

Riemannian Geometry

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Miao L-Chieh. 

Lecture 1, §1. Smooth Manifold

Def 1.1 - Suppose M is Hausdorff, T_2 Topological space
with $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$

(1) $\{U_\alpha\}_{\alpha \in A}$ is a cover of M

(2) $\varphi_\alpha: U_\alpha \rightarrow \mathbb{R}^n$ homeomorphism

(3) For any $U_\alpha \cap U_\beta \neq \emptyset$, $\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$
is C^k .

then M is a C^k mfd.

$k=0$, topological mfd

$k=\infty$ smooth mfd.

$k=\omega$ analytic mfd.

Remark: (1) (Whitney) C^1 -mfd admits C^∞ -structure

(2) (Freedman, Kervaire) Topological mfd may not
admit C^∞ -structure.

(3) (Milnor) S^7 admits 28 C^∞ -structure.

(4) (Smooth Poincaré Conjecture)

S^4 admits unique C^∞ -structure?
or infinite

$T_p M$?

is called germ of func at p

Def 1.3 let $\Sigma(p) = \{ \text{Equivalent class of smooth func defined on } U, p \in U, U \text{ is open} \}$.

" $\sim$ ": for any $f: U \rightarrow \mathbb{R}, g: V \rightarrow \mathbb{R}, p \in U \cap V$.

$$f \sim g \Leftrightarrow \exists W \subset U \cap V, f|_W = g|_W.$$

Def 1.4 let $\mathcal{U}_p = \{ \text{germs of } U: \mathbb{R} \rightarrow M^n, u(0)=p \}$.

$$T_p M := \mathcal{U}_p / \sim$$

$$" \sim ": u \sim v \Leftrightarrow \left. \frac{d}{dt} (f \circ u) \right|_{t=0} = \left. \frac{d}{dt} (f \circ v) \right|_{t=0}.$$

$$[f] \in \Sigma(p), u, v \in \mathcal{U}_p$$

$$(u_\alpha, \varphi_\alpha, \{x^i\}_{i=1}^n)$$

$$\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \leftarrow \text{basis.}$$

$$\frac{\partial}{\partial x^i} f = \frac{\partial}{\partial x^i} (f \circ \varphi_\alpha) \Big|_p$$

$$\bigcup_{p \in U} T_p M \approx U_\alpha \times \mathbb{R}^n$$

$$(u_\beta, \varphi_\beta, \{y^j\}_{j=1}^n)$$

$$\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \leftarrow \mathbb{R}^n \text{ vector field.}$$

$$\frac{\partial}{\partial x^i} = \left[\frac{\partial y^j}{\partial x^i} \right] \frac{\partial}{\partial y^j} \left. \vphantom{\frac{\partial}{\partial x^i}} \right\} \text{span}$$

$$\bigcup_{p \in U} T_p M \approx U_\beta \times \mathbb{R}^n$$

粘起来得到 TM .

Def 1.5 (Vector bundle)

(E, B, π) is called a vector bundle

E
 $\downarrow \pi$ (1) E is a topological manifold.

B (2) a C^∞ map, $\pi: E \rightarrow B$

(3) $\forall b \in B$, $\exists$ an open set $U \subset B$ and h s.t.

① $h: U \times \mathbb{R}^n \rightarrow \pi^{-1}(U)$ is homeomorphism

② $h(b, \cdot): \mathbb{R}^n \rightarrow \pi^{-1}(b)$ is an isomorphism between

vector space.

(h is called local trivialization)

E is total space, B is the base space

π is projection map.

Def 1.6 (Connection) $\pi: E \rightarrow M$ vector bundle

$\forall X, Y \in T(TM)$ and $s_1, s_2 \in T(E)$, $f, g \in C^\infty(M)$

∇ is called a connection if $\nabla_X s \in T(E)$ and

(1) $\nabla_{Xf} s = f \nabla_X s$, $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$ (f -linear)

(2) $\nabla_X(fs) = X(f)s + f \nabla_X s$, $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$

(标架)

不同点处切空间的对应

Def section

(E, B, π) is a vector bundle, s is a section of E

$$\pi \circ s = \text{id}_B$$

$\Gamma(E)$ is the set of smooth sections



§2. Riemannian metric.

Def 2.1 M^n is a smooth manifold, g is a Riemannian metric

on M if $g: T(TM) \times T(TM) \rightarrow \mathbb{R}$ is an inner product

$$\langle u, v \rangle := g(u, v), \quad u, v \in T_p M$$

$$(\textcircled{1} \langle u, u \rangle_g \geq 0 \quad "=" \text{ holds iff } u=0$$

$$\textcircled{2} \langle u, v \rangle = \langle v, u \rangle$$

$$\textcircled{3} \langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle)$$

If $(u_\alpha, \varphi_\alpha, \{x^i\}_{i=1}^n)$ is a local chart.

$\{ \frac{\partial}{\partial x^i} \}$ is a basis of $T u_\alpha$.

$$g_{ij}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$$

g_{ij} is a positive definite, symmetric matrix

$$u \in T_p M, \quad \|u\| = \sqrt{\langle u, u \rangle}$$

Example: $(\mathbb{R}^n, g_E)$ $g_E = \delta_{ij} dx^i dx^j, \quad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$
 $= \sum_i (dx^i)^2$

Fact: Any smooth manifold admits a smooth metric

$$1 = \sum_{\alpha \in I} \varphi_\alpha$$

$$(u_\alpha, \varphi_\alpha, g_\alpha) \quad g = \sum_{\alpha \in I} \varphi_\alpha g_\alpha$$

Fact: If $\Phi: M^n \rightarrow N^m$ is an immersion, (N^m, g_N) is Riem. mfd. Then $\Phi^* g_N$ is a Riem. mfd on M^n

$$(\Phi^* g_N)(u, v) = g_N(\Phi_* u, \Phi_* v) \quad u, v \in T(M)$$

e.g. $S^n \subset \mathbb{R}^{n+1}$, $S^n = \{(x^1, x^2, \dots, x^{n+1}) \mid (x^1)^2 + \dots + (x^{n+1})^2 = 1\}$
 $(\mathbb{R}^{n+1}, g_E)$

Def 2.2 $\Phi: (M^n, g_M) \rightarrow (N^m, g_N)$ is called isometric immersion if $\Phi^*(g_N) = g_M$.

If $f \in C^\infty(M_1)$, (M_1, g_1) , (M_2, g_2) are two Riem. mfd's.

We can define $M_1 \times_f M_2 = (M_1 \times M_2, g_1 \oplus f g_2)$

It is called warped product. $p = (x_1, x_2) \in M_1 \times M_2$

$$f(p) = f(x_1).$$

Let $p = (x_1, x_2) \in M_1 \times M_2$, $u_1, u_2 \in T_{x_1} M_1$

$$v_1, v_2 \in T_{x_2} M_2$$

$$g_1 \oplus f g_2 \Big|_{(x_1, x_2)} ((u_1, v_1), (u_2, v_2)) = (g_1) \Big|_{x_1} (u_1, u_2) + f(x_1) \Big|_{x_2} (v_1, v_2)$$

$$\mathbb{R}^+ \times_f S^{n-1}$$

构造 new metric. conv. $\rightarrow$, 0, 1. May not complete

$f \equiv 1$. $(M_1 \times M_2, g_1 \oplus g_2)$ is called the product of

(M_1, g_1) and (M_2, g_2) .

常曲率空间分类

(a: M has metric. N ?)

A basic group act on M + some properties

Def 2.3 (Riem. covering) $n=m$ $\pi: M^n \rightarrow N^n$ is a covering

map. Then for any g_N on N , $\pi^* g_N$ is a Riem. metric.

$(M^n, \pi^* g_N)$ is called a Riem. covering map.



Def 2.4 (Riem. submersion) $\pi: M^n \rightarrow N^m$ a submersion ($n > m$)

$\pi^*(g_N)$ is not a metric on M^n

$V_p = \ker(\pi_*)|_p \subset T_p M$ is called a vertical space at p of π .

(M, g_M) is a Riem. mfd.

$$V_p \oplus H_p = T_p M \quad \text{w.r.t. } g_M.$$

H_p is called the Horizontal at p of π .

$\pi: (M, g_M) \rightarrow (N, g_N)$ is called a Riem. submersion

$$\text{if } g_M|_{T_p} = \pi^* g_N|_{T_p}$$

• Connection of (M, g)

Thm 2.5 let (M^n, g) be a Riem. manifold. Then $\exists$ a unique connection ∇ on M satisfy $\forall x, Y, Z \in T(TM)$

$$(1) (\nabla g = 0) \quad x \langle Y, Z \rangle = \langle \nabla_x Y, Z \rangle + \langle Y, \nabla_x Z \rangle$$

$$(2) [\text{Torsion free}] \quad \nabla_x Y - \nabla_Y X - [X, Y] = 0.$$

Proof: $x \langle Y, Z \rangle = \langle \nabla_x Y, Z \rangle + \langle Y, \nabla_x Z \rangle \quad (1)$

$$Y \langle Z, X \rangle = \langle \nabla_Y Z, X \rangle + \langle Z, \nabla_Y X \rangle \quad (2)$$

$$Z \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle \quad (3)$$

$(1) + (2) - (3)$, we have. (use (2)).

$$\begin{aligned} \langle \nabla_x Y, Z \rangle &= \frac{1}{2} (x \langle Y, Z \rangle + Y \langle Z, X \rangle - Z \langle X, Y \rangle \\ &\quad + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle \\ &\quad + \langle [Z, X], Y \rangle). \end{aligned} \quad (*)$$

$\exists$ and uniqueness follow from $(*)$

□

Def The connection " ∇ " in Thm 2.5 is called,
 Levi-Civita Connection.

In local chart (loc. $\varphi, \{x^i\}_{i=1}^n$).

$$\forall X \in \Gamma(TM) \quad X = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} = \sum_{i=1}^n x^i \partial_i$$

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

Γ_{ij}^k is called the Christoffel symbol.

Remark (1) Γ_{ij}^k is not the entries of a tensor

$\tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k$ is not 0.

$$(2) \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

$$\text{For } X = \sum x^i \partial_i, \quad Y = \sum Y^j \partial_j$$

$$\nabla_X Y = \sum_k \left(X(Y^k) + x^i Y^j \Gamma_{ij}^k \right) \partial_k.$$

$\Gamma \in \Gamma(TM)$ not pointwise defined.

$$\begin{array}{c|c} Y(p) = \bar{Y}(p) & X(p) = \bar{X}(p) \\ \nabla_X Y(p) \neq \nabla_X \bar{Y}(p) & \nabla_X Y|_p = \nabla_X \bar{Y}|_p \quad \forall Y \in \Gamma(TM) \end{array}$$

这个位置可任意定义

$$\star \quad \forall v \in T_p M. \quad \nabla_v Y = \nabla_X Y|_p \quad X \in \Gamma(TM) \quad X(p) = v$$

Furthermore,

For any smooth curve $\gamma: I \rightarrow M$,

$$\dot{\gamma}(t) := \gamma_* \left(\frac{d}{dt} \right)$$

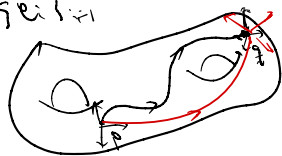
$\nabla_{\dot{\gamma}(t)} Y$ is well-defined for any $Y \in T(TM)$ ($\because *$).

Def 2.7 $\gamma: I \rightarrow M$ is smooth curve. A vector field.

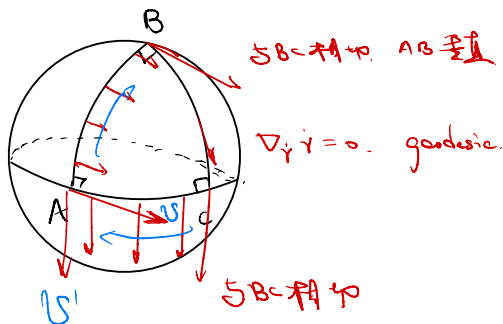
$Y(t)$ along γ is called parallel if $\nabla_{\dot{\gamma}} Y = 0 \quad \forall t \in I$

$$\text{1st order ODE} \Leftarrow \frac{dY^k(t)}{dt} + \Gamma_{ij}^k \dot{\gamma}^i Y^j = 0.$$

$\{e_i\}_{i=1}^n$ o.n.b. still o.n.b.



$$\nabla_{\dot{\gamma}}(e_i, u) \cdot e_j(t) \geq 0.$$



沿闭路作平行移动一般不是恒同映射

Lecture 2.

Def 2.8 (Covariant Differential)

For $(s, 0)$ -tensor, we define covariant differentiations as

$$\nabla_x : \Gamma(\otimes^s TM) \rightarrow \Gamma(\otimes^s TM)$$

$$\nabla_x (Z_1 \otimes Z_2 \otimes \cdots \otimes Z_s) := \sum_{i=1}^s Z_1 \otimes \cdots \otimes \nabla_x Z_i \otimes \cdots \otimes Z_s$$

For (s, r) -tensor α , we define co-diff. as

$$\begin{aligned} \nabla_x \alpha (Z_1 \otimes \cdots \otimes Z_r) &:= \nabla_x (\alpha (Z_1 \otimes \cdots \otimes Z_r)) \\ &\quad - \sum_{i=1}^r \alpha (Z_1 \otimes \cdots \otimes \nabla_x Z_i \otimes \cdots \otimes Z_r) \end{aligned}$$

$$\nabla : \begin{array}{cc} (s, r) & (s, r+1) \\ \alpha & \mapsto \nabla \alpha \end{array}$$

$$(\nabla \alpha)_x (Z_1, \dots, Z_r) := (\nabla_x \alpha)(Z_1 \otimes \cdots \otimes Z_r)$$

Def 2.9 For any $f \in C^\infty(M)$, we define ∇f as

$$\langle \nabla f, X \rangle := Xf$$

in local chart $\{x^i\}_{i=1}^n$, ∂_i

$$\nabla f = \sum_i (\nabla f)^i \partial_i$$

$\Downarrow$

$$\begin{aligned} \langle \nabla f, \partial_j \rangle &= \sum_i (\nabla f)^i \langle \partial_i, \partial_j \rangle \\ &= \sum_i (\nabla f)^i g_{ij} \end{aligned}$$

By the def of ∇f

$$\langle \nabla f, \partial_i \rangle = \partial_i f \quad (g^{ij}) \text{ is the inverse}$$

$$\Rightarrow (\nabla f)^i = \sum_j g^{ij} \partial_j f \quad \text{matrix of } (g_{ij})$$

$$\nabla f = \sum_{i,j} g^{ij} \partial_j f \partial_i$$

$$(\mathbb{R}^n, g_E) \quad g_E = \delta_{ij} \quad , \quad \nabla f = \sum_i \partial_i$$

$$f \quad df$$

$$"b" \quad b: T_p M \rightarrow T_p^* M, \quad v \mapsto v^b, \quad v^b(\omega) := \langle v, \omega \rangle_g$$

$$" \# " \quad \#: T_p^* M \rightarrow T_p M \quad \omega \mapsto \omega^\#, \quad \langle \omega^\#, v \rangle_g = \omega(v)$$

$$(\nabla f)^b = df, \quad (df)^\# = \nabla f$$

df not, but $\nabla f, b, \#$ depends on g

• Hess f : defined as $\nabla \nabla f = \nabla df$

$$\begin{aligned} \text{Hess } f(x, Y) &:= \langle \nabla_x \nabla f, Y \rangle \\ &= x(Yf) - \langle \nabla_x Y, \nabla f \rangle \\ &= (\nabla(df))(x, Y) \end{aligned}$$

$$= \underbrace{\nabla_x (df(Y))}_{X(Yf)} - \underbrace{(\nabla_x Y) f}_{\langle \nabla_x Y, \nabla f \rangle}$$

In local chart,

$$\text{Hess } f(\partial_i, \partial_j) = \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k}$$

- div of $X \in \Gamma(TM)$

$$\text{div}(X) := \text{Trace} (Y \mapsto \nabla_Y X)$$

in local chart, $\{\partial_i\}_{i=1}^n$ $\sum_s X^s \Gamma_{is}^l \partial_l$

$$X = \sum_i X^i \partial_i \quad //$$

$$\nabla_{\partial_i} X = \nabla_{\partial_i} \left(\sum_s X^s \partial_s \right) = \sum_s \left(\frac{\partial X^s}{\partial x^i} \partial_s + X^s \nabla_{\partial_i} \partial_s \right)$$

$$\Rightarrow \text{div}(X) = \sum_i \left(\frac{\partial X^i}{\partial x^i} + \sum_s \Gamma_{is}^i X^s \right) \partial_l$$

$$\sum_i \Gamma_{is}^i = \frac{1}{2} \sum_{t,i} g^{it} (\partial_s g_{it}) = \frac{\partial_s \sqrt{G}}{\sqrt{G}} \quad (1)$$

⌈ Note: for a matrix $A(s) = (A_{ij})_{n \times n}$ $G = \det(g_{ij})$

$$\frac{d}{ds} \log \det A = \sum_{i,j} (A^{-1})^{ij} \frac{d}{ds} A_{ij} \quad (2) \quad \rfloor$$

① + ②, we have

$$\operatorname{div}_Y(X) = \sum_s \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} X^s)}{\partial x^s}, \quad X = \sum_s X^s \partial_s$$

• Laplace of f $\Delta f = \operatorname{div}(\nabla f)$

$$\Delta f = \sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(g^{ik} \sqrt{g} \frac{\partial f}{\partial x^k} \right)$$

$$(\mathbb{R}^n, g_E) \quad g^{ik} = \delta_{ik}, \quad \sqrt{g} = 1.$$

$$\Delta f = \sum_i \frac{\partial^2 f}{(\partial x^i)^2}$$

§ 3. Curvature

Def 3.1 Riem. curv. tensor $(1,3)$ -tensor, defined by

$$R(x, Y)Z = \nabla_x \nabla_Y Z - \nabla_Y \nabla_x Z - \nabla_{[X, Y]} Z$$

$$\forall x, Y, Z \in T(TM)$$

Remark: (1) $R(fx, Y)Z = R(x, fY)Z = R(x, Y)(fZ)$
 $= fR(x, Y)Z$

(2) R can be regarded as $(0,4)$ tensor

$$R(x, Y, Z, W) = \langle R(x, Y)Z, W \rangle$$

Thm 3.2 (Properties of R)

$$(1) R(X, Y)Z = -R(Y, X)Z$$

$$(2) R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$

(1st Bianchi identity)

$$(3) \langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$$

$$(4) \langle R(X, Y)Z, W \rangle = -\langle R(X, Y)W, Z \rangle$$

$$(5) (\nabla_X R)(Y, Z)U + (\nabla_Y R)(Z, X)U + (\nabla_Z R)(X, Y)U = 0.$$

(2nd Bianchi identity)

Prove by direct computation.

