Suppose  $(\frac{\partial}{\partial x_i})$  is  $g$  orthonormal at  $p \sim \underline{E}_x$ .  
 $*$  is well-defined.

Properties of  $(*)$ .

$$\underline{E}_x: \underbrace{(\alpha \wedge * \beta)}_{n\text{-form}} = (\alpha, \beta)_g dV_g \text{ if } \alpha, \beta \text{ are } p\text{-form}$$

$$= \beta \wedge * \alpha.$$

$$* * = (-1)^{p(n-p)} \text{ on } A^p$$

$$(* \alpha, * \beta)_g = (\alpha, \beta)_g$$

$$d^* = (-1)^{n-p+n+1} * d * \text{ on } \underline{E}_x.$$

$$d \Delta = \Delta d, \quad d^* \Delta = \Delta d^*$$

e.g.,  $(\mathbb{R}^n, g, \text{flat}) \quad d^*$

$$d^* f_{i_1 i_2 \dots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p}$$

$$= C(n, p) * d * (f_{i_1 i_2 \dots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_p})$$

$$i \in \{i_1, \dots, i_p\} = C_1(n, p) * \left( \sum_i \frac{\partial f_{i_1 \dots i_p}}{\partial x_i} dx_i \wedge (dx_{i_1} \wedge \dots \wedge dx_{i_p}) \right)$$

$$= C_2(n, p) \sum_i \frac{\partial f_{i_1 \dots i_p}}{\partial x_i} * (dx_i \wedge (dx_{i_1} \wedge \dots \wedge dx_{i_p}))$$

$$= C_3(n, p) \sum_k \frac{\partial f_{i_1 \dots i_p}}{\partial x_k} (-1)^k dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_k}} \wedge \dots \wedge dx_{i_p}$$

Ex.  $\Delta(f_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}) = - \sum_{i=1}^n \frac{\partial^2 f_{i_1 \dots i_p}}{\partial x_i^2} dx_{i_1} \wedge \dots \wedge dx_{i_p}$

smooth p-form

Hodge Thm  $(X^n, g) \quad \alpha \stackrel{?}{\in} A^p(X)$ , then.

$$\alpha = Id \cdot \alpha = \mathcal{H}(\alpha) + d^*(G\alpha) + dd^*(G\alpha) \rightsquigarrow Id - \mathcal{H} = \Delta G$$

where  $\mathcal{H}(\alpha)$  is a harmonic p-form

$\}$   
kernel of  $\Delta$ .

$G$  is the Green operator:  $A^p \rightarrow A^p$   $\mathcal{H} = \ker \Delta$

$$\alpha = \mathcal{H}(\alpha) \oplus \text{Im } d \oplus \text{Im } d^*$$

$\Delta$  operator 核掉  $\ker \Delta$   
 $\exists$  inverse.

$L^2$  decomposition

$$\alpha = \mathcal{H}(\alpha) \oplus \text{Im } d \oplus \text{Im } d^*$$

$$d: ? \rightarrow ?$$

$$d^*: ? \rightarrow ?$$

## Lecture 8.

Hodge Thm:  $(X, g)$  Rie. mfd.

Let  $\alpha \in A^i(X)$ . then  $\alpha = \mathcal{H}(\alpha) + d d^* G(\alpha) + d^* d G(\alpha)$ .

$$G: A^p \rightarrow A^p(X) = \text{Ker } \Delta \oplus \text{Im } d \oplus \text{Im } d^*$$

Indeed, It is an orthogonal decomposition w.r.t.  $(\cdot, \cdot)$  inner product on forms

$$L^2(X, \wedge^p T^*M) = \text{Ker } \Delta \oplus \text{Im } d \oplus \text{Im } d^*.$$

$$\alpha - \text{Ker } \Delta = \Delta G(\alpha). \quad \text{if } \alpha \perp (\text{Ker } \Delta)$$

$$G|_{\text{Ker } \Delta} = 0.$$

$$\alpha \in (\text{Ker } \Delta)^\perp \stackrel{?}{\subset} \text{Im } \Delta$$

$$G|_{(\text{Ker } \Delta)^\perp} = \Delta^{-1}.$$

$$G\alpha := \Delta^{-1}\alpha$$

$$\Delta(G\alpha) = \alpha$$

Outline the proof.

Step 1. Let  $\xi \in W^{1,2}(X, \wedge^p T^*X)$

Define  $\|\xi\|_{\text{new}}^2 = (\xi, \xi) + (d\xi, d\xi) + (d^*\xi, d^*\xi)$

Garding Inequality:  $\|\xi\|_{\text{new}} \sim W^{1,2}$  norm.

Define a quadratic norm  $\mathcal{D}$  on  $W^{1,2}(X, \wedge^p T^*X)$



$$\begin{aligned}
\mathcal{D}(\xi, \eta) &= (\xi, \eta) + (d\xi, d\eta) + (d^*\xi, d^*\eta) \\
&= (\xi, \eta) + (\xi, \Delta\eta) \\
&= (\xi, (\text{Id} + \Delta)\eta).
\end{aligned}$$

Step 2. Fix  $\eta \in L^2(X, \Lambda^p T^*M)$

Let  $\xi \in A^p(X)$

$$(\eta, \xi) \leq \|\eta\| \|\xi\| \leq C_\eta \|\xi\|_{\text{new}}$$

where  $C_\eta$  independent of  $\xi$ .

$\leadsto \eta$  defines as functional (Bounded + Linear)

$$A^p(X) \subset W^{1,2}(X, \Lambda^p T^*X) \rightarrow \mathbb{C}$$

$$\xi \longmapsto (\eta, \xi)$$

$\exists \varphi := T(\eta) \in W^{1,2}(X, \Lambda^p T^*X)$  s.t.

$$\mathcal{D}(\varphi, \xi) = (\eta, \xi). \quad (\text{Riesz-Rep}).$$

$$\mathcal{D}(T(\eta), \xi)$$

Embedded.

$$T: L^2(X, \Lambda^p T^*X) \longrightarrow W^{1,2}(X, \Lambda^p T^*X) \xhookrightarrow{i} L^2(X, \Lambda^p T^*M)$$

$T$  is a compact operator.