Def 3.3 (Sectional Curv.)

For any $p \in M$, $u \in T_p M$, $v \in T_p M$

$$\text{Sec}(u, v) = \frac{\langle R(u, v)v, u \rangle}{\|u \times v\|} \stackrel{= K(u, v)}{=} \frac{K(u, v)}{\|u\|^2 \|v\|^2 - \langle u, v \rangle^2}$$

$\text{Sec}(u, v)$ is called the sectional curvature of the plane $\sigma = \text{span}\{u, v\}$, $\text{Sec}(\sigma)$. (not defined - u, v)

Remark: $\langle R(u, v)w, z \rangle = \frac{1}{6} \frac{\partial^2}{\partial s \partial t} \bigg|_{(s, t) = (0, 0)} (K(u + sz, v + tw) - K(u + sw, v + tz)).$

Def 3.4 For $p \in M$, Ricci curvature tensor is the trace of R , $R(u, v) := \text{trace}(x \mapsto R(x, u)v)$.

If $\{e_1, \dots, e_n\}$ is O.N.B. of $T_p M$

$$\begin{aligned} \text{Ric}(u, v) &= \sum_{i=1}^n \langle R(e_i, u)v, e_i \rangle \\ &= \text{Ric}(v, u) \end{aligned}$$

Hence, Ric is a bilinear symmetric form

Def 3.5 (Scalar curvature)

Scalar curvature S is the trace of Ric in local chart.

$$\begin{aligned} S(p) &:= \sum_{i \neq j} \text{sec}(e_i, e_j) \\ &\parallel \\ &\sum_{j=1}^n \text{Ric}(e_j, e_j) \end{aligned}$$

Def 3.6 (Curvature Operator)

$\Lambda^2 M$: space of bivectors

If $\{e_i\}_{i=1}^n$ O.N.B. of $T_p M$, $\{e_i \wedge e_j\}_{i \neq j}$ form O.N.B. of $\Lambda^2 M$

$$\mathcal{R} : \Lambda_p^2 M \times \Lambda_p^2 M \longrightarrow \mathbb{R}$$

$$\mathcal{R}(\underbrace{X \wedge Y}_{w_1}, \underbrace{V \wedge W}_{w_2}) := \mathcal{R}(X, Y, V, W)$$

(by properties, it's well-defined)

using g_{ij} , we can change $\mathcal{R} : \Lambda_p^2 M \rightarrow \Lambda_p^2 M$

$\mathcal{R}$ is a self-adjoint operator.

In some meaning, it's stronger than section curv.

like this 1. $\mathcal{R} > 0$, $w \in \Lambda_p^2 M$

$$\mathcal{R}(w, w) > 0.$$

$$2. \text{ sec} > 0. \quad \mathcal{R}(e_i, e_j, e_j, e_i) > 0$$

$$\Rightarrow \mathcal{R}(e_i, e_j, e_i, e_j) > 0$$

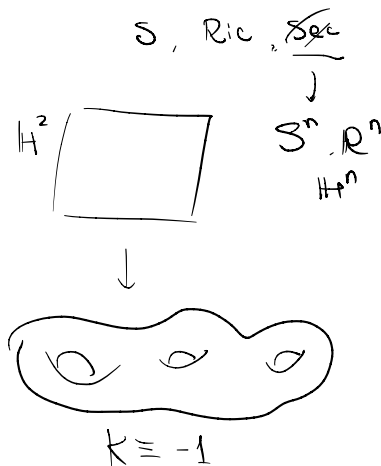
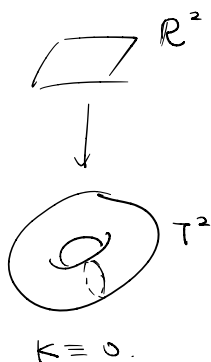
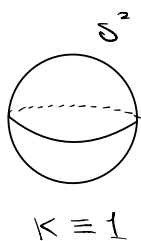
$\Downarrow \times$

$$\mathcal{R}(w, w) > 0, \quad \forall w \in \Lambda_p^2 M$$

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} * () > 0 \text{ positive definite.}$$

I. Existence of metric with constant curvature.

dim 2



$$n = \dim_M \geq 3,$$

- Yamabe (Richard Schoen)

For any compact (M^n, g) , $\exists f \in C^\infty(M)$,

$\tilde{g} = e^{2f} g$, $(M, \tilde{g})$ has constant scalar curvature.

- (Conjecture) Any compact manifold admits a metric with constant Ric, i.e. $Ric = \lambda g$ for $\lambda \in \mathbb{R}$

($n \geq 3$, if $f \in C^\infty(M)$, $Ric = f g \Rightarrow f$ is a constant)

Kähler case	compact	S^n
$C_1(M) < 0$	Aubin, Yau	$Sec \equiv 1$
$C_1(M) = 0$	Yau	$\exists$ Einstein metric
$C_1(M) > 0$	T-T-D	not unique for $n \gg 1$

II. Curvature and Topology

- noncompact + $\text{sec} > 0 \Rightarrow \text{diffeo to } \mathbb{R}^n$

$\text{sec} \geq 0 \Rightarrow \text{Soul theorem (Petersen)}$

- 3D compact $\text{Ric} > 0 \Rightarrow \text{Diffeo to } S^3/\sim$ (Hamilton)
- 4D curvature operator $> 0 \Rightarrow \text{Diffeo to } S^4/\sim$
- $n \geq 5$ ----- $> 0 \Rightarrow \text{Diffeo to } S^n/\sim$ (Bohn-Wilky)
- $\text{sec curvature} \underset{>0}{-\frac{1}{4} \text{ pinched}} \Rightarrow \text{diffeo to } S^n$

$$\frac{\lambda_{\max}}{\lambda_{\min}} \in (1, 4) \quad (\text{Schoen} = \text{Brendle})$$

- $\text{sec} > 0 + \text{compact} \quad ?$

$$S^4, \mathbb{CP}^2$$

- (Hopf conjecture) $S^2 \times S^2$ doesn't exist a metric with positive sectional curvature.

Riem. submanifold.

$$S^0(r) \hookrightarrow \mathbb{R}^{n+1}$$

$$\nabla \sim (M^n, g) \xrightarrow{\varphi} (\bar{M}^N, \bar{g}) \quad n < N, \quad \varphi \text{ is}$$

an isometric embedding i.e. $\varphi^* \bar{g} = g$

identify M with $\varphi(M)$ via φ .

Let $V_p M = T_p M^\perp$ be the orthogonal complement

$$\text{of } \varphi_* (T_p M) \subset T_{\bar{p}} \bar{M}, \quad \bar{p} = \varphi(p)$$

$\parallel$
 $T_p M$

$$T_{\bar{p}} \bar{M} = T_p M \oplus V_p M \quad \text{w.r.t. } \bar{g}$$

$$VM := \bigcup_{p \in M} T_p M^\perp, \quad T_{\bar{p}} \bar{M} = T_p M \oplus V_p M$$

$$V = V^T + V^\perp \quad (M \hookrightarrow \bar{M})$$

Thm 3.7 Let $\bar{\nabla}$ and ∇ be the Levi-Civita connection of $(\bar{M}, \bar{g})$ and (M, g) , then $\forall p \in M, v \in T_p M, \gamma \in \Gamma(TM)$.

$$(1) \quad \nabla_v \gamma = (\bar{\nabla}_v \gamma)^T$$

(2) $\exists$ a symmetric bilinear map

$$\mathbb{I}: T_p M \times T_p M \longrightarrow V_p M$$

$$(u, v) \longmapsto \mathbb{I}(u, v)$$

$$\mathbb{I}(u, v)|_p := (\bar{\nabla}_u \gamma)^\perp|_p \quad \text{where } \gamma|_p = v$$

$$(3) \quad \mathbb{I}_g(u, v) := \langle \mathbb{I}(u, v), \xi \rangle$$

$$S_g : \langle S_g(u), v \rangle := \mathbb{I}_g(u, v)$$

$$\text{then } S_g = -(\bar{\nabla}_u \gamma)^\top, \quad \gamma \in \mathcal{P}(\underline{VM}), \quad \gamma|_p = \xi.$$

法丛

proof: (1) $(\bar{\nabla}_u \gamma)^\top = \frac{\bar{\nabla}_u \gamma - (\bar{\nabla}_u \gamma)^\perp}{\text{wts}} = \nabla_u \gamma.$

let $D_u \gamma = \bar{\nabla}_u \gamma - (\bar{\nabla}_u \gamma)^\perp$, then "D" satisfies

$$\textcircled{1} Dg = 0$$

$$\textcircled{2} \text{ torsion free}$$

$$D \text{ is the Levi-Civita connect. of } (M, g) \Rightarrow D = \nabla.$$

$$(2) \quad u, v \in T_p M, \quad x, \gamma \in \mathcal{P}(TM), \quad x|_p = u, \quad \gamma|_p = v$$

$$\Rightarrow [x, \gamma] \in \mathcal{P}(TM)$$

$$\begin{aligned} \mathbb{I}(u, v) - \mathbb{I}(v, u) &= (\bar{\nabla}_u \gamma)^\perp - (\bar{\nabla}_v \gamma)^\perp \\ &= (\bar{\nabla}_x \gamma - \bar{\nabla}_\gamma x)^\perp|_p \\ &= ([x, \gamma]^\perp)|_p \\ &= 0. \end{aligned}$$

$$(3) \quad u, v \in T_p M, \quad \xi \in V_p M, \quad \gamma \in P(V_p M), \quad \gamma(p) = \xi$$

$$Z \in T(T_p M), \quad Z(p) = v, \quad \Rightarrow \langle \gamma, Z \rangle = 0.$$

$$\langle \mathbb{H}(u, v), \xi \rangle = \langle (\bar{\nabla}_u Z)^\perp, \xi \rangle$$

$$= \langle \bar{\nabla}_u Z, \gamma \rangle|_p$$

$$= \underbrace{\bar{\nabla}_u \langle Z, \gamma \rangle}_0|_p - \langle Z, \bar{\nabla}_u \gamma \rangle|_p$$

$$= - \langle Z, (\bar{\nabla}_u \gamma)^\top \rangle|_p$$

$$\Rightarrow S_\xi(u) = -(\bar{\nabla}_u \gamma)^\top$$

□

Lecture 3

Def 3.8 The eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_n$ of Π_g on S_g are called principle curvatures w.r.t. g (法向 $\frac{1}{2}$)

Thm 3.9 (Gauss formula)

Let R and $\bar{R}$ be the Riemannian curvature tensor of (M, g) and $(\bar{M}, \bar{g})$. Then $\forall p \in M, u, v, w, z \in T_p M$

$$(1) R(u, v)w = (\bar{R}(u, v)w)^T - S_{\Pi(u, w)}v + S_{\Pi(v, w)}u$$

$$(2) \langle R(u, v)w, z \rangle = \langle \bar{R}(u, v)w, z \rangle + \langle \Pi(u, z), \Pi(v, w) \rangle$$

$$- \langle \Pi(u, w), \Pi(v, z) \rangle$$

$$(3) \text{Sec}(u, v) = \bar{\text{Sec}}(u, v) + \frac{\langle \Pi(u, w), \Pi(v, v) \rangle - \|\Pi(u, v)\|^2}{\|u\|^2 \|v\|^2 - \langle u, v \rangle^2}$$

Proof: (1) $(\bar{R}(u, v)w)^T = \left(\bar{\nabla}_u \bar{\nabla}_v w - \bar{\nabla}_v \bar{\nabla}_u w - \bar{\nabla}_{[u, v]} w \right)^T$

$$= \left(\bar{\nabla}_u (\nabla_v w + \Pi(v, w)) - \bar{\nabla}_v (\nabla_u w + \Pi(u, w)) \right. \\ \left. - \nabla_{[u, v]} w - \Pi([u, v], w) \right)^T$$

$\bar{\nabla} = \nabla + \Pi$

$$= \left(\underbrace{\nabla_u \nabla_v w + \Pi(u, \nabla_v w)}_{\text{red}} \right) + \bar{\nabla}_u \Pi(v, w) \\ - \underbrace{\nabla_v \nabla_u w + \Pi(v, \nabla_u w)}_{\text{red}} - \bar{\nabla}_v \Pi(u, w) \\ - \underbrace{\nabla_{[u, v]} w + \Pi([u, v], w)}_{\text{red}} \Big)^T$$

$\parallel$
 0

$$= R(u, v, w) - S_{\text{II}(v, w)} u + S_{\text{II}(u, w)} v \quad \square$$

Example (Curvature of $S^n(r)$)

$$S^n(r) = \left\{ \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\} \xrightarrow{\varphi} (\mathbb{R}^{n+1}, g_E)$$

$\forall p \in S^n(r) \subset \mathbb{R}^{n+1}$, $\forall_p S^n(r)$ is 1-dim vector space

$$\text{spanned } \left\{ \frac{\vec{p}}{r} \right\} \quad \downarrow \quad \text{is a.k.a. } p = (x^1(p), \dots, x^{n+1}(p))$$

$$\vec{p} = \sum x^i(p) \partial_i$$

Shape operator $S_{\frac{\vec{p}}{r}}$

$$\forall Y \in T(VS^n(r)) \quad Y(p) = \frac{\vec{p}}{r}, \quad S_{\frac{\vec{p}}{r}}(W) \Big|_p = -(\bar{\nabla}_W Y)^T \Big|_p$$

$W \in T(TM).$

$$\text{Let } Y = \frac{1}{r} (x^1, x^2, \dots, x^{n+1}) = \sum_{i=1}^n \frac{1}{r} x^i \partial_i, \quad W = \sum w^i \partial_i$$

$$Y \Big|_p = \frac{\vec{p}}{r}$$

connection
on $\mathbb{R}^{n+1}$

$$\bar{\nabla}_W Y = WY = \frac{W}{r} \Rightarrow S_{\frac{\vec{p}}{r}}(W) \Big|_p = \frac{W}{r}$$

By Gauss formula

$$\langle R(u, v) v, u \rangle = \frac{1}{r^2} (\langle u, v \rangle \langle u, u \rangle - \langle u, u \rangle \langle u, v \rangle)$$

$$\Rightarrow \text{Sec}(S^n(r)) = \frac{1}{r^2}$$

$$\ddot{\gamma}(t) = \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \equiv 0 \quad \text{geodesic} \quad (\bar{\nabla}_{\dot{\gamma}} \ddot{\gamma})^T = 0$$

$$\nabla_{\dot{\gamma}} \dot{\gamma} \in VM, \quad \ddot{\gamma}(t) \in VM, \quad \ddot{\gamma}(t) = f(t) \gamma(t)$$

$$\gamma(p) = \frac{\vec{p}}{r}, \quad \forall p \in \bar{S}(r)$$

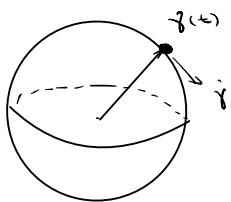
$$\nabla_{\dot{\gamma}} \langle \dot{\gamma}, \dot{\gamma} \rangle = 2 \langle \nabla_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma} \rangle \stackrel{\equiv 0}{=} 0 \Rightarrow \langle \dot{\gamma}, \dot{\gamma} \rangle \equiv \text{const.} \quad (\text{Assum} = 1, \text{ by tran})$$

$$0 = \frac{d}{dt} \langle \dot{\gamma}, \dot{\gamma} \rangle = \langle \ddot{\gamma}, \dot{\gamma} \rangle + \langle \dot{\gamma}, \ddot{\gamma} \rangle \quad \gamma = \vec{\gamma}(t)$$

$$\stackrel{\parallel}{=} 1$$

$$\ddot{\gamma}(t) = f(t) \cdot \frac{\dot{\gamma}(t)}{r}$$

$$\langle \vec{\gamma}, \vec{\gamma} \rangle = r^2$$



$$f(t) = -\frac{1}{r}$$

$$\therefore \ddot{\gamma}(t) + \frac{1}{r^2} \gamma(t) = 0.$$

$$\dot{\gamma} \in VM$$

$$\dot{\gamma} \in TM.$$

$$\vec{\gamma}(t) = (x^1(t), x^2(t), \dots, x^n(t))$$

$$\Rightarrow \gamma(t) = \cos\left(\frac{t}{r}\right) \vec{u}_1 + \sin\left(\frac{t}{r}\right) \vec{u}_2, \quad \|\vec{u}_1\| = \|\vec{u}_2\| = r, \quad \vec{u}_1 \perp \vec{u}_2$$

$$(\text{radius} = r, \text{ large circle.}) \quad \in S^0(r)$$

$$(M, g) \xrightarrow{\varphi} (\bar{M}, \bar{g}) \quad \varphi(p) = \bar{p}$$

$$\overset{\bar{\nabla}}{T}_{\bar{p}} \bar{M} = \overset{\nabla}{T}_p M \oplus \overset{\nabla^\perp}{V}_p M$$

$$VM = \bigcup_p V_p M, \quad \nabla_x^\perp \gamma = (\bar{\nabla}_x \gamma)^\perp.$$

法丛上联络定义.