$T$  is self-adjoint i.e.  $\mathcal{D}(T(\eta), \xi) = \mathcal{D}(\eta, T(\xi))$

$$\overset{||}{(\eta, \xi)} = \overset{||}{(\xi, \eta)}$$

Step 3 By spectrum decomp. of  $T$

$$L^2(X, \wedge^p T^* M) := \bigoplus_{m=0}^{+\infty} E_m$$

$E_m :=$  eigenspace of  $T$  with eigenvalue  $\rho_m$

$$\rho_0 = 1 > \rho_1 > \dots > \rho \rightarrow 0$$

Harmonic form is  $E_0$

$$\Delta E_m = \left(\frac{1}{\rho_m} - 1\right) E_m$$

$$\Delta E_m = \frac{\rho_m}{1 - \rho_m} E_m$$

$$\underline{T = (Id + \Delta)^{-1}}$$

$$\mathcal{D}(\xi, \varphi) = (\xi, (Id + \Delta)T\eta)$$

$||$  By def of  $\varphi$

$$(\xi, \eta)$$

$$\Rightarrow (Id + \Delta)T = Id$$

$$T \sim (Id + \Delta)^{-1}$$

## § 5.2. Hodge Thm in the Hermitian case

$(X, J, g)$  Hermitian metric

In hodge Decom if  $\alpha$  is  $d$ -closed.

$$\alpha = \mathcal{H}(\alpha) + dd^*G(\alpha) + \underline{\quad}$$

Thm. Let  $\alpha \in A^{p,q}(X)$ , then  $\alpha = \mathcal{H}_{\Delta_{\bar{\partial}}}(\alpha) + \bar{\partial} \bar{\partial}^* G_{\bar{\partial}}(\alpha) + \bar{\partial}^* \bar{\partial} G_{\bar{\partial}}(\alpha)$

$$\text{Here } \Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$$

" $\bar{\partial}$ " replace the role of " $d$ "

$$(\bar{\partial}^* \alpha, \beta) = (\alpha, \bar{\partial} \beta)$$

$$\Delta_{\bar{\partial}} : A^{p,q} \longrightarrow A^{p,q}$$

$$\underline{\text{Ex}} = (\mathbb{C}^n, g \text{ flat metric } \omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i)$$

$$\text{Compute } \bar{\partial} \text{ and } \Delta_{\bar{\partial}} \quad (d, \Delta_d)$$

Generalization : Vector Bundle Valued form.

$(X, J, g)$  Hermitian mfd.

$(E, h_E) \longrightarrow X$  holo VB + Hermitian metric.

Inner product on  $A^{p,q}(E)$

$\bar{\partial}_E$  with inner product  $\leadsto \bar{\partial}_E^*$

$$\Delta_{\bar{\partial}_E} := \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E$$

Ex:  $\Delta_{\bar{\partial}_E}$  is an elliptic operator.

Thm Let  $\alpha \in A^{p,q}(E)$ , then  $\alpha = \mathcal{H}(\alpha) + \bar{\partial}_E \bar{\partial}_E^* G(\alpha) + \bar{\partial}_E^* \bar{\partial}_E G(\alpha)$ .

$$\bar{\partial}_E : A^{p,q}(E) \rightarrow A^{p,q+1}(E)$$

### § 5.3 Applications of Hodge Thm

Real case, • Poincaré dual ( $X^n$ )

$$H_{dR}^1(X, \mathbb{R}) \xrightarrow[\text{iso}]{*} H_{dR}^{n-1}(X, \mathbb{R}) \quad (\otimes).$$

$$*: A^1(X) \rightarrow A^{n-1}(X)$$



Ex:  $\otimes$  is well-defined

$$Q = *g_1 = *g_2 ?$$

Once  $\otimes$  is well-defined,

$$(**) = \text{Id} \cdot C(n, i) \neq 0.$$

$\leadsto *$  is invertible

• Kodaira - Serre duality

$$H_{\bar{\partial}}^{p,q}(E) \xrightarrow[\text{iso}]{\bar{*}_E} H_{\bar{\partial}}^{n-p, n-q}(E^*)$$

Anti- $\mathbb{C}$ -linear

Ex.  $\bar{*}_E$  is well-defined

## § 5.4 Hodge Decomposition on Kähler mfd

$$(X, J, g) \rightsquigarrow \omega(\cdot, \cdot) = g(J\cdot, \cdot) \quad \downarrow \text{Better!}$$

Kähler  $\Leftrightarrow d\omega = 0 \Leftrightarrow \exists$  holomorphic normal coordinate

Fix  $p$ ,  $\exists (Z_1, \dots, Z_n)$  near  $p$  s.t.

$$g_{i\bar{j}} = (\partial_i, \partial_{\bar{j}})_h = \delta_{ij} \text{ at } p.$$

$$dg = 0 \text{ at } p$$

Kähler mfd has more symmetry on  $H_{\bar{2}}^{p,q}$

Def: Lefschetz Operator  $L$

$$L\eta = \omega \wedge \eta$$

$$L: A^{p,q} \rightarrow A^{p+1,q+1} \quad (L \text{ is real operator})$$

Def:  $\Lambda = L^*$

$$\overline{(L\eta)} = L\bar{\eta}$$

Def:  $d^c = \sqrt{-1}(\bar{\partial} - \partial)$

Check:  $\frac{1}{2} dd^c = \sqrt{-1} \partial \bar{\partial}$

★ Kähler Identity

$$\bullet \Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$$

$$\bullet [\Delta_d, \Lambda] = 0, [\Delta_d, L] = 0.$$

Lemma On  $A^{p,q}(X)$  form, we have

$$[\Lambda, \partial] = \sqrt{-1} \bar{\partial}^*$$

Take conjugate.  $[\Lambda, \bar{\partial}] = -\sqrt{-1} \partial^*$

Sketch of proof:

Step 1:  $\mathbb{C}^n$ ,  $g$  flat,  $\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$

$$g = \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i)$$

In this case prove the Kähler identity

$$[\Lambda, \partial] = \sqrt{-1} \bar{\partial}^* \quad (\text{Ex})$$

Step 2. Using normal holo. coord. at given  $p$ .

$\omega(p) = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i(p)$ ,  $[\Lambda, \partial] = \sqrt{-1} \bar{\partial}^*$  can be reduced to step 1.

Essential Point:  $[\Lambda, \partial] = \sqrt{-1} \bar{\partial}^*$  involves at most

1st order derivatives at  $g_{i\bar{j}}$ , which is 0 under normal holomorphic coordinate at  $p$ .