By Gauss formula, we have

Thm 3.11 (Codazzi formula)

$$\mathbb{I}(u, v) \in T(VM).$$

$$(\bar{R}(u, v)w)^\perp = (\nabla_u^\perp \mathbb{I})(v, w) - (\nabla_v^\perp \mathbb{I})(u, w), \quad u, v, w \in T_p M$$

$$\text{where } (\nabla_u^\perp \mathbb{I})(v, w) := \nabla_u^\perp (\mathbb{I}(v, w)) - \mathbb{I}(\nabla_u v, w)$$

$$- \mathbb{I}(v, \nabla_u w)$$

(Ex)

□

• Geometry of level set $\{r=c\}$, r is a distance func.

Def 3.11 $r: U \rightarrow \mathbb{R}$ is distance func. if $\|\nabla r\| = 1$.

$\Sigma_c \hookrightarrow (M, g)$

Σ_c

$$\partial \Sigma_c = \nabla r$$

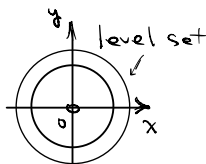
If $\{r=c\} \neq \emptyset$, then it is a submanifold of M

$$\|\nabla r\| \neq 0 \quad (\Sigma_c, \varphi^* g) \stackrel{||}{=} g_c$$

Example

$$(\mathbb{R}^2, g_E)$$

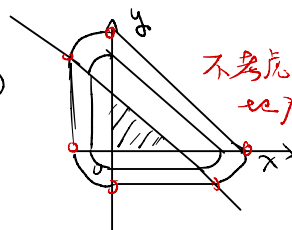
①



$$r(x) = d(o, x)$$

$$= \|\vec{x}\| = \sqrt{(x^1)^2 + (x^2)^2}$$

②



$$r_Q(x) = d(Q, x)$$

$$\{r=c\}$$

Π of (Σ_c, g_c)

$\mathcal{V}_p M$ spanned by $\{\partial_r\}$.

$$\mathbb{I}_{\partial_r}(u, v) = (\nabla_u v)^\perp = \langle \nabla_u v, \partial_r \rangle \partial_r \quad \begin{array}{l} v \in TM \\ \partial_r \in \mathcal{V}M \end{array}$$

$$\langle \nabla_u v, \partial_r \rangle = \nabla_u \langle v, \partial_r \rangle - \langle v, \nabla_u \partial_r \rangle$$

$$\stackrel{0}{=} -\text{Hess } r(u, v)$$

$$S(u) = \nabla_u \partial r$$

Further, $\langle S_{\partial_r}(u), v \rangle = -\text{Hess } r(u, v)$

$$S(x) := S_{\partial_r}(x) \quad , \quad \langle S(u), v \rangle = \text{Hess } r(u, v).$$

Thm 3.12 (Ricatti Eq. of S) $U \subset M$, r is dist formula on U .

Then $\partial_r S + S^2 + R(\cdot, \partial_r) \partial r = 0$

Proof: $\forall x \in P(TU)$

$$\begin{aligned} (\nabla_{\partial_r} S)(x) &= \nabla_{\partial_r}(S(x)) - S(\nabla_{\partial_r} x) \\ &= \nabla_{\partial_r}(\nabla_x \partial r) - \nabla_{\nabla_{\partial_r} x} \partial r. \end{aligned} \quad ①$$

$$S^2(x) = S(S(x)) = S(\nabla_x \partial r) = \nabla_{\nabla_x \partial r} \partial r \quad ②$$

①+②, we obtain

$$(\nabla_{\partial_r} S)(x) + S^2(x) = \nabla_{\partial_r}(\nabla_x \partial r) - \nabla_{[x, \partial r]} \partial r$$

Claim:

$$\boxed{+\nabla_x \nabla_{\partial r} \partial r} = 0, \text{ i.e. } \nabla_{\partial_r} \partial r = 0.$$

$$\forall \gamma \in \Gamma(U).$$

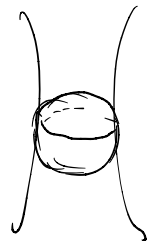
$$\begin{aligned} \langle \nabla_{\partial r} \partial r, \gamma \rangle &= \text{Hess } r(\partial r, \gamma) \\ &= \text{Hess } r(\gamma, \partial r) \\ &= \langle \nabla_{\gamma} \partial r, \partial r \rangle \\ &= \frac{1}{2} \nabla_{\gamma} \underbrace{\langle \partial r, \partial r \rangle}_{=1} \\ &= 0. \end{aligned}$$

Remark: $\nabla_{\partial r} \partial r \equiv 0 \Rightarrow$ The integral curve of ∇r is a geodesic.

Example: curvature of warped product. $I \times_{\varphi} S^{n-1}$
 $(0, +\infty)$

$$g = dr^2 + \varphi^2(r) g_{S^{n-1}}$$

We can get $K \equiv \pm 1, 0$.



$r(x)$ is a natural distance function

$$\begin{aligned} \{n=n_0\} \quad g_{n_0}(R^{\nabla}(x, \gamma) V, W) &= \frac{1}{\varphi^2(r)} g_{n_0}(x \wedge \gamma, V \wedge W) \\ \parallel \\ \Sigma_{r_0} \quad x, \gamma, V, W &\in T(\Sigma_{r_0}) \end{aligned}$$

$$\forall f \in C^\infty(M), \quad 2 \text{Hess } f = \mathcal{L}_{\nabla f} g$$

$$\begin{aligned} \text{II} &= \text{Hess } r = \frac{1}{2} \mathcal{L}_{\partial r} g \\ &= \frac{1}{2} \mathcal{L}_{\partial r} (dr^2 + \varphi^2(r) g_{S^{n-1}}) \end{aligned}$$

$$\begin{aligned} \Pi &= \underbrace{\mathcal{L}_{\partial r}}_0 dr + \varphi'(r) \varphi(r) g_{S^{n-1}} + \frac{1}{2} \varphi^2(r) \underbrace{\mathcal{L}_{\nabla r} g_{S^{n-1}}}_0 \\ &= \varphi(r) \varphi'(r) g_{S^{n-1}} = \frac{\varphi'}{\varphi} g_{r_0}. \end{aligned}$$

$$u, v \in T(TM) \rightsquigarrow g|_{T\Sigma_{r_0} \times T\Sigma_{r_0}}$$

$$S(x) = \frac{\varphi'}{\varphi} x, \quad x \in T\Sigma_{r_0}.$$

If $x, Y \in T(T\Sigma_{r_0})$ By Gauss formula,

$$g(R(x, Y)Y, x) = \frac{1 - (\varphi')^2}{\varphi} g_{r_0}(x \wedge Y, x \wedge Y)$$

By Codazzi formula

$$g(R(x, Y, Z), \partial r) = -(\nabla_x \Pi)(Y, Z) + (\nabla_Y \Pi)(X, Z)$$

$$x, Y, Z \in T\Sigma_{r_0}$$

Π depends only on r .

$$\nabla_x \Pi = \nabla_Y \Pi = 0.$$

By Riccati Eq and $S(x) = \frac{\varphi'}{\varphi} x$

$$\Rightarrow R(x, \partial r) \partial r = -\frac{\varphi''}{\varphi} x$$

$$\Rightarrow \text{Sec}(x, \partial r) = -\frac{\varphi''}{\varphi}$$

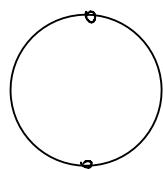
$$\text{Ric}(x, x) = (n-2) \frac{1 - (\varphi')^2}{\varphi^2} - \frac{\varphi''}{\varphi}, \quad x \in T(T\Sigma_{r_0})$$

$$\text{Ric}(\partial r, \partial r) = (n-1) \frac{\varphi''}{\varphi} \quad \text{取法向}$$

$$S = -2(n-1) \frac{\varphi''}{\varphi} + (n-1)(n-2) \frac{1-(\varphi')^2}{\varphi^2}$$

$$\text{Sec}(M) \equiv K$$

$$-\frac{\varphi''}{\varphi} \equiv K, \quad \varphi'' + K\varphi = 0.$$



$K > 0$ 时,
metric 并不完备, 有了两个端点

$$\varphi(r) = s_K(r) = \begin{cases} \sinh(\sqrt{|K|}r) & K < 0 \\ r & K = 0 \\ \frac{1}{\sqrt{K}} \sinh(\sqrt{K}r) & K > 0 \end{cases}$$

§ 4. Exponential Map

$$\exp_p: T_p M \rightarrow M$$

Def 4.1 A C^2 -curve $\gamma: [a, b] \rightarrow M$ is called a geodesic

if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ for all t

In local chart, $(x^1(t), \dots, x^n(t)) = (x^1(\gamma(t)), \dots, x^n(\gamma(t)))$

$$\frac{d^2 x^k}{dt^2} + \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0.$$

geodesics satisfy a 2nd order ODE (non-linear)

Given $\dot{\gamma}(a) = v$, $\gamma(a) = p$ (*)

$\exists!$ a unique geodesic $\gamma(t)$ with initial data (*)
denote by $\gamma_v(t)$ $t \in [a, b]$

• Exp. map

$$\forall p \in M, \quad \mathcal{U}_p(M) = \{ v \in T_p M \mid \|v\| = 1 \} \cong S^{n-1}$$

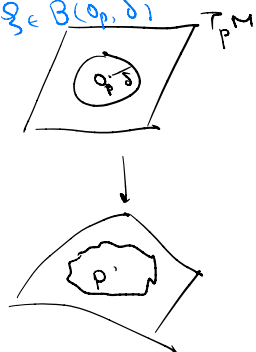
$\exists \delta = \delta(p)$, s.t. $\mathcal{U}_\delta(v)$ is well-defined for all $\xi \in \mathcal{U}_p(M)$

$$t \in (-\delta, \delta) \quad \exp_t(\xi) = \gamma_\xi(t) \quad |t| \leq 1, \quad \forall \xi \in \mathcal{B}(0_p, \delta)$$

Def 4.2 For $p \in M$, define exp map

$$\exp_p : \mathcal{B}(0_p, \delta) \rightarrow M$$

$$u \mapsto \gamma_u(1)$$



$$\mathcal{B}(0_p, \delta) = \{ v \in T_p M \mid \|v\| < \delta \}$$

$$(\exp_p)_* \Big|_{0_p} : T_{0_p}(T_p M) \longrightarrow T_p M$$

SI
 $T_p M$

By def ...

$$(\exp_p)_* \Big|_{0_p} (u) = \frac{d}{dt} (\exp_p(tu)) \Big|_{t=0} = \frac{d}{dt} (\gamma_u(t)) \Big|_{t=0} = u$$

$= \text{Id}_{T_p M}$

$\exists \varepsilon > 0$, $\exp_p \Big|_{\mathcal{B}(0_p, \varepsilon)}$ is a diffeo onto $\exp_p(\mathcal{B}(0_p, \varepsilon))$

(B, φ) becomes a local chart, φ is the inverse map of

$\exp_p$. $(B, \varphi, \text{axis})$ is called normal coordinate

Lecture 4.

$$\|u\| > 0. \quad (\exp_p)_*|_u = T_u(T_p M) \longrightarrow T_{\gamma_u(t)} M$$

$$\text{Fix } \xi \in T_u(T_p M)$$

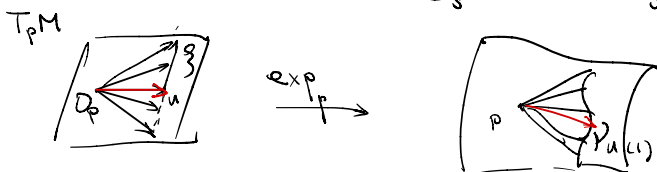
$$l: s \mapsto u + s\xi \quad \text{a line in } T_p M$$

$$\text{Define } \alpha: [0, 1] \times (-\epsilon, \epsilon) \longrightarrow M$$

$$(t, s) \mapsto \exp_p(t(u + s\xi)).$$

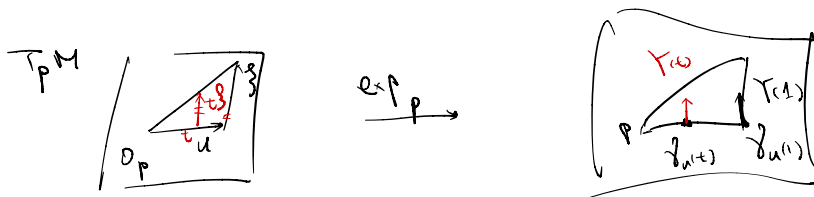
$$\text{Fix } s \in (-\epsilon, \epsilon) \quad C_s(\cdot) = \alpha(\cdot, s)$$

$$\dot{C}_s(0) = u + s\xi$$



$$\text{Fix } t \in [0, 1], \quad t \mapsto \left. \frac{\partial \alpha}{\partial s} \right|_{(t, 0)} \quad \text{is vector field along } \gamma_u(t)$$

ii
 $\Upsilon(t)$



$$\left. \frac{\partial (t(u + s\xi))}{\partial s} \right|_{s=0} = t\xi$$

$$\Upsilon(t) = (\exp_p)_*|_u (\xi)$$

$$\Upsilon(t) = (\exp_p)_*|_{t u} (t\xi) = t(\exp_p)_*|_{t u} (\xi).$$

非退化 $\leadsto$ 微分同胚,

• Equation of $\gamma(t)$

Def 4.3: Suppose $\alpha: [a, b] \times (-\epsilon, \epsilon) \rightarrow M$ is C^2 .

For any fixed $s \in (-\epsilon, \epsilon)$, $C_s(\cdot) = \alpha(\cdot, s)$

$\{C_s\}$ is called the variation of C_0 via s .

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \xrightarrow{\alpha_*} \alpha_* \left(\frac{\partial}{\partial t} \right) \quad \alpha_* \left(\frac{\partial}{\partial s} \right)$$

$$\parallel \quad \parallel$$

$$\frac{\partial \alpha}{\partial t} \quad \frac{\partial \alpha}{\partial s}$$

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = 0 \Rightarrow \left[\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s} \right] = 0 \quad (1)$$

$$(2) \quad R \left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s} \right) X = \nabla_{\frac{\partial \alpha}{\partial t}} \nabla_{\frac{\partial \alpha}{\partial s}} X - \nabla_{\frac{\partial \alpha}{\partial s}} \nabla_{\frac{\partial \alpha}{\partial t}} X - \nabla_{\left[\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right]} X$$

curvature term.

$C_s(t) = \exp_p(t(u + s\dot{\gamma}))$, $C_s(t)$ is a geodesic, $\nabla_{\dot{C}_s(t)} \dot{C}_s(t) = 0$.

$$\dot{\gamma}(t) = \nabla_{\frac{\partial \alpha}{\partial t}} \gamma(t) = \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s} \stackrel{By (1)}{=} \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial t}, \quad \frac{\partial \alpha}{\partial s} = \gamma(t),$$

$$\frac{\partial \alpha}{\partial t} = \dot{\gamma}_u(t).$$

$$\ddot{\gamma}(t) = \nabla_{\frac{\partial \alpha}{\partial t}} \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial t} \stackrel{By (2)}{=} \nabla_{\frac{\partial \alpha}{\partial s}} \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial t} + R \left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial s} \right) \frac{\partial \alpha}{\partial t}$$

(t,0) (t,0)

$$= -R(\gamma(t), \dot{\gamma}_u(t)) \dot{\gamma}_u(t)$$

$$\ddot{\gamma}(t) + R(\gamma(t), \dot{\gamma}_u(t)) \dot{\gamma}_u(t) = 0. \quad (*)$$

2nd order linear ODE.

Def 4.4 If $Y(t)$ is a vector field along $\gamma_u(t)$ and satisfies (*), then Y is called a Jacobi field along $\gamma_u(t)$.

$$(\exp_p)_* \Big|_{t=0} \{ \dot{\gamma} \} = \frac{1}{t} Y(t)$$

Remark: ① $(\exp_p)_*$ is non-degenerate at $t=0$ if for any

ξ and $Y(t)$ with $Y(0)=0$, $\dot{Y}(0)=\xi$, $Y(t) \neq 0$ for $t>0$,

② $\dim \text{Jac}(\gamma_u) = 2n$. $\text{Jac}(\gamma_u)$ is the set of all Jacobi field along γ_u .

$Y(0) = a = 0$
initial data

③ $\dim \text{Jac}^0(\gamma_u) = n$, $\text{Jac}^0(\gamma_u) = \{ J \in \text{Jac}(\gamma_u) \mid J(0) = 0 \}$

Def 4.5 If γ_u is a geodesic in M , and $\exists$ Jacobi field Y along γ_u , $Y(0)=Y(1)=0$, $Y(t) \neq 0$. Then we call $\gamma_u(1)$ is conjugate to $\gamma_u(0)$ along γ_u .

if $\gamma_u(1)$ is conj. to $\gamma_u(0)$, then $\gamma_u(0)$ conj. to $\gamma_u(1)$.

By Remark ① and ③

Cor 4.6 $(\exp_p)_*$ is non-degenerate at u iff

$p = \gamma_u(0)$ has no conjugate point along γ_u .

Def 4.7 $\dim \ker [(\exp_p)_*|_u]$ is called the multiplicity of the conj. point.
 $= \dim \{ J \in \text{Jac}(\gamma_u) \mid J(0) = J(1) = 0 \}$

$$\ddot{\gamma}(t) + R(\gamma(t), \dot{\gamma}_u(t)) \dot{\gamma}_u(t) = 0$$

$$\dot{\gamma}(t) = \begin{matrix} \dot{\gamma}_u(t) \\ \dot{\gamma}_v(t) \end{matrix} = \begin{matrix} \dot{\gamma}_u(t) \\ \dot{\gamma}_u^\perp(t) \end{matrix}$$

$$\text{If } Z(t) \in \text{Jac}(\gamma_u), \quad Z(t) = f(t) \dot{\gamma}_u(t)$$

$$\ddot{Z}(t) + R(f(t) \dot{\gamma}_u(t), f(t) \dot{\gamma}_u(t)) f(t) \dot{\gamma}_u(t) = 0$$

$$\ddot{Z}(t) = 0.$$

$$\Downarrow \text{Since } \nabla_{\dot{\gamma}_u(t)} \dot{\gamma}_u(t) = 0,$$

$$\ddot{f}(t) = 0.$$

$$\Downarrow$$

$$f(t) = at + b$$

Lemma 4.8 If $Z(t) \parallel \dot{\gamma}_u(t)$, $Z(t) \in \text{Jac}(\gamma_u)$, then $Z(t) = (a+bt) \dot{\gamma}_u(t)$

$$\langle Z(0), \dot{\gamma}_u(0) \rangle = a, \quad \langle Z'(0), \dot{\gamma}_u(0) \rangle = b.$$

Lemma 4.9 If $\gamma(t) \in \text{Jac}(\gamma_u)$, then $\langle \gamma(t), \dot{\gamma}_u(t) \rangle = a+bt$

$$\langle \gamma(0), \dot{\gamma}_u(0) \rangle = a, \quad \langle \dot{\gamma}(0), \dot{\gamma}_u(0) \rangle = b,$$

$$\text{Proof.} \quad \frac{d}{dt} \langle \gamma(t), \dot{\gamma}_u(t) \rangle = \langle \gamma'(t), \dot{\gamma}_u(t) \rangle + \langle \gamma(t), \nabla_{\dot{\gamma}_u(t)} \dot{\gamma}_u(t) \rangle$$

$$= \langle \gamma'(t), \dot{\gamma}_u(t) \rangle \quad \Downarrow$$

$$\frac{d^2}{dt^2} \langle \gamma(t), \dot{\gamma}_u(t) \rangle$$

$$= \langle \gamma''(t), \dot{\gamma}_u(t) \rangle = -\langle R(\gamma(t), \dot{\gamma}_u(t)) \dot{\gamma}_u(t), \dot{\gamma}_u(t) \rangle = 0.$$

□

By Lemma 4.8 and Lem 4.9,

Prop 4.10 If $Y(t) \in \text{Jac}(\gamma_u)$, then $Y(t) = (a+bt) \dot{\gamma}_u(t) + Y^\perp(t)$

$Y^\perp(t) \perp \dot{\gamma}_u(t)$ for all t . where $a = \langle Y(0), \dot{\gamma}_u(0) \rangle$, $b = \langle Y'(0), \dot{\gamma}_u(0) \rangle$

Def 4.11 $\text{Jac}^\perp(\gamma_u) = \{ J \in \text{Jac}(\gamma_u) \mid J(t) \perp \dot{\gamma}_u(t) \forall t \}$.

$\forall J \in \text{Jac}^\perp(\gamma_u)$, J is called normal Jacobi field.

• $\text{Sec}(M) \equiv K$,

$$R(\dot{\gamma}, x) \dot{\gamma} = -Kx$$

Let $\{e_i(t)\}$ be a o.n.b along $\gamma(t)$ and parallel along $\gamma(t)$,

$$\text{wlog. } e_1(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$$

If $J(t) \in \text{Jac}^\perp(\gamma)$, then $J = \sum_{i=2}^n J^i(t) e_i(t)$

$$\ddot{J}^i(t) + K J^i(t) = 0,$$

$$\Rightarrow J^i(t) = c^i \sin_K(t) + d^i \cos_K(t), \quad \cos_K(t) = \sin'_K(t)$$

• Taylor formula of $J(t)$, " $\langle J_1, J_2 \rangle$ "

Recall for $\alpha(t,s) = \exp_p[t(u+s\dot{\gamma})]$

$$Y(t) = \frac{\partial}{\partial s} \Big|_{(t,0)} \text{ is a Jacobi field.}$$

$$\text{and } Y(t) = t \left(\exp_p \right)_* \Big|_{t u} (\dot{\gamma})$$

$$\begin{aligned} \gamma'_{(0)} &= \lim_{t \rightarrow 0} \frac{\gamma(t) - 0}{t} = \lim_{t \rightarrow 0} (\exp_p)_* \Big|_{t u} (\xi) \\ &= (\exp_p)_* \Big|_{0_p} (\xi) = \xi. \end{aligned}$$

We get

Cor 4.12 If $\gamma(t) \in \text{Jac}^0(\gamma)$, then

$$\gamma(t) = (\exp_{\gamma(0)})_* \Big|_{t \gamma'(0)} (t \gamma'_{(0)})$$

Let $v, w \in \text{Jac}^0(\gamma)$. $v'_{(0)} = X_p \in T_p M$, $w'_{(0)} = Y_p \in T_p M$

$$f(t) = \langle v(t), w(t) \rangle$$

$$f(0) = \langle v(0), w(0) \rangle = 0$$

$$f'_{(0)} = \langle \nabla_{\gamma'(0)} v(0), w(0) \rangle + \langle v(0), \nabla_{\gamma'(0)} w(0) \rangle = 0$$

$$f''_{(0)} = \langle \nabla_{\gamma'(0)} \nabla_{\gamma'(0)} v(0), w(0) \rangle + 2 \langle \nabla_{\gamma'(0)} v(0), \nabla_{\gamma'(0)} w(0) \rangle$$

$$+ \langle v(0), \nabla_{\gamma'(0)} \nabla_{\gamma'(0)} w(0) \rangle$$

$$= 2 \langle X_p, Y_p \rangle$$

$$f'''_{(0)} = \sum_{l=0}^3 \binom{3}{l} \langle \nabla_{\gamma'(0)}^{(3-l)} v, \nabla_{\gamma'(0)}^l w \rangle = 0.$$

$$l=0, w(0)=0; \quad l=3, v(0)=0.$$

$$l=1. \quad \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V = V'' = -R(\dot{V}, \dot{\gamma}) \dot{\gamma} = 0 \quad (V(0)=0)$$

$l=2$ Similar to " $l=1$ "

$$f^{(4)}_{(0)} = \sum_{l=0}^4 \binom{4}{l} \langle \nabla_{\dot{\gamma}}^{(4-l)} V, \nabla_{\dot{\gamma}}^l W \rangle$$

$l=1, 3$, may not be zero

$$= 4 \langle \nabla_{\dot{\gamma}(0)}^3 V, \nabla_{\dot{\gamma}(0)}^1 W \rangle + 4 \langle \nabla_{\dot{\gamma}(0)}^1 V, \nabla_{\dot{\gamma}(0)}^3 W \rangle$$

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V = R(\dot{\gamma}, V) \dot{\gamma}$$

$$\nabla_{\dot{\gamma}} (\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V) = \nabla_{\dot{\gamma}} (R(\dot{\gamma}, V) \dot{\gamma})$$

$\Downarrow$

$$\begin{aligned} \nabla_{\dot{\gamma}(0)}^{(3)} V &= (\nabla_{\dot{\gamma}} R)(\dot{\gamma}, V) \dot{\gamma} - R(\dot{\gamma}, V) \nabla_{\dot{\gamma}} \dot{\gamma} - R(\nabla_{\dot{\gamma}} \dot{\gamma}, V) \dot{\gamma} \\ &\quad - R(\dot{\gamma}, \nabla_{\dot{\gamma}} V) \dot{\gamma} = 0. \end{aligned}$$

$$f^{(4)}_{(0)} = -8 \langle R(\dot{\gamma}(0), x_p) \dot{\gamma}(0), \gamma_p \rangle$$

$$\text{Hence } \langle V(t), W(t) \rangle = f(t) = \langle x_p, \gamma_p \rangle t^2 - \frac{1}{3} R(\dot{\gamma}(0), x_p, \dot{\gamma}(0), \gamma_p) t^4 + O(t^5)$$

$$V(t) = W(t), \quad \|x_p\| = 1, \quad + O(t^5)$$

$$\|V(t)\|^2 = t^2 - \frac{1}{3} R(\dot{\gamma}(0), x_p, \dot{\gamma}(0), \gamma_p) t^4 + O(t^5).$$

Thm 4.13 In the normal coordinate of p

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + O(|x|^3).$$

Proof: Let $x_p = C^i \partial_i$, $\{x^i\}_{i=1}^n$ is the local coordinate.

$$\gamma(t) = (tc^1, tc^2, \dots, tc^n) \quad \text{red arrow } t \rightarrow h$$

$$\alpha_i(t, s) \exp_p(tc^1, \dots, t(c^i + s), tc^n)$$

$$V_i(t) = \left. \frac{\partial \alpha_i}{\partial s} \right|_{(t, 0)} \text{ is a Jacobi field.}$$

$$V_i'(t) = \partial_i$$

$$g_{ij}(tc^1, \dots, tc^n) = \langle \partial_i, \partial_j \rangle \Big|_{\gamma(t)} \quad T_p M$$

$$\begin{aligned} V_i(t) &= \exp_{p*} \Big|_{(t \partial_i)} \\ &= \frac{1}{t^2} \langle V_i^{(t)}, V_j^{(t)} \rangle \\ &= \frac{1}{t^2} \left(\delta_{ij} t^2 - \frac{1}{3} R(\partial_i, x_p, \partial_j, \gamma_p) t^4 + o(t^3) \right) \\ &= \delta_{ij} - \frac{1}{3} R(\partial_i, x_p, \partial_j, \gamma_p) t^2 + o(t^3) \\ &= \delta_{ij} - \frac{1}{3} R(\partial_i, \partial_k, \partial_j, \partial_l) (C^k t) (C^l t) \\ &\quad + o(t^3) \end{aligned}$$

$$\text{let } x^i = tc^i, \sqrt{(C^1)^2 + \dots + (C^n)^2} = 1.$$

$$\Rightarrow g_{ij}(x^1, \dots, x^n) = \delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l + o(|x|^3) \quad \square$$

Cor 4.14 $\sqrt{\det(g_{ij})} = 1 - \frac{1}{6} R_{kl} x^k x^l + o(|x|^3)$ ← normal coord.

Cor 4.15 $\text{Vol}(B(p, r)) = \omega_n r^n \left(1 - \frac{S(p)}{6(n+2)} r^2 + o(r^3) \right)$

$$\int_{B(p, r)} \sqrt{\det(g_{ij})} dx^1 \dots dx^n$$

Thm 4.16 (Gauss Lemma)

$\gamma_u(t)$ is a geodesic with $\gamma_u(0) = u$, then

$$\langle (\exp_p)_*|_{t_u}(\xi), \dot{\gamma}_u(t) \rangle = \langle u, \xi \rangle$$

Proof: $\alpha(t, s) = \exp_p[t(u + s\xi)]$, $\Upsilon(t) = \frac{\partial}{\partial s} \alpha(t, s)|_{(t, 0)}$

$$\Rightarrow \Upsilon(t) \in T_{\alpha(t, 0)}, \quad \Upsilon'(0) = \xi$$

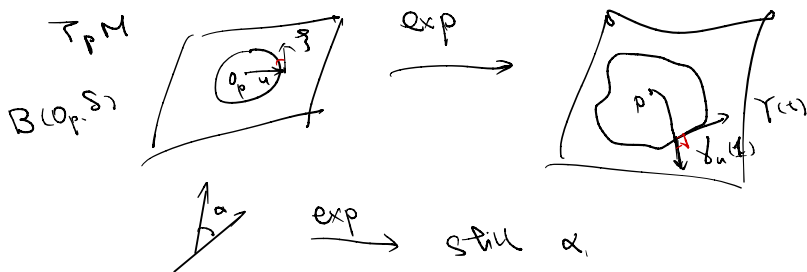
By Lemma 4.9, $\langle \Upsilon(t), \dot{\gamma}_u(t) \rangle$ is a linear function

$$\Rightarrow \langle \Upsilon(t), \dot{\gamma}_u(t) \rangle = t \langle \xi, u \rangle$$

$$(\exp_p)_*|_{t_u}(\xi) = \Upsilon(t)$$

$\parallel$

$$t(\exp_p)_*|_{t_u}(\xi) \Rightarrow \langle (\exp_p)_*|_{t_u}(\xi), \dot{\gamma}_u(t) \rangle = \langle \xi, u \rangle, \quad \square$$



Remark ① $\langle \xi, u \rangle = 0$. $\partial B(p, \delta) \perp \dot{\gamma}_u(1)$

where $B(p, \delta) = \exp_p(B(p, \delta))$

← actually, it's a geodesic ball.

② $(\exp_p)_*$ preserve norm along $\dot{\gamma}_u(t)$

• Length of γ

$\gamma: [a, b] \rightarrow M$, smooth curve ($\|\dot{\gamma}\| > 0$)

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt.$$

$S(t) = \int_a^t \|\dot{\gamma}(u)\| du$ is the arc-parameter.

Length can be defined on piecewise smooth curve.

$$d(p, q) = \inf \{ L(\gamma) \mid \gamma(a) = p, \gamma(b) = q, \gamma \text{ is piecewise smooth} \}.$$

Prop 4.17 (M^n, d) is metric space.

$$\textcircled{1} \quad d(p, q) \geq 0 \quad " = " \text{ iff } p = q$$

$$\textcircled{2} \quad d(p, q) = d(q, p)$$

$$\textcircled{3} \quad d(p, q) + d(q, r) \geq d(p, r).$$

$$B(p, \delta) = \{ x \in M \mid d(p, x) < \delta \}.$$

$$T_p M \quad \{x^1, \dots, x^n\} \quad r = \sqrt{\sum_{i=1}^n (x^i)^2}$$

$$\exp_{p*} \left(\frac{\partial}{\partial r} \right) = \nabla r, \quad r(x) = d(p, x).$$

Lecture 5

Prop 4.18 (local minimizing)

$C: [0,1] \rightarrow T_p M$ with $C(t) = tu$, $u \in T_p M$, $\exp_p$ defined on U open.

$C([0,1]) \subset U$. Let $\varphi: [0,1] \rightarrow U$, $\varphi(0) = 0 \in T_p M$.

$\varphi(1) = u$. Let γ_u be the geodesic $\exp_p(C(t))$,

$\tilde{\gamma}(t) = \exp(\varphi(t))$. Then $L(\tilde{\gamma}) \geq L(\gamma_u)$

 (exp is regular) " = " iff $\tilde{\gamma}$ is a reparameterization of γ_u .

Proof: Let $r(t) = \|\varphi(t)\|$, then $\varphi(t) = r(t) e(t)$

$e(t) \in T_p M$

$$\dot{\varphi}(t) = \dot{r}(t) e(t) + r(t) \dot{e}(t)$$

$e(t)$ is the radial derivation.

By Gauss Lemma,

$$\left\langle (\exp_p)_* \Big|_{\varphi(t)} (e(t)), (\exp_p)_* \Big|_{\varphi(t)} (\dot{e}(t)) \right\rangle = \langle e(t), \dot{e}(t) \rangle = 0.$$

$$\| (\exp_p)_* \Big|_{\varphi(t)} (e(t)) \| = \| e(t) \| = 1$$

$$\|\dot{\tilde{\gamma}}(t)\| = \| (\exp_p)_* \Big|_{\varphi(t)} (\dot{\varphi}(t)) \|$$

$$= \| (\exp_p)_* \Big|_{\varphi(t)} (\dot{r}(t) e(t) + r(t) \dot{e}(t)) \|$$

$$\geq \| (\exp_p)_* |_{\varphi(t)} (\dot{r}(t) e(t)) \| = |\dot{r}(t)|$$

$$L(\tilde{\gamma}) = \int_0^1 \|\dot{\tilde{\gamma}}(t)\| dt \geq \int_0^1 |\dot{r}(t)| dt \geq \int_0^1 \frac{d}{dt} \| \gamma_u(t) \| dt$$

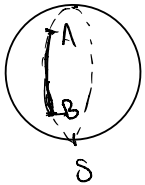
$$= L(\gamma_u)$$

$$r(t) \dot{e}(t) = 0$$

$$\Rightarrow e(t) \equiv e$$

$$\Rightarrow \varphi(t) = r(t) e \quad \text{3rd ~~last~~ term is 0. up to.}$$

geodesic may be ~~not~~ shortest
the line.



§ 5. Variation of Length and Energy

$$\gamma: [a, b] \rightarrow M$$

$$\frac{d}{ds} \Big|_{s=0} L(\gamma_s(t)) = \int \dots dt = 0.$$

$$C_{p,q} = \{ \text{piecewise, smooth } \gamma, \text{ from } p \text{ to } q \}. \quad [a, b]$$

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt$$

$$E(\gamma) = \frac{1}{2} \int_a^b \|\dot{\gamma}(t)\|^2 dt$$

By Cauchy ineq.

$$= 2(b-a) E(\gamma)$$

$$L^2(\gamma) = \left(\int_a^b \|\dot{\gamma}(t)\| dt \right)^2 \leq (b-a) \int_a^b \|\dot{\gamma}(t)\|^2 dt$$

$$= \text{iff } |\dot{\gamma}(t)| = \text{const.}$$

Prop 5.1 $\gamma \in C_{p,q}$ minimize $E(\gamma) \Leftrightarrow \gamma$ minimize $L(t)$ and

$$|\dot{\gamma}| = \text{const}$$

Thm 5.2 (1st Variation of $E(\gamma)$)

Given any variation α of a smooth curve $\gamma: [a,b] \rightarrow M$

$$\alpha: [a,b] \times (-\epsilon, \epsilon) \rightarrow M, \quad \alpha(t, 0) = \gamma(t), \quad \alpha(t, \epsilon) = \gamma_\epsilon(t).$$

$$\begin{aligned} \frac{d}{ds} E(\gamma_s) &= \frac{1}{2} \frac{d}{ds} \int_a^b \left\langle \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle dt \\ &= \int_a^b \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right\rangle dt. \\ &= \int_a^b \left\langle \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle dt, \\ &= \int_a^b \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle - \left\langle \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial t} \right\rangle dt \\ &= \left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle \Big|_{t=a}^{t=b} - \int_a^b \left\langle \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial t} \right\rangle dt \end{aligned}$$

$$\frac{d}{ds} \Big|_{s=0} E(\gamma_s) = - \int_a^b \left\langle v(t), \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \right\rangle dt$$

$$v(t) = \frac{\partial \alpha}{\partial s} \Big|_{(t,0)}$$

$$- \langle v(a), \dot{\gamma}(a) \rangle + \langle v(b), \dot{\gamma}(b) \rangle$$



Consider 定端变分. $v(a) = v(b) = 0$

if $\alpha(a, s) = \gamma(a)$, $\alpha(b, s) = \gamma(b) \quad \forall s$

$$\text{Then } \frac{d}{ds} \Big|_{s=0} E(\gamma_s) = - \int_a^b \langle V(t), \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \rangle dt$$

γ is critical curve of $E(\gamma_s)$ iff $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \quad \forall t$.