Proof:  $\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$ .

$$\begin{aligned}\Delta_d &= dd^* + d^*d = (\bar{\partial} + \partial)(\bar{\partial}^* + \partial^*) + (\bar{\partial}^* + \partial^*)(\bar{\partial} + \partial) \\ &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} + \partial\partial^* + \partial^*\partial \\ &\quad + \bar{\partial}\partial^* + \partial^*\bar{\partial} + \underbrace{\partial\bar{\partial}^* + \bar{\partial}^*\partial}_0 \\ &= \Delta_{\bar{\partial}} + \Delta_{\partial} + ( )\end{aligned}$$

Claim 1:  $\bar{\partial}\partial^* + \partial^*\bar{\partial} = 0$ .

$$\Leftrightarrow \bar{\partial} \cancel{\sqrt{-1}[\Lambda, \bar{\partial}]} + \cancel{\sqrt{-1}[\Lambda, \bar{\partial}]} \bar{\partial} = 0$$

$$\Leftrightarrow \bar{\partial} \Lambda \bar{\partial} - \bar{\partial} \bar{\partial} \Lambda + \Lambda \bar{\partial} \bar{\partial} - \bar{\partial} \Lambda \bar{\partial} = 0.$$

Claim 2:  $\Delta_{\bar{\partial}} = \Delta_{\partial}$ .

$$\begin{aligned}\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^* &= -\sqrt{-1}[\Lambda, \bar{\partial}]\bar{\partial} - \sqrt{-1}\bar{\partial}[\Lambda, \bar{\partial}] \\ &= -\sqrt{-1}[\Lambda\bar{\partial}\bar{\partial} - \bar{\partial}\Lambda\bar{\partial} + \bar{\partial}\Lambda\bar{\partial} + \bar{\partial}\bar{\partial}\Lambda]\end{aligned}$$

Take Conjugate.  $\sqrt{-1}(\Lambda\bar{\partial}\bar{\partial} - \bar{\partial}\Lambda\bar{\partial} + \bar{\partial}\Lambda\bar{\partial} + \bar{\partial}\bar{\partial}\Lambda) = \Delta_{\partial}$

using  $\partial\bar{\partial} = -\bar{\partial}\partial \leadsto \Delta_{\bar{\partial}} = \Delta_{\partial}$ . □

Thm. Hodge Decomposition

$$H_{dR}^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H_{\bar{\partial}}^{p,q}(X) \quad \text{and}$$

$$H_{\bar{\partial}}^{p,q}(X) \subseteq H_{\bar{\partial}}^{q,p}(X)$$

proof: Suppose  $\alpha \in H_{dR}^i(X, \mathbb{C})$  is  $\Delta_d$  harmonic rep.

$$\leadsto \alpha = \sum_{p+q=i} \alpha^{p,q} \text{ as a form}$$

$$\leadsto \Delta_d \alpha = \sum_{p+q=i} \Delta_d (\alpha^{p,q}) = \sum_{p+q=i} 2 \Delta_{\bar{\partial}} (\alpha^{p,q})$$

Note  $\Delta_{\bar{\partial}} \alpha^{p,q}$  is still a  $(p,q)$  form  $\leadsto \Delta_{\bar{\partial}} \alpha^{p,q} = 0$

i.e.  $\alpha^{p,q}$  is  $\Delta_{\bar{\partial}}$  harmonic

$$\leadsto H_{dR}^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H_{\bar{\partial}}^{p,q}(X), \quad (\text{Hodge Decom.})$$

Ex. Hodge decom is independent of metric  $g$ .

Take  $\alpha \in H_{\bar{\partial}}^{p,q}(X)$  is  $\Delta_{\bar{\partial}}$  harmonic rep.

$$\bar{\alpha} \text{ is a } (q,p) \text{ form, } \Delta_{\bar{\partial}} \alpha = 0 \Leftrightarrow \Delta_{\partial} \bar{\alpha} = 0$$

$$\bar{\alpha} \text{ is also } \Delta_{\bar{\partial}} \text{ harmonic} \Leftrightarrow \Delta_{\bar{\partial}} \bar{\alpha} = 0$$

$$(?) \quad \overline{H_{\bar{\partial}}^{p,q}(X)} = \overline{H_{\bar{\partial}}^{q,p}(X)}$$

is

$$H^q(X, \Omega^p) \quad H^p(X, \Omega^q)$$



( $\partial\bar{\partial}$  lemma)

Let  $\omega$  is real (1,1) form and  $d$  exact, then

$$\exists \varphi \in C^\infty(X, \mathbb{R}) \text{ s.t. } \omega = \sqrt{-1} \partial\bar{\partial} \varphi = \frac{1}{2} d(d^c \varphi)$$

proof.  $\omega = d\gamma = \bar{\partial}\gamma^{1,0} + \partial\gamma^{0,1}$

$$\gamma^{1,0} = \mathcal{H}(\gamma^{1,0}) + \underbrace{\bar{\partial}\bar{\partial}^* G(\gamma^{1,0})}_{\bar{\partial}\text{-closed}} + \bar{\partial}^* \bar{\partial} G(\gamma^{1,0})$$

Hodge decomp.

$$\bar{\partial}\gamma^{1,0} = \underbrace{\bar{\partial}\bar{\partial}^* \bar{\partial} G(\gamma^{1,0})}_{\text{Im } \partial}$$

$$(\text{Im } \partial^* \oplus \mathcal{H}_\partial)^\perp = \text{Im } \partial$$

$$\bar{\partial}^* \bar{\partial} G(\gamma^{1,0}) \perp (\mathcal{H}_\partial = \mathcal{H}_\perp = \mathcal{H}_{\bar{\partial}})$$

$$\omega = \underbrace{\bar{\partial}\bar{\partial}^* \bar{\partial}(\gamma^{1,0})}_{\text{Im } \partial} \cdot \bar{\partial}^*(G_{\bar{\partial}} \gamma) \perp \text{Harmonic } (d, \bar{\partial}, \partial)$$

Im  $\partial$

$$\bar{\partial}^*(G_\partial \gamma) = \underline{\Delta_\partial G_\partial} \bar{\partial}^* G_\partial(\gamma)$$

$$\leadsto \bar{\partial}^*(G_\partial \gamma) = \partial\bar{\partial}^* G_\partial \bar{\partial} G_{\bar{\partial}}(\gamma)$$

$$+ \underbrace{\bar{\partial}\bar{\partial}^* G_{\bar{\partial}} \bar{\partial} G_{\bar{\partial}}(\gamma)}_{=0}$$

# Lecture 9.

( $\partial\bar{\partial}$  lemma)

If  $\omega$  is real (1,1) form and  $d$  exact, then

$$\exists f \in C^\infty(X, \mathbb{R}) \text{ s.t. } \omega = \sqrt{-1} \partial\bar{\partial} f$$

proof: By assumption 
$$\begin{cases} \omega = \bar{\partial} \gamma^{1,0} + \partial \gamma^{0,1} \\ \omega = d\gamma^{1,0} + d\gamma^{0,1} \end{cases} \quad \gamma = \gamma^{0,1} + \gamma^{1,0}$$

$$\underbrace{\omega = d\gamma^{1,0} + d\gamma^{0,1}} \rightsquigarrow \begin{cases} \partial \gamma^{1,0} = 0 \\ \bar{\partial} \gamma^{0,1} = 0 \end{cases}$$

$$\gamma^{1,0} = \overline{\gamma^{0,1}}$$

$$\omega = \bar{\partial} \gamma^{1,0} + \partial \gamma^{1,0} = 2\operatorname{Re}(\bar{\partial} \gamma^{1,0})$$

By Hodge Theory,

$$\Delta_{\partial} \gamma^{1,0} = \mathcal{H}(\gamma^{1,0}) + \partial\partial^* G(\gamma^{1,0}) + \partial^* \partial G(\gamma^{1,0})$$

$$\text{By } \partial \gamma^{1,0} = 0 \rightsquigarrow \partial^* \partial G(\gamma^{1,0}) = 0$$

$$\gamma^{1,0} = \mathcal{H}_{\partial}(\gamma^{1,0}) + \partial\partial^* G(\gamma^{1,0})$$

$$\bar{\partial} \gamma^{1,0} = \bar{\partial} \mathcal{H}_{\partial}(\gamma^{1,0}) + \bar{\partial} \partial\partial^* G(\gamma^{1,0})$$

$$\text{By Kähler, } \mathcal{H}_{\partial} = \mathcal{H}_{\bar{\partial}} = \mathcal{H}_d \rightsquigarrow \bar{\partial} \mathcal{H}_{\partial}(\gamma^{1,0}) = 0$$

$$\bar{\partial} \gamma^{1,0} = \bar{\partial} \partial[\partial^* G(\gamma^{1,0})] = \bar{\partial} \circ g$$

$$\omega = 2\operatorname{Re}(\bar{\partial} \gamma^{1,0}) = 2\operatorname{Re} \sqrt{-1} \bar{\partial} \partial (g \sqrt{-1}) = 2\sqrt{-1} \bar{\partial} \partial [\operatorname{Re}(\sqrt{-1}) g] \quad \square$$

Cor. If  $\omega_1, \omega_2$  Kähler metric in a  $H^{1,1}(X, \mathbb{R})$ , then  $\omega_1 - \omega_2 = \sqrt{-1} \partial\bar{\partial} \varphi$

## § 6. Kodaira vanishing and embedding $(X, \omega)$ Kähler

Recall:  $H^q(X, \Omega^p) = H_{\bar{\partial}}^{p,q}(X)$   $\xrightarrow{\text{Hodge } \Delta_{\bar{\partial}}}$   $\exists$  a unique harmonic rep. of  $H_{\bar{\partial}}^{p,q}(X)$ .

$H_{\bar{\partial}}^{p,q}(X, E)$ ,  $H_{\bar{\partial}}^{p,q}$  is hard to compute.

$\uparrow$   
 $H^q(X, \Omega^p(E))$  It's good to know when  $H_{\bar{\partial}}^{p,q}(X) = 0$ .

Bocher technique: Harmonic rep + Curvature assumption

Force the vanishing of harmonic rep to be 0.

(Roughly say).

Only discuss  $E$  is a line bundle.

Def:  $(L, h)$  be a Hermitian Line bundle,  $\Theta_h = -\partial\bar{\partial} \log h(e, e)$   $\swarrow$  curvature.

We say  $(L, h)$  is positive if  $\sqrt{-1} \Theta_h > 0$ .

( $L$  is positive if  $\exists$  metric  $\tilde{h}$  s.t.  $\sqrt{-1} \Theta_{\tilde{h}} > 0$ .)

(Kodaira-Nakano Vanishing)

Bundle 上的曲率假设  
而非 mfd 上作假设

$(L, h) \rightarrow X$  is positive i.e.  $\sqrt{-1} \Theta_h > 0$ .

$(X, \omega)$  is compact Kähler mfd, then

$$H_{\bar{\partial}}^{p,q}(X, L) = 0 \text{ if } p+q > n.$$

In particular,  $H^1(X, K_X \otimes L) = 0$ ,  $\gamma > 0$ .

$$K_X = \Lambda^n T^*X \quad \text{canonical line bundle} \\ = \Omega^n$$

Lemma: Let  $L: A^{p,q}(E) \xrightarrow{\Lambda \omega} A^{p+1,q+1}(E)$  be the Laplace operator.

$\Lambda = L^*$ ,  $h_E$  be a Hermitian metric on  $E \rightarrow$  Chern Connection

$$\text{Then } [\Lambda, \bar{\partial}] = -\sqrt{-1} (D^{1,0})^* \quad D = D^{1,0} + \bar{\partial}$$

$$[\Lambda, D^{1,0}] = \sqrt{-1} (\bar{\partial})^* \quad \text{If there is no } E, \\ \text{this is Kähler identity } D^{1,0} = \bar{\partial}$$

Ex. Prove the identity.

Normal holomorphic coordinate of  $\omega$ , And.  
normal holo. frame, of  $E$

Consequence: 2 Laplacian Comparison  $\Delta_{\bar{\partial}} - \Delta_{D^{1,0}} = [\sqrt{-1} (\bar{\partial} + \Lambda), \Lambda]$

Rmk: There's no curvature of  $(\omega)$  involved.

Proof of Consequence

$$\begin{aligned} -\sqrt{-1} \Delta_{D^{1,0}} &= -\sqrt{-1} (D^{1,0} (D^{1,0})^* + (D^{1,0})^* D^{1,0}) \\ &= D^{1,0} [\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}] D^{1,0} \end{aligned}$$

$$= \cancel{D^{1,0} \wedge \bar{\partial}} - \cancel{D^{1,0} \bar{\partial} \wedge} + \wedge \bar{\partial} D^{1,0} - \bar{\partial} \wedge D^{1,0}$$

$$\begin{aligned} \sqrt{-1} \Delta_{\bar{\partial}} &= \sqrt{-1} (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) \\ &= \bar{\partial} [\wedge, D^{1,0}] + [\wedge, D^{1,0}] \bar{\partial} \\ &= \bar{\partial} \wedge D^{1,0} - \bar{\partial} D^{1,0} \wedge + \wedge D^{1,0} \bar{\partial} - \cancel{D^{1,0} \wedge \bar{\partial}} \end{aligned}$$

$$\sqrt{-1} \Delta_{\bar{\partial}} - \sqrt{-1} \Delta_{D^{1,0}} = \wedge [\bar{\partial} D^{1,0} + D^{1,0} \bar{\partial}] - [D^{1,0} \bar{\partial} + \bar{\partial} D^{1,0}] \wedge$$

$$\begin{aligned} \left[ \bar{\partial} D^{1,0} + D^{1,0} \bar{\partial} = (\bar{\partial} + D^{1,0})^2 = D^2 = \mathbb{H} \right] \\ = [\wedge, \mathbb{H}] \end{aligned}$$

$$\leadsto \Delta_{\bar{\partial}} - \Delta_{D^{1,0}} = [\sqrt{-1} \mathbb{H}_h, \wedge] \quad \square$$

Proof of Kodaira Vanishing ( $\sqrt{-1} \mathbb{H}_h > 0$ ).