Thm (1st Variation of Length)

Given any variation α , $\gamma: [a, b] \rightarrow M$

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} L(\gamma_s) &= - \int_a^b \langle V(t), \nabla_{\frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}} \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} \rangle dt \\ &\quad - \langle V(a), \frac{\dot{\gamma}(a)}{\|\dot{\gamma}(a)\|} \rangle + \langle V(b), \frac{\dot{\gamma}(b)}{\|\dot{\gamma}(b)\|} \rangle \end{aligned}$$

For α with fixed endpoints, then

$$\begin{aligned} \frac{d}{ds} \Big|_{s=0} L(\gamma_s) &= - \int_a^b \langle V(t), \nabla_{\frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}} \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} \rangle dt \\ \nabla_{\frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}} \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|} &\equiv 0. \end{aligned}$$

Cor 5.4 If γ is a piecewise smooth curve $\gamma: [a, b] \rightarrow M$

i.e. $a = t_0 < t_1 < \dots < t_k < t_{k+1} = b$

γ is smooth $[t_i, t_{i+1}]$, $\alpha(t, s)$ is a variation γ .

s.t. $\gamma_s(t) = \alpha(t, s)$ is also piecewise smooth for all s .

$s \mapsto \alpha(t, s)$ is smooth. $V(t) = \frac{\partial \alpha}{\partial s} \Big|_{(t, 0)}$

$$\left. \frac{d}{ds} \right|_{s=0} E(\gamma_s) = - \int_a^b \langle V(t), \nabla_{\dot{\gamma}} \dot{\gamma} \rangle dt - \langle V(a), \dot{\gamma}(a) \rangle + \langle V(b), \dot{\gamma}(b) \rangle \\ - \sum_{i=1}^k \langle V(t_i), \dot{\gamma}(t_i^+) - \dot{\gamma}(t_i^-) \rangle$$

$$\left. \frac{d}{ds} \right|_{s=0} L(\gamma_s) = - \int_a^b \langle V(t), \nabla_{\dot{\gamma} / \|\dot{\gamma}\|} \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \rangle dt - \langle V(a), \frac{\dot{\gamma}(a)}{\|\dot{\gamma}(a)\|} \rangle \\ + \langle V(b), \frac{\dot{\gamma}(b)}{\|\dot{\gamma}(b)\|} \rangle - \sum_{i=1}^k \langle V(t_i), \frac{\dot{\gamma}(t_i^+)}{\|\dot{\gamma}(t_i^+)\|} - \frac{\dot{\gamma}(t_i^-)}{\|\dot{\gamma}(t_i^-)\|} \rangle$$

where $\dot{\gamma}(t_i^+) = \lim_{t \rightarrow t_i^+} \dot{\gamma}(t)$

Prop 5.5 If γ is a critical curve of $E(\gamma_s)$ for any variation $\alpha(t, s)$ with $\alpha(a, s) = \gamma(a)$, $\alpha(b, s) = \gamma(b)$.

Then $\gamma(s)$ is C^1 and is a geodesic.

 Proof: Step 1: Choose α s.t. $V(t_i) = 0$ for any $V(t)$

$$0 = \left. \frac{d}{ds} \right|_{s=0} E(\gamma_s) = - \int_a^b \langle V(t), \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \rangle dt$$

$$\Rightarrow \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \equiv 0, \quad \forall t \in (t_i, t_{i+1})$$

Step 2 By the result in step 1

$$0 = \left. \frac{d}{ds} \right|_{s=0} E(\gamma_s) = - \sum_{i=1}^k \langle V(t_i), \dot{\gamma}(t_i^+) - \dot{\gamma}(t_i^-) \rangle$$

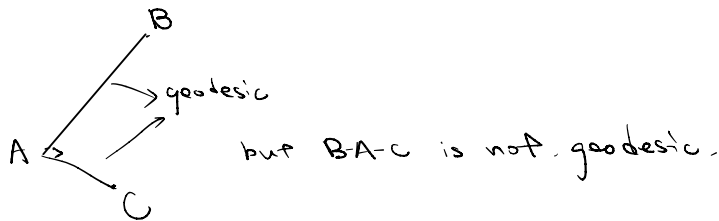
Choose $V(t_i)$, $V(t_i) = \dot{\gamma}(t_i^+) - \dot{\gamma}(t_i^-)$

$$\Rightarrow \dot{\gamma}(t_i^+) = \dot{\gamma}(t_i^-)$$

$$\Rightarrow \gamma \text{ is } C^1.$$

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) \equiv 0 \quad \forall t \in [a, b]$$

□



Thm 5.6 (2nd variation of $E(\gamma_s)$).

Suppose $\gamma: [a, b] \rightarrow M$ is a geodesic, variation $\alpha(t, s)$ satisfies $\alpha(t, 0) = \gamma$. Then

$$\begin{aligned} \frac{d^2}{ds^2} \Big|_{s=0} E(\gamma_s) &= \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \dot{\gamma} \right\rangle \Big|_{t=a}^{t=b} + \int_a^b \langle R(\dot{\gamma}, V) \dot{\gamma}, V \rangle + \langle \nabla_{\dot{\gamma}} V, \nabla_{\dot{\gamma}} V \rangle dt \\ &= \left(\left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \dot{\gamma} \right\rangle + \langle V, \nabla_{\dot{\gamma}} V \rangle \right) \Big|_{t=a}^{t=b} \quad (*) \\ &\quad - \int_a^b \langle V, \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V - R(\dot{\gamma}, V) \dot{\gamma} \rangle dt \end{aligned}$$

If α has fixed endpoints $\Rightarrow V(a) = 0, V(b) = 0, \quad V = \frac{\partial \alpha}{\partial s} \Big|_{(t, 0)}$

$$\begin{aligned} \Rightarrow \frac{d^2}{ds^2} \Big|_{s=0} E(\gamma_s) &= \int_a^b \left(\langle R(\dot{\gamma}, V) \dot{\gamma}, V \rangle + \langle \nabla_{\dot{\gamma}} V, \nabla_{\dot{\gamma}} V \rangle \right) dt \\ &= \int_a^b \langle V, -\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V + R(\dot{\gamma}, V) \dot{\gamma} \rangle dt \end{aligned}$$

Proof: $\frac{d^2}{ds^2} \Big|_{s=0} E(\gamma_s) = \int_a^b \frac{\partial}{\partial s} \left\langle \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle dt$

$$= \int_a^b \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle + \left\langle \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial t} \right\rangle dt$$

$$= \int_a^b \left\langle R\left(\frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t}\right) \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle dt$$

$$+ \int_a^b \left\langle \nabla_{\frac{\partial \alpha}{\partial t}} \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle dt \quad \text{II}$$

$$+ \int_a^b \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s} \right\rangle dt \quad \text{III} \quad (1)$$

$$\text{II} = \int_a^b \left[\frac{\partial}{\partial t} \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle - \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial t} \right\rangle \right] dt$$

$$= \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle \Big|_{t=a}^{t=b} - \int_a^b \left\langle \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s}, \underbrace{\nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial t}}_{=0 \text{ when } s=0} \right\rangle dt \quad (2)$$

By ① and ②, $\Rightarrow$ the first equality of (*).

$$\text{III} = \int_a^b \left[\frac{\partial}{\partial t} \left\langle \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s} \right\rangle - \left\langle \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s} \right\rangle \right] dt$$

$$= \left\langle \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s} \right\rangle \Big|_{t=a}^{t=b} - \int_a^b \left\langle \frac{\partial \alpha}{\partial s}, \nabla_{\frac{\partial \alpha}{\partial t}} \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s} \right\rangle dt \quad (3)$$

By ①, ② and ③ $\Rightarrow$ the 2nd equality of (*).

Thm 5.7 (Synge Theorem)

Suppose (M, g) has positive sectional curvature. M is compact. (1) If $\dim M = 2n$ and M is oriented, M is simply connected.

(2) If $\dim M = 2n+1$, then M is orientable.

Fact 5.8 Suppose (M, g) is compact mfd. If $\pi_1(M)$ is nontrivial. $\exists$ a minimal closed geodesic $\gamma \in [L]$, if $[L] \neq 0$. 闭路等价类构成的群

thm 5.7 中, 不必取定闭曲线

$$\alpha(s, t) = \exp_{\gamma(t)}(s V(t))$$

$$\left. \frac{\partial \alpha}{\partial s} \right|_{(t, 0)} = V(t)$$



$\gamma(t)$, geodesic. $V(t)$ is a parallel vector field along γ .

$$\nabla_{\dot{\gamma}(t)} V(t) \equiv 0. \quad \text{Fixed } t. \quad \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial s} \equiv 0.$$

Lemma 5.9 Suppose (M, g) is oriented, $\gamma: [a, b] \rightarrow M$ smooth loop consider the parallel translate: $P_{a,b}^\gamma: T_p M \rightarrow T_p M$.

$$\text{Then } \det(P_{a,b}^\gamma) = 1$$

$$\text{Proof: } P_{a,b}^\gamma \in O(T_p M) \Rightarrow \det(P_{a,b}^\gamma) = 1 \text{ or } -1$$

$$\text{Target: } \det(P_{a,b}^\gamma) > 0.$$

(M, g) is oriented $\Rightarrow \exists$ n -form ω , s.t. $\omega(e_1, \dots, e_n) > 0$.

for $p \in M$ $\{e_1, \dots, e_n\} \in T_p M$

$$\text{let } e_i(t) = P_{a,t}^\gamma(e_i)$$

Then $\omega(e_1(t), \dots, e_n(t)) \neq 0$ for all t

$\omega(e_1(t), \dots, e_n(t))$ smooth and is positive at $t=a$

Notice $\omega(e_1(b), \dots, e_n(b)) = (\det P_{a,b}^T) \omega(e_1, \dots, e_n)$

$\Rightarrow \det(P_{a,b}^T) > 0.$

□

(1) If M is not simply connected

$\exists$ a minimal geodesic $\gamma, \gamma: [0,1] \rightarrow M$ $2n$

By Lemma 5.9 $\det(P_{0,1}^T) = 1, P_{0,1}^T \in SO(T_p M).$

all real eigenvalue of $P_{0,1}^T$ is $1, -1$ or imaginary roots

$P_{0,1}^T(\dot{\gamma}(0)) = \dot{\gamma}(0), \dot{\gamma}(0)$ is an eigen ^{even-dim} vector of 1. (pairwise)

$\dim(\text{eigen space of } -1) = \text{even}$ $X_p \perp \dot{\gamma}(0)$

$\Rightarrow \dim(\text{eigen space of } 1) = \text{even}$ $\exists X_p, X_p$ is also an eigen vector of 1 .

Let $X(t)$ the parallel translate of X_p along γ .

$$\alpha(t,s) = \exp_{\gamma(t)}(sX(t))$$

$$\left. \frac{d^2}{ds^2} \right|_{s=0} E(\gamma_s) = \int_0^1 \langle X, -\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X(t) + R(\dot{\gamma}, X) \dot{\gamma}(t) \rangle dt$$

$$= \int_0^1 \langle R(\dot{\gamma}, X) \dot{\gamma}, X \rangle dt \quad \dot{\gamma} \perp X$$

$$< 0$$

Contradiction!

(2) If M is not orientable, $\exists \{e_1(t), \dots, e_n(t)\}$

$$e_1(1) \wedge \dots \wedge e_n(1) = -e_1(0) \wedge \dots \wedge e_n(0)$$

$(-1)^{2n+1} = -1$

$$\text{let } \bar{e}_i(t) = p_{0,t}^{-1}(e_i(0))$$

$$\bar{e}_1(t) \wedge \dots \wedge \bar{e}_n(t) = f(t) e_1(t) \wedge \dots \wedge e_n(t)$$

$$\dim M = 2n+1, \quad \Rightarrow \quad \det(p_{0,1}^{-1}) = -1, \quad \left(\begin{array}{l} \text{Assume} \\ [f] \text{ is minimal geo.} \end{array} \right)$$

$$p_{0,1}^{-1}(\dot{\gamma}(0)) = \dot{\gamma}(0)$$

$$p_{0,1}^{-1}|_{E_p}$$

$$E_p \oplus \{\dot{\gamma}(0)\} = T_p M$$

eigenvalue $1, -1$ imaginary roots
 odd odd pairwise even

$$\therefore \exists x_p \neq 0 \in E_p$$

x_p is the eigen vector of 1.

□

Cor 5.8 (M,g) positive sectional curv. $\dim M = 2n$

M is not oriented. Then $\pi_1(M) = \mathbb{Z}_2$.

Proof. $\downarrow \exists \tilde{M} \xrightarrow{\pi} M$ $\mathbb{Z}_2$ -covering map.

$$\tilde{M} \cong \mathbb{R}P^n$$

$(\tilde{M}, \pi^*g)$ oriented, + sec > 0. Singer thm $\Rightarrow$ simply connected

$$S^{2n-1} \xrightarrow{\quad} \mathbb{R}P^{2n}$$

quoto. space.

Remark: $\mathbb{R}P^2 \times \mathbb{R}P^2$ doesn't admit a metric with sec > 0
 its $\downarrow$
 fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$

Lecture 6

We consider a general variation $\gamma_{s,r} = \alpha(t, s, r)$

$$X(t) = \left. \frac{\partial \alpha}{\partial s} \right|_{(t,0,0)}, \quad Y(t) = \left. \frac{\partial \alpha}{\partial r} \right|_{(t,0,0)}$$

$$\begin{aligned} \left. \frac{\partial^2}{\partial s \partial r} \right|_{s=r=0} E(\gamma_{s,r}) &= \int_0^b \left(\langle R(\dot{\gamma}, X) \dot{\gamma}, Y \rangle + \langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} Y \rangle \right) dt \\ &= \int_a^b \langle R(Y, X) \dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X, Y \rangle dt \\ &\quad + \langle \nabla_{\dot{\gamma}} X, Y \rangle \Big|_a^b \end{aligned}$$

Hessian of E .

Def 5.1 (Index Form) $\gamma: [a, b] \rightarrow M$ geodesic

$V_\gamma = \{ \text{all piecewise smooth vector fields along } \gamma \}$. Then

$$\begin{aligned}
 I(\gamma, Y) &= \int_0^b (\langle R(\gamma, x) \dot{\gamma}, Y \rangle + \langle \nabla_{\dot{\gamma}} X, \nabla_{\dot{\gamma}} Y \rangle) dt \\
 &\quad \int_a^b \langle R(\gamma, x) \dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} X, Y \rangle dt \\
 &\quad + \langle \nabla_{\dot{\gamma}} X, Y \rangle \Big|_a^b - \sum_{i=1}^k \langle \nabla_{\dot{\gamma}(t_i^+)} X - \nabla_{\dot{\gamma}(t_i^-)} X, Y \rangle
 \end{aligned}$$

$$a < t_1 < \dots < t_k < b$$

$$\text{Let } V_Y^\circ = \{ x \in V_Y \mid x(a) = x(b) = 0 \}$$

Lemma 5.10 Let $V \in V_Y$. Then $V \in J_{ac}(V) \iff I(V, X) = 0$
 $\forall X \in V_Y^\circ$

Proof: " $\Rightarrow$ " trivial

" $\Leftarrow$ " Step 1. Suppose $f: [a, b] \rightarrow \mathbb{R}$, smooth $f \geq 0$.

$$f(t_i) = 0, \quad f(t) > 0, \quad t \neq t_i$$

$$\text{Let } X = f(t) (R(\gamma, v) \dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V)$$

$$\text{Then } X \in V_Y^\circ$$

$$\Rightarrow 0 = I(V, X) = \int_a^b f(t) \|R(\gamma, v) \dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V\|^2 dt$$

$$\Rightarrow R(\gamma, v) \dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V = 0$$

Step 2 Choose $Y \in V_Y^\circ$, $Y(a) = Y(b) = 0$

$$Y(t_i) = \nabla_{\dot{\gamma}(t_i^+)} V - \nabla_{\dot{\gamma}(t_i^-)} V, \quad 1 \leq i \leq k$$

By Step 1.

$$\Rightarrow 0 = I(V, Y) = - \sum_{i=1}^k \|\nabla_{\dot{\gamma}(t_i^+)} V - \nabla_{\dot{\gamma}(t_i^-)} V\|^2$$

$$\Rightarrow \nabla_{\dot{\gamma}(t_i^+)} V = \nabla_{\dot{\gamma}(t_i^-)} V$$

$$V \in C^1$$

$$\Rightarrow V \in \text{Joc}(\gamma).$$

□

func. , $p(\nabla f|_p = 0)$.

$$\textcircled{1} \text{Hess } f|_p > 0. \Rightarrow f|_p \text{ is minimal.}$$

$$\textcircled{2} \exists X, \text{Hess } f_p(X, X) < 0. \Rightarrow f_p \text{ is not a extreme value}$$

$$\textcircled{3} \text{Hess } f|_p \geq 0. \exists X \neq 0. \text{Hess } f_p(X, X) = 0 \Rightarrow \text{unknown.}$$

Thm 5.11 Suppose $\gamma: [a, b] \rightarrow M$ geodesic from p to q .