Let  $\omega = \sqrt{-1} \mathbb{H}_h$  be the Kähler metric, on  $X$ .

$$\Delta_{\bar{\partial}} \geq [\sqrt{-1} \mathbb{H}_h, \wedge] \Leftrightarrow (\Delta_{\bar{\partial}} s, s) \geq ([\sqrt{-1} \mathbb{H}_h, \wedge] s, s)$$

where  $s \in A^{p,q}(L)$

Suppose  $S$  is harmonic  $L$ -valued form  $\leadsto 0 \geq ([\sqrt{-1} \mathbb{H}_h, \wedge] s, s)$

The key is to compute  $[\sqrt{-1} \mathbb{H}_h, \wedge]$

Ex. if  $L$  is Line bundle,  $\omega = \sqrt{-1} \mathbb{H}_h$ , then  $[\sqrt{-1} \mathbb{H}_h, \wedge] = (p+q-n) \text{Id}$ .

Kodaira embedding  $(X, \omega)$  Kähler  $(L, h)$  positive, then

$\exists m$  large,  $|L^m|: X \rightarrow \mathbb{CP}^N$  is embedding

$$N+1 = \dim_{\mathbb{C}} H^0(X, L).$$

Suppose  $\{S_0, \dots, S_N\}$  is a basis of  $H^0(X, L)$

We can define a meromorphic  $|L^m|: X \rightarrow \mathbb{CP}^N$

$e$  is a local frame of  $L$ .  $x \mapsto \left[ \frac{S_0(x)}{e(x)}, \dots, \frac{S_N(x)}{e(x)} \right]$

Note:  $|L^m|$  is not well-defined at  $\{S_0(x) = S_1(x) = \dots = S_N(x) = 0\}$

$|L^m|: X \rightarrow \mathbb{CP}^N$  is embedding

1:  $|L^m|$  is everywhere defined i.e.

$$H^0(X, L^m) \rightarrow H^0(X, L^m \otimes \mathcal{O}_X / \mathfrak{m}_p) =: L_p \text{ surjective}$$

$\mathfrak{m}_p$  is sheaf of holo. function vanish at  $p$

(stronger than 1)  $S \rightarrow S(p)$

↓ 2:  $|L^m|(p) \neq |L^m|(q)$  for any  $(p, q)$

$$H^0(X, L^m) \rightarrow L_p \oplus L_q \text{ is surjective}$$

$$\parallel$$
$$H^0(X, L^m \otimes \mathcal{O}_X / \mathfrak{m}_{p,q})$$

$(\mathfrak{m}_{p,q}$  sheaf of holo. func.

vanish at  $p, q$ )

3.  $|L^m|$  sep tangents  $T_p X$  at  $p$

$$H^0(X, L^m) \rightarrow H^0(X, L^m \otimes \mathcal{O}_X / \mathfrak{m}_p^2) \text{ is surjective}$$

Sketch the proof 1, (厚天大 sheaf?)

$$H^0(X, L^m) \rightarrow H^0(X, L^m \otimes \mathcal{O}_X / \mathfrak{m}_p) \text{ is surjective.}$$

$$\text{Consider } 0 \rightarrow \mathfrak{m}_p \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X / \mathfrak{m}_p \rightarrow 0.$$

$$0 \rightarrow L^m \otimes \mathfrak{m}_p \rightarrow L^m \rightarrow L^m \otimes \mathcal{O}_X / \mathfrak{m}_p \rightarrow 0.$$

$$H^0(X, L^m) \rightarrow H^0(X, L^m \otimes \mathcal{O}_X / \mathfrak{m}_p) \rightarrow H^1(X, L^m \otimes \mathfrak{m}_p) \rightarrow \dots$$

It suffices to show  $H^1(X, L^m \otimes \mathfrak{m}_p) = 0$ .

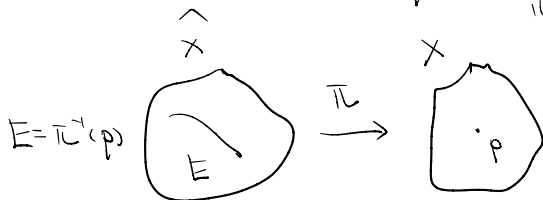
Recall:  $H^1(X, K_X \otimes L) = 0$ .

Problem 1:  $\mathfrak{m}_p$  is not a line bundle (locally free sheaf).

Kodaira: Blow up.  $\exists \hat{X} \xrightarrow{\pi} X$  holomap. s.t.

$\pi$  is iso at  $(\hat{X} \setminus \pi^{-1}(p))$

and  $\pi^{-1}(p)$  is a divisor.



Fact:  $\pi_* \mathcal{O}_{\hat{X}}(-E) = \mathfrak{m}_p$ ,  $R^i \pi_* \mathcal{O}_{\hat{X}}(-E) = 0$ ,  $i \geq 1$ .

Derived  
push-forward  $\rightarrow R\pi_* \mathcal{O}_{\hat{X}}(-E) = \mathcal{m}_p$

$$\leadsto \frac{H^1(\hat{X}, \pi^* L \otimes \mathcal{O}(-E)) \rightarrow H^1(X, L \otimes \mathcal{m}_p)}{\uparrow}$$

Trouble 2:  $\pi^* L$  is not positive

$$(\pi^* L, \pi^* h) \quad C_1(\pi^* L) = [\pi^* \sqrt{-1} \partial \bar{\partial} \log h]$$

But  $\pi^* \sqrt{-1} \partial \bar{\partial} \log h$  is not positive  $\in H^{1,1}(\hat{X}, \mathbb{R})$

pull back  
 $\downarrow$

at  $E$

( Somehow make it worse! )

Kodaira Lemma:  $\textcircled{E}$  (Line bundle) admits a smooth metric  $h_E$  s.t.  
divisor.

$$\sqrt{-1} \Theta_{h_E} = -\sqrt{-1} \partial \bar{\partial} \log h_E < 0 \text{ in a neighbour of } E.$$

i.e.  $-E \leadsto$  line bundle is positive neighbor of  $E$



$E_X$ : It's related to the construction of Blow up.



$\pi^*(mL) \otimes (-E)$  is positive for  $m$  sufficiently large.

why?

$$H^1(\hat{X}, \pi^* mL \otimes (-E)) = H^1(\hat{X}, \underbrace{\pi^* mL \otimes (-E)}_{\uparrow} \otimes \underbrace{K_X^{-1}}_{\uparrow})$$

If  $\pi^* mL \otimes (-E) \otimes K_X^{-1}$  is positive, done!

Adjunction.  $K_{\hat{X}} = \pi^* K_X + (n-1)E \leadsto K_X^{-1} = \pi^* K_X^{-1} - (n-1)E$

$$\leadsto \pi^* mL \otimes (-E) \otimes K_X^{-1} = \underbrace{\pi^*(mL \otimes K_X^{-1})}_{\substack{\text{mx curv.} \\ m \text{ large enough}}} \otimes \underbrace{\mathcal{O}(-nE)}_{\substack{\text{curv. fixed} \\ \text{it can be positive.}}} > 0$$

Another method:

Direct proof  $H^0(X, mL) \rightarrow H^0(X, mL \otimes \mathcal{O}_X / \mathfrak{m}_p) \rightarrow 0$ .

Suppose: a local frame of  $e$  near  $p$  of  $L^m$  ( $p \in U$ )

Let  $\eta$  be a smooth function supported in  $U$ .

$\eta e$  is a smooth section of  $L^m$  and  $\eta e(p) \neq 0$ .

(0.1) -  $L^m$  valued

Idea: Find a solution  $u \in C^\infty(X, L)$  s.t.  $\bar{\partial} u = \bar{\partial}(\eta e)$

But not clear if

$\leadsto u - \eta e$  is holo.

$(u - \eta e)(p) \neq 0$  or not.

$\int$  estimate of 0. For example  $u|_D = 0$ . perfect!  
 $\|u\|_2 \leq \frac{1}{c} \|\bar{\partial} g\|_2$

Sol. is not enough. also need some good estimate

Thm  $(X, \omega)$  cpt Kähler.  $(L, h)$  is Hermitian  $\varphi \in L^1(X)$ ,

s.t.  $\sqrt{-1} \Theta_h + i \partial \bar{\partial} \varphi > c \omega$  where  $c > 0$  constant

Then  $\forall g \in L^2(X, \Lambda^{n,1} T^* X \otimes L)$ ,  $\exists u \in L^2(X, \Lambda^n T^* X \otimes L)$   
 and  $\bar{\partial} g = 0$

s.t.  $\bar{\partial} u = \bar{\partial} g$  and Moreover

$$\int |u|_{h,g}^2 e^{-\varphi} dV_g \leq \frac{1}{c} \int |g|_{h,g}^2 e^{-\varphi} dV_g$$

Key:  $\varphi$  is not  $C^2$

Rmk:  $\sqrt{-1} \Theta_h + i \partial \bar{\partial} \varphi > c \omega$  holds in weak sense.

Rmk: Define  $\tilde{h} = h e^{-\varphi}$  as singular metric on  $L$ .

$$\sqrt{-1} \Theta_{\tilde{h}} = \sqrt{-1} \Theta_h + i \partial \bar{\partial} \varphi > c \omega$$

$$\text{Take } F = L \otimes K_X \xrightarrow{\sim} \Lambda^{n,0} T^* X \otimes L = F$$

$$\forall g \in L^2(X, \Lambda^{0,1} T^* X \otimes F) \quad \bar{\partial} g, \exists u$$

$$\bar{\partial} u = g, \quad \|u\|_L^2 \leq \frac{1}{c} \|g\|_L^2$$

The Key Curvature Assumption for  $F$  is

$$\sqrt{F} \otimes_F + \frac{\text{Ric}(w)}{\sqrt{F} \otimes \otimes \varphi} \gamma \subset w,$$

$\swarrow$   $\searrow$  fixed  $\swarrow$   
 $F$ 's power  
 large enough.

- Take  $p=0$ .

Take  $\varphi = \log |Z|^N$ ,  $N$  large

$$e^{-\varphi} = e^{-\log |z|^N} = \frac{1}{|z|^N}$$

$$\int |u|^2 e^{-\phi} < +\infty \quad \longrightarrow \quad u(p) = 0.$$

Sketch of pf

$$\bar{\partial} u = g \quad ? \Leftrightarrow \quad (\bar{\partial} u, \beta) = (g, \beta)$$

$$\Rightarrow (u, \vec{\partial}^* \beta) = (g, \beta)$$

Define Linear Bounded functional  $l_g: \text{Im } \partial^* \rightarrow \mathbb{C}$

?  $\bar{\partial}^* \beta \rightarrow (g, \beta)$  (well-defined)

Extend  $L_g$  to  $L^2 \rightarrow \mathbb{C}$

Riesz  $\leadsto u = T(g)$  s.t.  $(u, \bar{\partial}^* \beta) = (g, \beta)$

Bounded means

$$(g, \beta) \leq \|g\| \|\beta\| \leq C_g \|\partial^* g\|$$

positive  $\rightarrow \| \beta \|^2 \leq (\| \bar{\alpha} \beta \|^2 + \| \alpha \beta \|^2) \frac{1}{c}$ .

# Lecture 10. Deformation of Complex Str.

- Given a Complex mfd  $(X, J)$ ,  $\exists$  other integrable Complex str.  $\tilde{J}$  on  $X$  s.t.  $(X, J)$  and  $(X, \tilde{J})$  are not biholomorphic.
- Can we find a parameter space for all different  $J$  on  $X$ .  
What's property of parameter <sup>(Moduli)</sup> space.
- Do we have universal or (semi-universal) family  $\mathcal{X} \rightarrow S$

Def: Deformation of  $X$

We say  $\mathcal{X} \xrightarrow{\pi} \underset{(t)}{\Delta}$  holomorphic submersion between complex manifold s.t.  $\pi^{-1}(0) = X$ ,  $X_t$  ( $t \neq 0$ )

Rmk:  $\mathcal{X} \overset{\text{dff}}{\simeq} X \times \Delta$

Suppose  $\mathcal{X} \xrightarrow{\pi} (\Delta, t)$ ,  $\Delta$  is a disk is a deformation of  $X$ .