I^γ is the index form. Then

$$(1) I > 0 \text{ in } V_\gamma^\circ \iff p \text{ has no conj. point along}$$

$$\gamma|_{[a, b]}$$

Moreover, γ is the unique minimal curve from p to q "nearby"

i.e. $\exists \varepsilon > 0$. $\tilde{\gamma}$ is any piecewise smooth from p to q .

$$\tilde{\gamma}: [a, b] \rightarrow M$$

$$d(\gamma(t), \tilde{\gamma}(t)) \leq \varepsilon. \forall t \in [a, b], \text{ then } L(\tilde{\gamma}) \geq L(\gamma).$$

$$(2) \exists x \in V_y^0, I(x, x) < 0 \Leftrightarrow \exists \bar{t}, \bar{t} \in (a, b), q = \gamma(\bar{t})$$

is a conj. point of p .

Moreover, γ is not a minimal geodesic.

$$(3) I \geq 0 \text{ in } V_y^0 \text{ and } \exists \text{ nonzero } X, I(X, X) = 0$$

$$\Leftrightarrow q \stackrel{\gamma(\bar{t})}{=} \text{is the first conj. point to } p \text{ along } \gamma.$$

Proof = (2) "Moreover part". If $I(x, x) < 0$, then $\exists$ variation

$$\alpha(t, s) = \gamma_s(t), \quad \left. \frac{d^2}{ds^2} \right|_{s=0} E(\gamma_s) < 0.$$

$$\text{Let } f(s) = E(\gamma_s), \quad f'(s) \big|_{s=0} = 0 \text{ (} \gamma \text{ is geodesic)}$$

$$f''(s) \big|_{s=0} < 0$$

$$\text{then } f(s) < f(0) \quad \forall s \in (-\delta, \delta) \text{ for some } \delta > 0$$

$$E(\gamma_s) < E(\gamma) \Rightarrow \gamma \text{ is not minimal.}$$

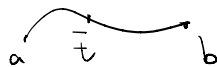
" $\Leftarrow$ " If $\bar{t} < b$, $q = \gamma(\bar{t})$ is conj. point to p

$$\text{Choose a nonzero } X \in \text{Jac}(\gamma) \text{ and } X(a) = X(\bar{t}) = 0$$

$$\nabla_{\dot{\gamma}(\bar{t})} X \neq 0.$$

Choose Y along γ .

$$Y(a) = Y(b) = 0, \quad Y(\bar{t}) = -\nabla_{\dot{\gamma}(\bar{t})} X$$



$$\text{Let } \gamma_1 = \gamma|_{[a, \bar{t}]}, \quad \gamma_2 = \gamma|_{[\bar{t}, b]}, \quad Y_1 = Y|_{[a, \bar{t}]}, \quad Y_2 = Y|_{[\bar{t}, b]}.$$

$\forall \eta > 0$, let

$$Z_\eta = \begin{cases} Z_\eta^1(t) = X(t) + \eta Y_1(t) & , \quad a \leq t \leq \bar{t} \\ Z_\eta^2(t) = \eta Y_2(t) & \bar{t} \leq t \leq b \end{cases}$$

$$I^1(Z_\eta(t), Z_\eta(t)) = -\eta \|\nabla_{\dot{X}(t)} X\|^2 + \eta^2 I^1(Y_1, Y_1) + \eta^2 I^2(Y_2, Y_2)$$

$$\eta > 0 \quad = -\left(\frac{\|\nabla_{\dot{X}(t)} X\|^2}{\eta_0} - \eta (I^1 + I^2) \right) \eta$$

$$\eta \rightarrow 0^+ \quad < 0, \\ \text{small enough } \eta.$$

(1) "Moreover Part" If p has no conj. point along γ .

$\exists$ a partition of $[a, b]$

$$a = t_0 < t_1 < \dots < t_k < t_{k+1} = b$$

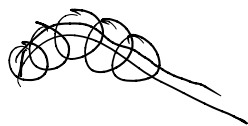
$$\exists V_i (\text{open}) \subset T_p M, \quad [t_i, t_{i+1}] \dot{\gamma}(t) \subset V_i$$

For small ε , we may assume $\tilde{\gamma}([t_i, t_{i+1}]) \subset U_i = \exp_p(V_i)$

$$\text{Define } \varphi(t) = \left(\exp_p|_{V_i} \right)^{-1}(\tilde{\gamma}(t)), \quad t_i \leq t \leq t_{i+1}$$

$$\varphi(t) \text{ is a curve in } T_p M, \quad \tilde{\gamma} = \exp_p(\varphi(t)).$$

By prop 4.18, $\Rightarrow L(\tilde{\gamma}) \geq L(\gamma)$



$$3). " \Leftarrow " \quad \forall c \in (a, b), \quad \forall X \in V_y^0 \quad X = \sum X^i e_i(c).$$

Let $\{e_i(c)\}$ be a D.N.B. along γ (parallel vector fields)

$$\text{Let } X^c(t) = X^i \left(\frac{(b-a)(t-a)}{c-a} + a \right) e_i(t)$$

$$X^c(a) = X^c(c) = 0.$$

$$\gamma^c = \gamma|_{[a, c]}, \quad X^c \in V_{\gamma^c}^0, \quad X^c \rightarrow X \text{ when } c \rightarrow b$$

p has no conj. point along γ^c

$$\text{By (1)} \Rightarrow I^{\gamma^c}(X^c, X^c) > 0$$

$$I^{\gamma}(X, X) = \lim_{c \rightarrow b} I^{\gamma^c}(X^c, X^c) \geq 0.$$

$$\gamma(b) \text{ conj. to } p. \Leftrightarrow \exists J, J(a) = J(b) = 0, J \text{ non zero}$$

$\Downarrow$

$$I(J, J) = 0.$$

Cor 5.12 (Fundamental Index Lemma)

$\gamma: [a, b] \rightarrow M$, geodesic, $p = \gamma(a)$ has no conj. point along γ .

$V \in \text{Jac}(\gamma)$, $x \in V_\gamma$, $V(a) = X(a)$, $V(b) = X(b)$

$$I(V, V) \leq I(X, X). \quad "=" \text{ iff } X = V.$$

Proof: $V - X \in V_\gamma^\circ$, $I(V - X, V - X) > 0$ by Thm 5.11(1) if $V - X \neq 0$.

$$\left. \begin{array}{l} I(V, X) = 0 \\ I(V - X, V - X) > 0 \end{array} \right\} \Rightarrow I(X, X) - I(V, V) > 0$$

□

Def 5.13 (Geodesic complete) (M, g) is geodesic complete.

if $\forall p \in M$, $U \in T_p M$, the geodesic $\gamma_U(t)$ with $\gamma_U(0) = p$.

$\gamma_U(0) = U$, is defined for all $t \in (-\infty, +\infty)$.

Def 5.14 (Metric complete). (M, d) is a complete metric

space i.e.

$\forall$ Cauchy sequence $\{p_i\} \subset M$, there exists a point

$$p_\infty \in M, \quad \lim_{i \rightarrow \infty} d(p_i, p_\infty) = 0$$

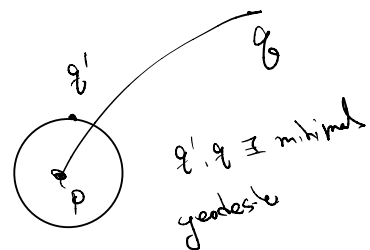
Lemma 5.15 (M.g) for $p, q \in M$, $\exists \delta > 0$ $(\exp_p)|_{B(0_p, \delta)}$ is diffeom.

Note $(\exp_p)(B(0_p, \delta)) = B(p, \delta)$.

Then $\exists q' \in \partial B(p, \delta)$ s.t. $d(p, q) = \delta + d(q', q)$

(Note $d(p, q') = \delta$, for small δ)

Proof: By Prop 4.18 $d(p, x) = \delta, \forall x \in \partial B(p, \delta)$



Now we show $d(q, \partial B(p, \delta)) = d(p, q) - \delta$

$\partial B(p, \delta)$ is close. $\Rightarrow \exists q' \in \partial B(p, \delta), d(q, q') = d(q, \partial B(p, \delta))$

$$d(q', q) \geq d(p, q) - d(p, q') = d(p, q) - \delta \quad (1)$$

$\forall \varepsilon > 0, \exists$ piecewise smooth γ from p to q .

$$L(\gamma) - d(p, q) < \varepsilon.$$

$$\exists c \in (a, b) \quad \gamma(c) \in \partial B(p, \delta)$$

$$L(\gamma|_{[a, c]}) + L(\gamma|_{[c, b]}) \leq d(p, q) + \varepsilon$$

$$\forall \delta \quad d(q, \partial B(p, \varepsilon)) = d(q, q')$$

$$\text{Let } \varepsilon \rightarrow 0. \quad \delta + d(q, q') \leq d(p, q) \quad (2)$$

$$\text{By } (1) + (2) \quad \delta + d(q, q') = d(p, q)$$

□

Lemma 5.16 If any geodesic through $p \in M$ is complete.

(i.e. $\gamma(t)$ is defined for all $t \in (-\infty, +\infty)$, if $\gamma(0) = p$)

Then for any $q \in M$, $\exists$ a smooth minimal ^{some p} geodesic from p to q .

Proof: By Lemma 5.15

$$d(p, q) = d(p, q') + d(q', q) \quad \text{and} \quad d(q', q) = \delta$$

where $q' \in \partial B(p, \epsilon)$

Suppose $\|v\| = 1$, $v \in T_p M$, $\gamma = \gamma_v(s)$ geodesic $\gamma(0) = v$

$\gamma(0) = p$, $A = \sup \{ t > 0 \mid d(p, q) = d(p, \gamma(t)) + d(\gamma(t), q), t \leq d(p, q) \}$

Need to show $A = d(p, q)$

Suppose $A < d(p, q)$. Let $A = t_0$.

$$d(\gamma(t_0), q) = d(p, q) - d(p, \gamma(t_0))$$

By Lem 5.15. $\exists q''$ and $\epsilon_2 > 0$

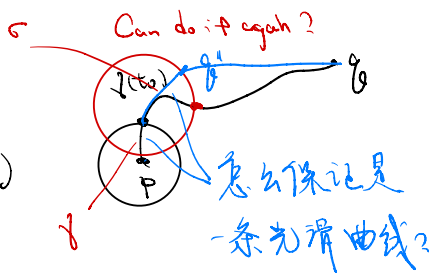
$$d(\gamma(t_0), q) = d(\gamma(t_0), q'') + d(q'', q)$$

$$d(p, q'') \geq d(p, q) - d(q'', q)$$

$$= [d(\cancel{v(t_0)}, q) + d(p, v(t_0))] - [d(\cancel{v(t_0)}, q) - d(v(t_0), q'')] =$$

$$= d(p, v(t_0)) + d(v(t_0), q')$$

$$\Rightarrow d(p, q'') = d(p, v(t_0)) + d(v(t_0), q')$$



Let σ be the minimal geodesic realizing $d(\gamma(t_0), q'')$

$$\gamma|_{[0, t_0]}$$

$\gamma|_{[0, t_0]} \cup \sigma$ realizing $d(p, q'')$

By the 1st variation of L $\frac{\partial L(\alpha(t, s))}{\partial s} \Big|_{s=0} = 0$

$\Rightarrow \gamma|_{[0, t_0]} \cup \sigma$ is a C^1 geodesic.

$\sup\{t_1, \dots\} \geq A + \varepsilon > A$ Contradiction!

□

Thm 5.17 (Hopf-Rinow) (M^n, g) Riem. manifold (Connected)

The following are equivalent.

① (M, d) is metric complete

② Any geodesic through p is complete

③ (M, g) is geodesic complete

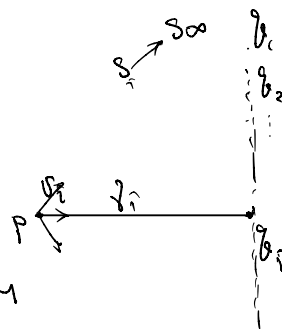
Proof: ③ $\Rightarrow$ ② trivial

② $\Rightarrow$ ① Suppose $\{q_i\}$ Cauchy sequence

Let $U_i = \dot{\gamma}_i(0)$ $\|U_i\| = 1$, $U_i \in T_p M$. $\exists U_\infty = \lim_{i \rightarrow \infty} U_i$

$$\gamma_i(s_i) = q_i$$

$$|s_i - s_j| = |d(p, \gamma_i(s_i)) - d(p, \gamma_j(s_j))|$$



$$\{S_i\} \text{ Cauchy sequence} \leq d(\gamma_i(S_{i-1}), \gamma_i(S_j)) = d(q_i, q_j) \leftarrow \text{Cauchy sequence.}$$

$$\exists S_\infty = \lim_{i \rightarrow \infty} S_i$$

$$\Rightarrow \gamma_{V_\infty}(S_\infty) = \lim_{i \rightarrow \infty} \gamma_{V_i}(S_i) = \lim_{i \rightarrow \infty} q_i$$

$$\textcircled{1} \Rightarrow \textcircled{3} \quad \forall p \in M, \quad v \in T_p M, \quad \|v\| \neq 0.$$

$$\gamma(t) \text{ geodesic} \quad \gamma(0) = p, \quad \dot{\gamma}(0) = v$$

$$A = \{t > 0 \mid \gamma(t) \text{ exist}\} \quad \underline{\text{Need } \sup A = +\infty.}$$

□

A is open by the short time existence.

$$\sup A = t_0.$$

$$\exists t_i \in (0, t_0), \quad t_i \rightarrow t_0, \quad \{t_i\} \text{ is Cauchy sequence.}$$

$$\text{By metric complete } \lim_{t_i \rightarrow \infty} \gamma(t_i) \text{ exists}$$

$$\left(d(\gamma(t_i), \gamma(t_j)) \leq L(\gamma|_{[t_i, t_j]}) = |t_i - t_j| \right)$$

$$\Rightarrow t_0 \in A, \quad \Rightarrow \exists \varepsilon > 0, \quad t_0 + \varepsilon \in A.$$

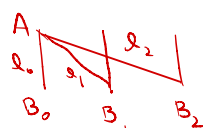
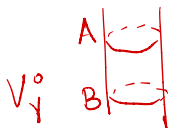
$$\Rightarrow \sup A = t_0 + \varepsilon > t_0.$$

Contradiction!

□

$S^1 \times \mathbb{R}$
no conj. point.

$$I(x, x) > 0.$$



γ_i geodesic
--- each one
minimal local.
but γ_0 is global.

Lecture 17.

• cut locus

Def 5.18 Let (M, g) complete Riem. mnfd. $p \in M$, γ is a geodesic $\gamma(0) = p$. Let $t_0 = \sup \{t \mid \gamma([0, t]) \text{ is a minimizing geodesic}\}$

If $t_0 < +\infty$, then $\gamma(t_0)$ is called a cut point of p

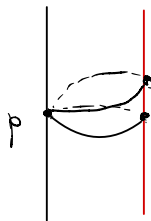
$\text{Cut}(p) = \{q \in M \mid q \text{ is a cut point of } p\}$ is called the cut locus of p

Example (1) $(\mathbb{R}^n, g_E)$, $\text{cut}(p) = \{\emptyset\}$.

(2) (S^n, g_{S^n}) $\text{cut}(p) = \{-p\}$.

(3) $S^1 \times \mathbb{R}$ $p = (e^{i\theta_0}, z_0)$

$$\text{cut}(p) = \{(e^{i(\theta_0 + \pi)}, z) \mid z \in \mathbb{R}\}$$



(4) $(\mathbb{H}^n, g_{\mathbb{H}^n})$, $\text{cut}(p) = \{\emptyset\}$

Lemma 5.19 (M, g) Complete $\forall q \notin \text{cut}(p)$, $\exists$ a unique minimal geodesic from p to q .

Proof: Suppose γ and σ two minimal geodesic from p to q .

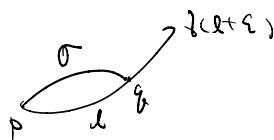
$$L(\gamma) = L(\sigma) = d(p, q) := d$$

$\gamma(s)$ $\sigma(s)$ are parameterizations.

By def of cut(p), $\exists \varepsilon > 0$, $\gamma = [0, l + \varepsilon] \rightarrow M$ still Minimizing

$$L(\gamma|_{[0, l + \varepsilon]}) = L(\gamma|_{[0, l]}) + L(\gamma|_{[l, l + \varepsilon]})$$

$$\stackrel{||}{=} d(\gamma(0), \gamma(l + \varepsilon)) = L(\sigma|_{[0, l]}) + L(\gamma|_{[l, l + \varepsilon]})$$



$\sigma|_{[0, l]} \cup \gamma|_{[l, l + \varepsilon]}$ realizing $d(\gamma(0), \gamma(l + \varepsilon))$ minimizing

By 1st variation of $L(\gamma)$ ($E(\gamma)$)

$\sigma|_{[0, l]} \cup \gamma|_{[l, l + \varepsilon]}$ is a C^1 geodesic.

$$\Rightarrow \sigma|_{[0, l]} = \gamma|_{[0, l]}$$

□

Thm 5.20 (M, g) Complete, $\gamma: [0, \infty) \rightarrow M$ is a geodesic.

$\gamma(t_0)$ is the first cut point of $p = \gamma(0)$ along γ .

Then one of the following holds

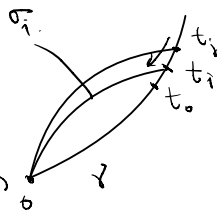
(1) $\gamma(t_0)$ is the first conj. point of p along γ

(2) $\gamma(t_0)$ is the first point along γ with property:

$\exists$ another geodesic σ from p to q and $L(\sigma) = t_0 = L(\gamma|_{[0, t_0]})$

Proof: $t_n \downarrow t_0$.

σ_i be the minimal geodesic from p to $\gamma(t_i)$



$$\Rightarrow \sigma_i \neq \gamma.$$

$$S_i = L(\sigma_i) < L(\gamma|_{[0, t_i]}) = t_i$$

$$\dot{\sigma}_i(0) = U_i, \quad \|U_i\| = 1, \quad \text{by taking a sequence } U_i \rightarrow U_p$$

$$\sigma(s) = \gamma_{U_p}(s).$$

Then σ is a minimizing geodesic from p to $\gamma(t_0)$.

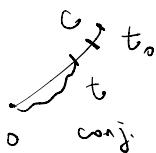
$$(1) \quad U_p = \dot{\gamma}(0), \quad \text{then } \sigma = \gamma|_{[0, t_0]}$$

$$S_i \dot{\sigma}_i(0) \neq t_i \dot{\gamma}(0)$$

$$\exp_p(S_i \dot{\sigma}_i(0)) = \sigma_i(S_i) = \gamma(t_i) = \exp_p(t_i \dot{\gamma}(0))$$

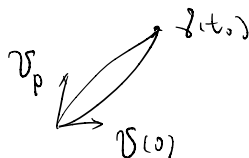
$\Rightarrow \exp_p$ is not a diffe.

By Lem 5.19, Thm 5.11(2)



$$C \in (t, t_0) \quad \angle(X, X) < 0 \Rightarrow \gamma|_{[0, t]} \text{ not minimizing}$$

$$(2) \quad U_p \neq \dot{\gamma}(0)$$



By Lem 5.19, it is unique $\gamma|_{[0, t]}$ $t < t_0$.

□

$$\text{Let } \Sigma_p = \{v \in T_p M \mid \gamma_{t,v} \notin \text{Cut}(p) \quad \forall t \in \mathbb{R} \text{ or } \exists T > 1. \\ \exp_p(Tv) \in \text{Cut}(p)\}$$

$\exp_p|_{\Sigma_p}$ is diffeomorphism.

$$\exp_p(\Sigma_p) = M \setminus \text{Cut}(p)$$

$$\text{Fix } p \in M. \quad r_p(x) = d(p, x)$$

Thm 5.21 $r(x)$ is smooth on $M \setminus (\text{Cut}(p) \cup \{p\})$

Moreover, $\forall q \in M \setminus (\text{Cut}(p) \cup \{p\})$ (By Lem 5.19) $\exists$ a unique minimizing geodesic γ from p to q .

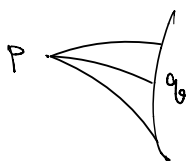
$$\nabla r|_q = \dot{\gamma}|_{r(q)}.$$

$$\text{Proof: } \exp_p(\Sigma_p \setminus \{0_p\}) = M \setminus (\text{Cut}(p) \cup \{p\})$$

$$r_q = L(\gamma|_{r(q)}) = \|\exp_p^{-1}(q)\| \quad \text{smooth}$$

$\sigma(0) = q, \dot{\sigma}(0) = z$. Let γ_s^q be the minimizing geodesic from p to $\sigma(s)$.

$$\gamma_s^q \text{ is a variation of } \gamma. \quad \alpha(t, s) = \gamma_s^q$$



$$V(t) = \frac{\partial}{\partial s} \Big|_{(t,0)} \gamma_s^q$$

$$V(t) \Big|_{q_t} = Z$$

By 1st Variation

$$\begin{aligned} \langle \underbrace{Z, \nabla r \Big|_{q_t}} \rangle &= \frac{d}{ds} \Big|_{s=0} r(r(s)) = \frac{d}{ds} \Big|_{s=0} L(\dot{\gamma}_s^q) \\ &= \langle \underbrace{Z, \dot{\gamma} \Big|_{r(q_t)}} \rangle \end{aligned}$$

□

$$\text{Cut}(p) \quad r_q(x)$$

$$|r_q(x) - r_q(y)| \leq d(x, y) \Rightarrow r_q \text{ is Lip-fnc.}$$

$\Rightarrow \text{Cut}(p)$ is Zero measure.

Hess $r = ?$

For $x \in M \setminus (\text{Cut}(p) \cup \{p\})$, $r(x)$ is smooth

$$\begin{aligned} (\nabla^2 r)_q(X_q, Y_q) &= \langle (\nabla_X \nabla r) \Big|_q, Y_q \rangle \\ &= \underbrace{\langle \nabla_{X_q} \dot{\gamma}^q, Y_q \rangle}_{\parallel \text{ goal.}} + Y_q \in T_q M. \end{aligned}$$

Jacobi

$X(t)$ is a vector field along γ

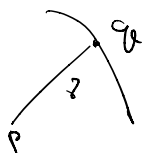
$$\langle \nabla_{\dot{\gamma}} X, Y_q \rangle = I(X, Y_q)$$

$$X(t_0) = X_{q_0}$$

variation

$$\alpha(t, s) = \gamma_s(t) = \beta_t(s)$$

$$\gamma(t) = \gamma_0(t)$$



$$\nabla_{\frac{\partial \alpha}{\partial t}} \alpha(t) = 0, \quad \nabla_{\frac{\partial \alpha}{\partial s}} \beta(s) = 0.$$

$$\alpha(t, s) = \exp_p(t(\dot{\gamma}(0) + \xi(s))), \quad \dot{\gamma}(0) + \xi(s) \in S_p M.$$

$$\Rightarrow \nabla_{\frac{\partial \alpha}{\partial t}} \frac{\partial \alpha}{\partial s} = \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial t} = 0.$$

$$\Rightarrow (\nabla^2 r)_{q_0}(X_{q_0}, Y_{q_0}) = \langle \nabla_{\dot{\gamma}} X, Y_{q_0} \rangle$$

Thm 5.22 Suppose $q_0 \notin \text{Cut}(p) \cup \{p\}$, $r: [0, r(q_0)] \rightarrow M$ is the unique geodesic from p to q_0 . $X \in \text{Jac}(\gamma)$ with $X(0) = 0$.

$$X(r(q_0)) = X_{q_0}. \text{ Then}$$

$$(\nabla^2 r)|_{q_0}(X_{q_0}, Y_{q_0}) = \langle \nabla_{\dot{\gamma}(r(q_0))} X|_{r(q_0)}, Y_{q_0} \rangle$$

Moreover

(1) If $X(t) = \dot{\gamma}(t)$, then

$$(\nabla^2 r)|_{q_0}(X_{q_0}, Y_{q_0}) = \langle \nabla_{\dot{\gamma}} \dot{\gamma}|_{r(q_0)}, Y_{q_0} \rangle = 0$$

(2) $\forall X \in \text{Jac}(\gamma)$, let $X_{q_0} = X(r(q_0))$, then

$$(\nabla^2 r)|_{q_0}(X_{q_0}, X_{q_0}) = \langle \nabla_{\dot{\gamma}(r(q_0))} X|_{r(q_0)}, X_{q_0} \rangle = I^{\dot{\gamma}}(X, X).$$

Thm 5.23 (Cartan - Hardmarc)

(M, g) Complete, $\text{Sec} \leq 0$

$(\exp_p)_*$ ~~非单射~~

(1) $\forall p \in M$, $\exp_p: T_p M \rightarrow M$ is a covering map

(2) If M is simply connected, then $\exp_p$ is diffeo.

Proof: $\forall p$, p has no conj. point.

Suppose p has a first conj. point q along $\gamma(t)$, $q = \gamma(t_0)$

$$\gamma|_{[0, t_0]} \quad \text{first}$$
$$I(\dot{\gamma}, \dot{\gamma}) \geq \underline{I(J, J)}_{>0}, \quad J \in \text{Jac}(\gamma) \cap V_{\gamma}^{\circ}$$

$(I(J, J) > 0 \text{ in } V_{\gamma}^{\circ} \iff p \text{ has no conj. point along } \gamma)$

$\exists$ nonzero $J \in \text{Jac}(\gamma) \cap V_{\gamma}^{\circ}$, $J(0) = J(t_0) = 0$.

$$I(J, J) = \int_0^{t_0} \left(\underbrace{\langle R(\dot{\gamma}, J)\dot{\gamma}, J \rangle + \|\nabla_{\dot{\gamma}} J\|^2}_{\geq 0 \text{ (since Sec} \leq 0)} \right) dt > 0.$$

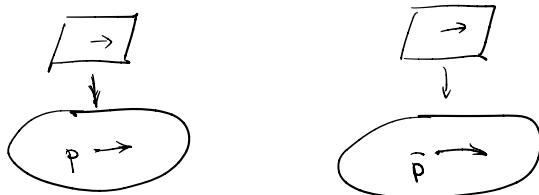
Since J is nonzero.

$$\Rightarrow \nabla_{\dot{\gamma}} J \neq 0$$

Contradiction!

□

§ 6. Comparison Theorem



Let $(M, g), (\bar{M}, \bar{g})$ Riemann manifold.

$\gamma: [0, a] \rightarrow M, \quad \bar{\gamma}: [0, a] \rightarrow \bar{M}$ geodesic

$$\gamma(0) = p$$

$$\bar{\gamma}(0) = \bar{p}$$

$$K^-(t) = \min \{ K(\pi_{\gamma(t)}) \mid \pi_{\gamma(t)} \subset T_p M \text{ is a 2-plane, } \dot{\gamma}(t) \in \pi_{\gamma(t)} \}$$

$$\bar{K}^+(t) = \max \{ \bar{K}(\pi_{\bar{\gamma}(t)}) \mid \pi_{\bar{\gamma}(t)} \subset T_{\bar{p}} \bar{M} \text{ is a 2-plane, } \dot{\bar{\gamma}}(t) \in \pi_{\bar{\gamma}(t)} \}$$

Def 6.1 $X_c \in T_{\gamma(t)} M, \quad \bar{X}_c \in T_{\bar{\gamma}(t)} \bar{M}$ are called almost

the same if $\|X_c\| = \|\bar{X}_c\|, \langle X_c, \dot{\gamma}(t) \rangle = \langle \bar{X}_c, \dot{\bar{\gamma}}(t) \rangle$

denoted by $X_c \approx \bar{X}_c$

Lem 6.2 $X \in \text{Jac}(\gamma), \quad \bar{X} \in \text{Jac}(\bar{\gamma}), \langle X(0), \dot{\gamma}(0) \rangle = \langle \bar{X}(0), \dot{\bar{\gamma}}(0) \rangle$

$$\text{If } X(a) \approx \bar{X}(a)$$

$$X = (c+dc) \dot{\gamma} + X^\perp, \quad \bar{X} = (\bar{c}+\bar{d}\bar{c}) \dot{\bar{\gamma}} + \bar{X}^\perp$$

$$\Rightarrow c = \bar{c}, \quad d = \bar{d}$$

限制在 $\gamma \perp$ 的向量场

Thm 6.3 (Index Comparison Thm)

$$X \in \text{Jac}(\gamma), \bar{X} \in \text{Jac}(\bar{\gamma}), \quad X(0) = \bar{X}(0) = 0, \quad X(a) \approx \bar{X}(a)$$

If $\gamma(0)$ has no conj. point in $[0, a]$, and $K^+(t) = K^-(t)$.

$\forall t \in [0, a]$, then

$$I(X, X) \leq I(\bar{X}, \bar{X}).$$

$$"=" \text{ iff } K^+ = K^- \quad \forall t \in [0, a]$$

Proof: By Lemma 6.2, we may assume $X \perp \dot{\gamma}$, $\bar{X} \perp \dot{\bar{\gamma}}$

$\{e_i(t)\}$, $\{\bar{e}_i(t)\}$ be parallel o.n.b. along γ and $\bar{\gamma}$

$$\text{with } \begin{cases} e_1(t) = \dot{\gamma}(t) \\ e_2(t) = \frac{X(a)}{\|X(a)\|} \end{cases} \quad \text{and} \quad \begin{cases} \bar{e}_1(t) = \dot{\bar{\gamma}}(t) \\ \bar{e}_2(t) = \frac{\bar{X}(a)}{\|\bar{X}(a)\|} \end{cases}$$

$$\|X(a)\| = \|\bar{X}(a)\| = \alpha.$$

$$\text{Then } X^i(0) = \bar{X}^i(0) = 0, \quad X^i(a) = \bar{X}^i(a) = \begin{cases} \alpha, & i=2 \\ 0, & i \neq 2 \end{cases}$$

$$X^i(t) = \bar{X}^i(t) = 0, \quad \forall t$$

$$\bar{X}(t) = \bar{X}^i(t) \bar{e}_i(t)$$

$$Y(t) = \bar{X}^i(t) e_i(t)$$

$$Y(0) = 0, \quad Y(a) = X(a). \quad \text{By Index Lemma}$$

$$I(X, X) \leq I(Y, Y)$$

$$I(\gamma, \gamma) = \int_0^a \|\nabla_i \gamma\|^2 + \langle R(\dot{\gamma}, \gamma) \dot{\gamma}, \gamma \rangle dt$$

$$K(\dot{\gamma} \wedge \gamma) \geq K^-(t) \\ \geq \bar{K}^+(t).$$

$$= \int_0^a \left(\sum_i (\bar{X}^i e_i(t))^2 - \sum_i (\bar{X}^i(t))^2 K(\dot{\gamma} \wedge \gamma) \right) dt.$$

$$\leq \int_0^a \left(\sum_i (\dots)^2 - \sum_i (\bar{X}^i(t))^2 \bar{K}^+(t) \right) dt$$

$$\leq \int_0^a \|\bar{\nabla}_i \bar{x}\|^2 + \langle \bar{R}(\dot{\gamma}, \bar{x}) \dot{\gamma}, \bar{x} \rangle dt.$$

$$= I(\bar{x}, \bar{x}), \quad \square$$

Thm 6.4 (local Hessian Comparison)

$\gamma: [0, a] \rightarrow M$, $\bar{\gamma}: [0, a] \rightarrow M$ both minimal geodesic

$$\gamma(0) = p, \quad \bar{\gamma}(0) = \bar{p} \quad \bar{K}^+(t) \leq K^-(t), \quad \forall t \in [0, a].$$

Fix $b \in (0, a)$, $q = \gamma(b)$, $\bar{q} = \bar{\gamma}(b)$, $X_{q\gamma} \approx \bar{X}_{q\bar{\gamma}}$, then

$$\nabla^2 r_p(X_{q\gamma}, X_{q\gamma}) \leq \bar{\nabla}^2 \bar{r}_{\bar{p}}(\bar{X}_{q\bar{\gamma}}, \bar{X}_{q\bar{\gamma}})$$

Proof.

$$\begin{matrix} X(t), & \bar{X}(t) \\ \text{Jac} & \text{Jac} \end{matrix}$$

$$X(0) = 0$$

$$X(b) = X_{q\gamma}$$

$$\bar{X}(0) = 0$$

$$\bar{X}(b) = \bar{X}_{q\bar{\gamma}}$$

$$\nabla^2 r_p(X_{q\gamma}, X_{q\gamma}) = I(X, X) \leq I(\bar{X}, \bar{X}) = \bar{\nabla}^2 \bar{r}_{\bar{p}}(\bar{X}_{q\bar{\gamma}}, \bar{X}_{q\bar{\gamma}}).$$

$\square$

By taking trace, $\Delta r_{\bar{g}}(x) \leq \bar{\Delta} \bar{r}_{\bar{g}}(\bar{x})$

Thm 6.5 (Rauch Comparison)

$X \in \text{Jac}(\gamma)$, $\bar{X} \in \text{Jac}(\bar{\gamma})$, $\gamma, \bar{\gamma}$ are two geodesic

$0 = X(0) = \bar{X}(0)$, $\nabla_{\dot{\gamma}(0)} X \approx \bar{\nabla}_{\dot{\bar{\gamma}}(0)} \bar{X}$. Suppose γ has no conj.

point in $[0, a]$, $K^+(t) \leq K^-(t) \forall t \in [0, a]$.