Suppose we have open cover  $\underline{U}$  of  $\mathcal{X}$ ,

$\underline{U} = \{U_\alpha\}_{\alpha \in \mathbb{N}}$  on each  $U_\alpha = (z_1^\alpha, z_2^\alpha, \dots, z_n^\alpha, t)$  are holomorphic coordinate of  $\mathcal{X}$ .

From 1-parameter Def we can have Čech 1-cocycle on  $X$

on  $U_\alpha \cap U_\beta \cap X$ ,  $\theta_{\alpha\beta} := \frac{\sum_{i=1}^n \frac{\partial f_{\alpha\beta}^i}{\partial t} \Big|_{t=0}}{\text{function}} \boxed{\frac{\partial}{\partial z_i^\alpha}}_{\text{vector}}$ , where

$z^\alpha = f_{\alpha\beta}(z^\beta, t)$

Trivial case  $\mathcal{X} = X \times \Delta$ ,  $f_{\alpha\beta}$  could be independent of  $t \leadsto \theta_{\alpha\beta} = 0$

Ex:  $\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} = 0$

In fact, if on  $U_\alpha$  define  $\eta_\alpha := \sum_{i=1}^n \frac{\partial z_i}{\partial t} \frac{\partial}{\partial z_i}$

on  $U_\alpha \cap U_\beta$ ,  $\theta_{\alpha\beta} = \eta_\alpha \cdot \eta_\beta$ ,  $\bar{\partial} \eta_\alpha = \bar{\partial} \eta_\beta$

$\leadsto \{\theta_{\alpha\beta}\} \leadsto H^1(X, TX) \xrightarrow{\bar{\partial}} \bar{\partial}\text{-closed}$   
 (TX valued 0,1 form)

somehow 反映了 Complex str. deformation.

Def Kodaira - Spencer map:

无形变, trivial.  $\rightarrow 0$ ,  
 injective  $\Rightarrow$  形变 everywhere.

Suppose  $\mathcal{X} \rightarrow S$  a holo. submersion

$KS: \underbrace{T_0 S}_{\Delta \text{ disk}} \rightarrow H^1(X, TX)$

Another point of view: A deformation of  $X$  is a family of integrable complex str. on a fixed mnd.  $X$ .

Almost complex str.  $J: T^{\mathbb{R}} X \rightarrow T^{\mathbb{R}} X$ ,  $J^2 = -1$ .

$\Leftrightarrow T^{\mathbb{C}} X = \underbrace{T^{1,0}}_{\sqrt{-1}} \oplus \underbrace{T^{0,1}}_{\sqrt{-1}} \quad \text{or} \quad T^{\mathbb{C}} X = \underbrace{\Omega^{1,0}}_{\sqrt{-1}} \oplus \underbrace{\Omega^{0,1}}_{-\sqrt{-1}}$

Integrable (Nirenberg - Nijlander)

$$[T^{1,0}, T^{1,0}] = T^{1,0} \text{ or } \bar{\partial}_j \bar{\partial}_j = 0$$

Suppose  $X'$  is new complex str.

$$T^{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} = T^{\mathbb{C}}X, T^{*\mathbb{C}}X \text{ only depends real str.}$$

$$T^{*\mathbb{C}}X = \Omega_{X'}^{1,0} \oplus \Omega_X^{0,1}$$

$$T^{*\mathbb{C}}X = \Omega_{X'}^{1,0} \oplus \Omega_{X'}^{0,1}$$

Def: Let  $\pi^{1,0}, \pi^{0,1}$  be the projection of  $T^{*\mathbb{C}}X$  to

$$\Omega_X^{1,0} \text{ and } \Omega_X^{0,1}$$

$$\pi^{1,0}|_{\Omega_{X'}^{1,0}} : \Omega_{X'}^{1,0} \xrightarrow{\text{iso}} \Omega_X^{1,0}$$

$$\text{Hence } \left(\pi^{1,0}|_{\Omega_{X'}^{1,0}}\right)^{-1} : \Omega_X^{1,0} \rightarrow \Omega_{X'}^{1,0}$$

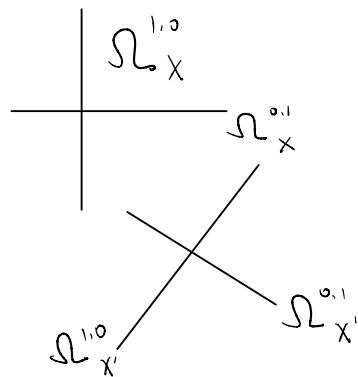
$$S = \pi^{0,1} \circ \left(\pi^{1,0}|_{\Omega_{X'}^{1,0}}\right)^{-1} : \Omega_X^{1,0} \rightarrow \Omega_X^{0,1}$$

$$\text{If } X' = X, S = 0.$$

$S$  is a  $C^\infty$  section of  $TX$  valued

$(0,1)$  form  $X$

Conversely, If give a  $TX$ -valued



$(0,1)$  form  $S$  on  $X$ .  $S$  determines an almost complex str.

In the following way 复化切空间的分解.

define  $\Omega_{X'}^{1,0} = (a, -S(a))$ ,  $a \in \Omega_X^{1,0}$

and define  $\Omega_{X'}^{0,1}$  to be the conjugate of  $\Omega_{X'}^{1,0}$ .

$S \rightsquigarrow$  almost complex str.  $\rightsquigarrow$  What's the integrability condition for  $S$

Locally  $S = S_{ij} \frac{\partial}{\partial z_j} \otimes d\bar{z}_i$  (where  $Z$  is holomorphic coordinate of  $X$ )

$$\forall k \quad \theta_k = dZ_k - S_{ij} \frac{\partial}{\partial z_j} (dZ_k) \otimes d\bar{z}_i = dZ_k - \underbrace{\sum_{i=1}^n S_{ik} d\bar{z}_i}_{\text{new}(1,0) \text{ form.}}$$

Apply Nirenberg - Newlander

$J$  is integrable  $\Leftrightarrow d\theta_k$  is no  $(0,2)$  part w.r.t.  
reduce to new complex str.  
(Ex)

Thm (Cartan-Maurer)  $\overset{S \text{ (above)}}{=}$

If a  $C^\infty$  section  $\phi$  of  $TX$  valued  $(0,1)$  form gives  
a integrable complex str. iff  $\bar{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0$ .

Thm (Kodaira - Spencer) If  $H^2(X, TX) = 0$ , then

$$X \longrightarrow B_{\mathbb{C}}^m, \quad m = \dim_{\mathbb{C}} H^1(X, TX)$$

1)  $X_0 = X$

2)  $KS: T_0 B_{\mathbb{C}}^m \longrightarrow H^1(X, TX)$  is

3)  $\exists \tilde{X} \longrightarrow T$  is another <sup>orientation</sup> def of  $X$ , then  $\exists \varphi: T \longrightarrow B_{\mathbb{C}}^m$ ,

$\varphi(0) = 0$ . s.t.  $\tilde{X} = \varphi^* X$

$$\begin{array}{ccc} \varphi^* X = \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & B_{\mathbb{C}}^m \end{array}$$

Sketch of proof:

Suppose  $\beta_1 \dots \beta_m$  is harmonic rep of  $H^1(X, TX)$

$$H_{\bar{\partial}}^{0,1}(X, TX)$$

1) construct  $TX$  valued  $(0,1)$  form  $\phi(t_1, \dots, t_m) =: \phi(t)$

$$\phi(t) = \sum_{\substack{K_1 + \dots + K_m \geq 1 \\ K_i = 0}} \phi_{K_1 \dots K_m} t_1^{K_1} t_2^{K_2} \dots t_m^{K_m} \quad \text{where } \phi_{K_1 \dots K_m} \text{ are } TX$$

valued  $(0,1)$  form.

$$\left\{ \begin{array}{l} \phi(0) = 0 \\ \bar{\partial} \phi + \frac{1}{2} [\phi, \phi] = 0, \end{array} \right.$$

$$\frac{\partial \phi}{\partial t_i} \Big|_{t_1 = \dots = t_m = 0} = \beta_i$$

How to solve?



2)  $\phi(t) \leadsto$  Integrable complex str.  
✓ Kodaira's student

(Kuranishi's proof) 全微分式

Consider Equation:

$$\phi(t) = \phi_1(t) + \frac{1}{2} \bar{\partial}^* G [\phi(t), \phi(t)]$$

$\phi_1(t)$  is the given first order term.

$\phi(t) = \phi_1(t) + \frac{1}{2} \bar{\partial}^* G [\phi(t), \phi(t)]$  has a formal solution.

recursively given  $\phi_2 \leadsto \phi_3$

Elliptic Estimate gives a convergent solution

$$\phi(t) = \sum_{k_1 \dots k_m} \phi_{k_1 \dots k_m} t_1^{k_1} \dots t_m^{k_m}$$

Lemma (Kuranishi) If  $\mathcal{H}[\phi(t), \phi(t)] = 0$ , then  $\bar{\partial} \phi + \frac{1}{2} [\phi, \phi] = 0$ .

↓  
TX valued (0,2) form.

Cor. In particular, if  $H^2(X, TX) \simeq H_{\bar{\partial}}^{0,2}(X, TX) = 0$ .

$\mathcal{H}([\phi(t), \phi(t)]) = 0$  always true.

proof of Lemma

$$\left\{ \begin{array}{l} \leftarrow (*) \phi(t) = \phi_1(t) + \frac{1}{2} \bar{\partial}^* G [\phi(t), \phi(t)] \\ \bar{\partial} \phi + \frac{1}{2} [\phi, \phi] = 0 \end{array} \right\} \Rightarrow \mathcal{H}([\phi, \phi]) = 0.$$

$$\bar{\partial}(\ast) \quad \bar{\partial}\phi = \frac{1}{2} \bar{\partial} \bar{\partial}^* G[\phi(t), \phi(t)] \sim \underbrace{[\phi, \phi]}_{\text{By Hodge theory}} = \bar{\partial} \bar{\partial}^* G[\phi, \phi]$$

$$\frac{1}{2} \|\phi, \phi\|$$

$$\mathcal{H}[\phi, \phi] = 0, \text{ and}$$

$$\bar{\partial}^* \bar{\partial} G[\phi, \phi] = 0$$

$$\rightarrow \text{Suppose } \phi(t) = \phi_1(t) + \frac{1}{2} \bar{\partial}^* G[\phi, \phi], \quad \mathcal{H}[\phi, \phi] = 0.$$

$$\leadsto \bar{\partial}\phi - \frac{1}{2} [\phi, \phi] = 0.$$

( $\|\phi\|$  is small, if  $t$  small)  $\stackrel{=}{=} \psi(t)$

$$\text{Goal: } \psi(t) = 0$$

$$\bar{\partial}\phi_1 = 0.$$

$$\leadsto \bar{\partial}\phi = \frac{1}{2} \bar{\partial} \bar{\partial}^* G[\phi, \phi]$$

$$= \frac{1}{2} ([\phi, \phi] - \bar{\partial}^* \bar{\partial} G[\phi, \phi])$$

$$= -\psi + \bar{\partial}\phi - \frac{1}{2} \bar{\partial}^* \bar{\partial} G[\phi, \phi]$$

$$\leadsto \psi = \frac{1}{2} \bar{\partial}^* G[\bar{\partial}\phi, \phi] \times 2$$

$$= \bar{\partial}^* G[\psi + \frac{1}{2} [\phi, \phi], \phi]$$

$$= \bar{\partial}^* G[\psi, \phi]$$

$$\text{i.e. } \psi = \bar{\partial}^* G[\psi, \phi] \leadsto \psi = 0.$$

Since small deformation,  $\phi$  has very small norm

$$|\psi|_{C^{k,\alpha}} \leq C |\psi|_{C^{k,\alpha}} |\phi|_{C^{k,\alpha}}$$

$$\Rightarrow | \mathcal{H} | = 0$$

□

Thm (Kuranishi) When  $H^2(X, TX) \neq 0$ .

$\exists \mathcal{X} \rightarrow S$ .  $S$  singular  $\subseteq H^1(X, TX)$ .

s.t.  $\tilde{\mathcal{X}} \rightarrow T$  is another def of  $X$ . then  $\exists \varphi: T \rightarrow S$

$$\tilde{\mathcal{X}} = \varphi^* \mathcal{X}$$

(Bogomolov-Tian-Todorov)  $X$  is Calabi-Yau.  $K_X$  is trivial line bundle, then the deformation of  $X$  is unobstructed.

$$S^1 = H^1(X, TX).$$

In other words.  $\mathcal{H}([\phi(t), \phi(t)]) = 0$  always true!

Rmk:  $H^2(X, TX)$  is not necessarily zero, (Mogib!)