Then $\bar{\gamma}$ has no conj. point.

$$\|X(t)\| \leq \|\bar{X}(t)\| \quad \forall t \in [0, a].$$

Proof: By Lemma 6.2, we assume $X \perp \dot{\gamma}(t)$, $\bar{X} \perp \dot{\bar{\gamma}}(t)$

$$\text{let } P(t) = \|X(t)\|^2, \quad \bar{P}(t) = \|\bar{X}(t)\|^2$$

$$\textcircled{1} \lim_{t \rightarrow 0} \frac{P(t)}{\bar{P}(t)} = 1$$

$$\gamma \text{ has no conj.} \Rightarrow P(t) \neq 0 \quad \forall t. \quad \textcircled{2} \frac{d}{dt} \left(\frac{\bar{P}}{P} \right) \geq 0.$$

By $X(t) = t \nabla_{\dot{\gamma}(0)} X + O(t^2)$ 法坐标下 Taylor Poly.

$$\bar{X}(t) = t \nabla_{\dot{\bar{\gamma}}(0)} \bar{X} + O(t^2)$$

$$\lim_{t \rightarrow 0} \frac{\bar{P}(t)}{P(t)} = \lim_{t \rightarrow 0} \frac{t^2 \|\bar{\nabla}_{\dot{\bar{\gamma}}(0)} \bar{X}\|^2 + O(t^4)}{t^2 \|\nabla_{\dot{\gamma}(0)} X\|^2 + O(t^4)} = \frac{\|\bar{\nabla}_{\dot{\bar{\gamma}}(0)} \bar{X}\|^2}{\|\nabla_{\dot{\gamma}(0)} X\|^2} = 1.$$

$$\frac{d}{dt} \left(\frac{\bar{P}}{P} \right) \geq 0 \Leftrightarrow \dot{\bar{P}}(t) P(t) - \dot{P}(t) \bar{P}(t) \geq 0$$

Note $X(t) \perp \dot{\gamma}(t)$, $\bar{X} \perp \dot{\bar{\gamma}}(t)$.

$$\frac{X(t)}{\|X(t)\|} \approx \frac{\bar{X}(t)}{\|\bar{X}(t)\|} \iff \|\bar{X}(t)\| \neq 0.$$

$$\text{Let } c = \sup \{ t \in [0, a] \mid p(t) > 0 \}.$$

$$\forall b \in (0, c), \quad X_b(t) = \frac{X(t)}{\|X(t)\|}, \quad \bar{X}_b(t) = \frac{\bar{X}(t)}{\|\bar{X}(t)\|}$$

By Index Comparison,

$$\begin{aligned} \langle \nabla_{\dot{\gamma}(b)} X_b, X_b(b) \rangle &= I(X_b(t), X_b(t)) \\ &\leq I(\bar{X}_b(t), \bar{X}_b(t)) = \langle \nabla_{\dot{\bar{\gamma}}(b)} \bar{X}_b, \bar{X}_b(b) \rangle \end{aligned}$$

$$\frac{1}{2} \frac{\dot{p}(b)}{p(b)} = \frac{\langle \nabla_{\dot{\gamma}(b)} X, X(b) \rangle}{\langle X(b), X(b) \rangle} = \frac{\langle \nabla_{\dot{\bar{\gamma}}(b)} \bar{X}, \bar{X}(b) \rangle}{\langle \bar{X}(b), \bar{X}(b) \rangle} = \frac{1}{2} \frac{\dot{\bar{p}}(b)}{\bar{p}(b)}$$

$$\left. \frac{d}{dt} \left(\frac{\bar{p}(t)}{p(t)} \right) \right|_{t=b} \geq 0 \quad \forall b \in (0, c)$$

$$\Rightarrow \|\bar{X}(c)\| \geq \|X(c)\| > 0$$

Since $\|X(t)\| > 0$ for all $t \in [0, a]$.

$$\Rightarrow c = a.$$

□

Thm 6.6 (Rauch Comparison - II)

Suppose $\gamma, \bar{\gamma}$ are two geodesic, $p = \gamma(0)$, $\bar{p} = \bar{\gamma}(0)$, $X_p \in T_p M$

$\bar{X}_{\bar{p}} \in T_{\bar{p}} \bar{M}$, $X_p \approx \bar{X}_{\bar{p}}$. Suppose γ has no conj. point in

$[0, a]$. $\bar{K}^+(t) \leq K^-(t) \quad \forall t \in [0, a]$. Then

$$\| (\exp_p)_* \Big|_{t \dot{\gamma}(0)} (X_p) \| \leq \| (\exp_{\bar{p}})_* \Big|_{t \dot{\bar{\gamma}}(0)} (\bar{X}_{\bar{p}}) \|$$

(By taking $X(t) = (\exp_p)_* \Big|_{t \dot{\gamma}(0)} (\bar{X}_{\bar{p}})$).

Lecture 8.

(M, g)

p

$(\bar{M}, \bar{g})$

$\bar{p}$

$\nabla^2 r_p(x)$

M_K^n

By Cartan - Ambrose - Hück's Thm

(S^n, g_{S^n})

$K > 0$

$(\mathbb{R}^n, g_E)$

0

$(H^n; g_{H^n})$

$K < 0$

• Hess r_p in M_K^n

Let $\gamma: [0, 1] \rightarrow M_K^n$, minimal

$\gamma(0) = p, \quad \gamma(1) = q, \quad q \notin \text{cut}(p) \cup \{p\}.$

$\forall X_q \in T_q M, \quad X_q \perp \dot{\gamma}(1), \quad X_q(t)$ parallel vector field along

By direct computation for $J(t) \in J_{ac}(\gamma)$ with $X_{q_1}(t) = X_{q_1}$
 $\Rightarrow J(t) = \frac{sn_K(t)}{sn_K(l)} X_{q_1}(t).$ $J(l) := X_{q_1}.$

For $Y_{q_1} \in T_{q_1}M$,

$$(\nabla^2 r_p)|_{q_1}(X_{q_1}, Y_{q_1}) = \langle \nabla_{\dot{r}(l)} J(t) |_{t=l}, Y_{q_1} \rangle = \frac{cn_K(t)}{sn_K(l)} \langle X_{q_1}, Y_{q_1} \rangle$$

Take o.n.b. $\{e_i\}$ at q_1 , $e_i = \dot{\gamma}(l)$

$$(\nabla^2 r_p)|_{q_1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \frac{cn_K(l)}{sn_K(l)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{cn_K(l)}{sn_K(l)} \end{pmatrix} \quad X_{q_1} \xrightarrow{e_i} Y_{q_1} \xrightarrow{e_j}$$

$$(\nabla^2 r_p)|_{q_1} = \frac{cn_K(t)}{sn_K(l)} (g - dr \otimes dr).$$

Hope $\nabla^2[F(r_p)] = 0 \cdot g.$

Let $S_K(t) = \int_0^t sn_K(\tau) d\tau$. $\dot{S}_K(t) \cdot \frac{cn_K(t)}{sn_K(t)} = \ddot{S}_K(t)$

By direct computation.

$$\underline{\nabla^2(S_K(r_p))}(X_{q_1}, Y_{q_1}) = \dot{S}_K(r_p) \cdot (\nabla^2 r_p)|_{q_1}(X_{q_1}, Y_{q_1})$$

$$\frac{cn_K(l)}{sn_K(l)} g(X_{q_1}, Y_{q_1})$$

$$+ \ddot{S}_K(r_p) \langle \dot{r}(l), X_{q_1} \rangle \langle \dot{r}(l), Y_{q_1} \rangle$$

$$= \dot{S}_K(r_p) [(\nabla^2 r_p)(x_q, \gamma_q) + \frac{C\eta_K(l)}{S\eta_K(l)} \langle \gamma(l), x_q \rangle \langle \gamma(l), \gamma_q \rangle]$$

Def 6.7 (Barrier func) Suppose f is a continuous func on

(M, g) 1) If $f \in C^2(U)$, U open, $p \in U$, s.t.

$$h(p) = f(p), \quad h(q) \underset{(\leq)}{\geq} f(q), \quad \forall q \in U$$

then h is called an upper (lower) barrier func of f at p .

2) Let $c \in \mathbb{R}$, If $\forall \varepsilon > 0$, $\exists$ barrier func. h_ε of f at p

$$\text{s.t.} \quad \Delta h_\varepsilon \leq c + \varepsilon.$$

support. func.

Then we say $\Delta f \leq 0$ in barrier sense

3) For any geodesic σ with $\sigma(0) = p$ if $(f \circ \sigma)'(0) \leq \frac{C}{\|\dot{\sigma}(0)\|}$

in Barrier sense, then we say

normalize

$$\|\dot{\sigma}(0)\| = 1$$

$$(\nabla^2 f)(q) \leq c g \quad \text{in barrier sense.}$$

Remark: If $f: (a, b) \rightarrow \mathbb{R}$ continuous

(1) If $f \in C^2$, then $f''(t_0) \leq C$ in barrier sense

$$\Leftrightarrow f''(t_0) \leq C$$

(2) If $f''(t_0) \leq C$ in barrier sense and f attains its minimizer at $t = t_0$, then $C \geq 0$

(3) If $f''(t) \leq 0$ in barrier sense, then f is a concave func.

(4) (Hopf-Calabi Strong Maximum Principle)

Suppose $U \subset M$, U open, $f \in C(M)$. $\Delta f \leq 0$ on U in barrier sense. If f attains its minimizer at $x_0 \in U \Rightarrow f \equiv c$ in U

Thm 6.8 (Global Hessian Comparison).

Suppose (M, g) satisfies $\sec(M) \geq K$. Then $\forall p \in M$ complete.

$$\nabla^2(S_K \circ r_p) \leq C \nabla_K \circ r_p \cdot g \text{ in barrier sense}$$

(Sine r_p may not diff. in cut $^+(p)$)

proof (1). Suppose r_p is smooth at q .

Let $\gamma: [0, l] \rightarrow M$ be a minimal geodesic from p to q .

It suffices to show

$$l = d(p, q) = r(q)$$

$$(S_K \circ r_p \circ \sigma)''(0) \leq C \nabla_K(l)$$

σ is any geodesic $\sigma(0) = p$. $\|\sigma'(0)\| = 1$.

$$(S_K \circ r_p \circ \sigma)''(0) = \nabla^2(S_K \circ r_p)(\sigma'(0), \sigma'(0)).$$

$$= S_{\nabla_K(l)}(\nabla^2 r_p)|_q(\sigma'(0), \sigma'(0)) + C \nabla_K(l) \langle \dot{\gamma}(l), \dot{\sigma}(0) \rangle^2$$

Compare γ with a geodesic $\bar{\gamma}$ in M_K^n

$$\bar{\gamma}(0) = \bar{p}, \quad \bar{\gamma}(l) = \bar{q}$$

$$\bar{X}_{\bar{q}} \approx X_q, \quad X_q = \dot{\sigma}(0)$$

By local Hessian Comparison

$$\begin{aligned} (S_K \circ r_p \circ \sigma)''(0) &\leq S\eta_K(l) (\bar{\nabla}^2 \bar{r}_{\bar{p}}) \Big|_{\bar{q}} (\bar{X}_{\bar{q}}, \bar{X}_{\bar{q}}) \\ &\quad + C\eta_K(l) \langle \dot{\gamma}(l), \bar{X}_{\bar{q}} \rangle^2 \\ &= \bar{\nabla}^2 (S_K \circ \bar{r}_{\bar{p}}) \Big|_{\bar{q}} (\bar{X}_{\bar{q}}, \bar{X}_{\bar{q}}) \\ &= C\eta_K(l) |\bar{X}_{\bar{q}}|^2 \\ &= C\eta_K(l). \end{aligned}$$

(2) If $q = p$. For any geodesic $\sigma: (-\infty, +\infty) \rightarrow M$

$$\sigma(0) = q = p, \quad \|\dot{\sigma}(0)\| = 1, \quad S_K \circ r_p \circ \sigma(t) = S_K(u(t)) = S_K(t)$$

$$\Rightarrow (S_K \circ r_p \circ \sigma(t))'' \Big|_{t=0} = C\eta_K(0)$$

(3) When $q \in \text{cut}(p)$, we need to construct barrier func.

$$\text{Let } \gamma: [0, l] \rightarrow M, \quad \gamma(0) = p, \quad \gamma(l) = q.$$

$$\sigma: [0, +\infty) \rightarrow M, \quad \sigma(0) = q, \quad \|\dot{\sigma}(0)\| = 1.$$

$$K > 0 \Rightarrow l \leq \frac{\pi}{\sqrt{K}} \quad \left(\text{If } l > \frac{\pi}{\sqrt{K}}, \exists \text{ conj. point} \Rightarrow \text{NOT minimal} \right)$$

$$K \geq 0.$$

Remark: Thm (Bonnet-Meyer) (M, g) Complete, $\exists K > 0$,

$\text{Ric} \geq (n-1)Kg$. Then M is compact.

$$\text{diam}(M) := \sup_{p, q \in M} d(p, q) \leq \frac{\pi}{\sqrt{K}}.$$

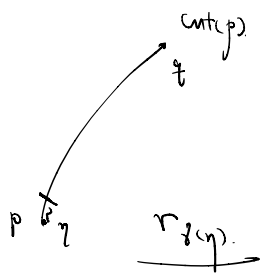
Can Be Proved By Thm 5.11. (2). $I(V, V) < 0$]

Let $r_\eta(x) = \eta + d(x, r(p))$, $\eta > 0$ is small, to be determined.

For $\eta > 0$, $r_\eta(x)$ is smooth around $q = r(c)$.

We will show

$$(S_K \circ r_\eta \circ \sigma)''(0) \leq C\eta_K(l) + \mathcal{O}(\eta).$$



By local Hessian comparison

$$(S_K \circ r_{\eta,p} \circ \sigma)''(0) = S\eta_K(l) (\nabla^2 r_{\eta,p})|_q (\dot{\sigma}(0), \dot{\sigma}(0))$$

$$+ C\eta_K(l) \langle \dot{\gamma}(l), \dot{\sigma}(0) \rangle^2$$

$$\leq S\eta_K(l) \cdot \frac{C\eta_K(l-\eta)}{S\eta_K(l-\eta)} \cdot |\dot{\sigma}^\perp(0)|^2 + C\eta_K(l) \langle \dot{\gamma}(l), \dot{\sigma}(0) \rangle^2$$

$$\dot{\sigma}^\perp(0) = \dot{\sigma}(0) - \langle \dot{\sigma}(0), \dot{\gamma}(l) \rangle \dot{\gamma}(l).$$

$$= \frac{S\eta_K(\eta)}{S\eta_K(l-\eta)} |\dot{\sigma}^\perp(0)|^2 + C\eta_K(l)$$

Remark: Thm (Cheng, 吴嘉超). (M, g) Complete. $\text{Ric} \geq (n-1)Kg > 0$

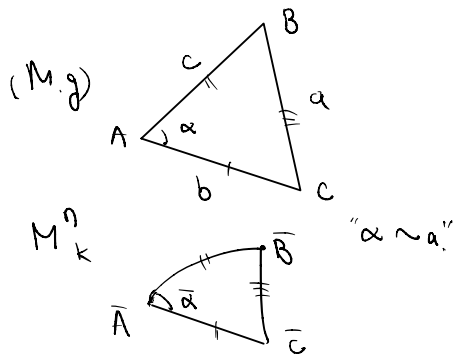
If $\text{diam}(M, g) = \frac{\pi}{\sqrt{K}}$, then (M, g) is isometric to S_K^n]

Thm 6.9 (Cosine Thm in M_K^n).

Let $\triangle ABC$ be a geodesic triangle in M_K^n . The sides

$a, b, c \rightsquigarrow$ corners $\angle A, \angle B, \angle C$

$a, b, c < \frac{\pi}{\sqrt{K}}$ if $K > 0$, Then



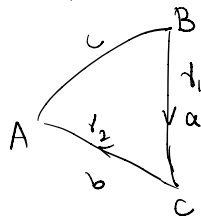
$$Cn_K(c) := Cn_K(a) Cn_K(b) + K Sn_K(a) Sn_K(b) \cos \angle C.$$

Proof: Let $\gamma_1 = [0, a] \rightarrow M_K^n$, $\gamma_1(0) = B$, $\gamma_1(a) = C$.

$\gamma_2 = [0, a] \rightarrow M_K^n$, $\gamma_2(0) = C$, $\gamma_2(a) = A$

Let $\varphi(t) = S_K \circ r_B \circ \gamma_2(t)$, Then

$$\varphi(0) = S_K(a), \quad \varphi(b) = S_K(c)$$



Recall $\nabla^2(S_K \circ r_B)|_q = Cn_K(r_B(q)) g$.

By def. of S_K , $Cn_K(t) = 1 - K \cdot S_K(t)$

$$\varphi''(t) = 1 - K S_K(r_B(\gamma_2(t))) = 1 - K \varphi(t)$$

Thm 6.21 $\varphi'(0) = Sn_K(a) \langle \dot{\gamma}_1(0), \dot{\gamma}_2(0) \rangle = -Sn_K(a) \cos \angle C$

$$\Rightarrow \varphi(t) = ? \quad \varphi(b) = S_K(c).$$

$$\Rightarrow Cn_K(c) = Cn_K(a) Cn_K(b) + K Sn_K(a) Sn_K(b) \cos \angle C.$$

□

Def 6.10 Let (M, g) complete

(1) A geodesic triangle $\triangle ABC$ consists of vertices A, B, C and minimal geodesics $\gamma_{AB}, \gamma_{BC}, \gamma_{CA}$.

(2) A geodesic angle $\angle BAC$ consists of vertex $A \in M$ and two minimal geodesics γ_{AB}, γ_{AC} .

Def 6.11 Let (M, g) complete $\sec(M) \geq K$

(1) Suppose $\angle BAC$ is a geodesic angle

It is easy to check the Existence of geodesic angle

$\angle \bar{B}\bar{A}\bar{C}$ in M_K^n with the same degree, related sides with the same length, $\angle \bar{B}\bar{A}\bar{C}$ is called comparison angle

(2) Let $\triangle ABC$ be the geodesic triangle in M with $\gamma_{AB}, \gamma_{BC}, \gamma_{CA}$, $\exists \triangle \bar{A}\bar{B}\bar{C}$ in M_K^n ,

$$L(\gamma_{\bar{A}\bar{B}}) = L(\gamma_{AB}), L(\gamma_{\bar{B}\bar{C}}) = L(\gamma_{BC}), L(\gamma_{\bar{C}\bar{A}}) = L(\gamma_{CA}).$$

$\triangle \bar{A}\bar{B}\bar{C}$ is called comparison triangle

$\rightarrow \pi$ 生成元 对应的地球夹角 $\Rightarrow$ 有限性

Thm 6.12 (Toponogov Comparison)

α 会大于一定角度

Let (M, g) be complete $\sec(M) \geq K$

(1) Suppose $\angle BAC$ is the geodesic angle, $\angle \bar{B}\bar{A}\bar{C}$ is the comparison angle in M_K^n . Then $d(B, C) \leq \bar{d}(\bar{B}, \bar{C})$.

(2) Suppose $\triangle ABC$ is the geodesic triangle $\triangle \bar{A}\bar{B}\bar{C}$ is the comparison triangle. Then $\angle A \geq \angle \bar{A}$, $\angle B \geq \angle \bar{B}$, $\angle C \geq \angle \bar{C}$.

Lemma 6.13 Suppose $f: [0, l] \rightarrow \mathbb{R}$ is continuous, $f(0) = 0$.

and $f'(0) \leq 0$. If $k > 0$, we also assume $l < \frac{\pi}{\sqrt{k}}$.

Suppose $f''(t) + k f(t) \leq 0$ in Barrier sense.

Then $f(t) \leq 0$, $\forall t \in [0, l]$.

Proof: $\forall \varepsilon > 0$, Let $f_\varepsilon(t) = f(t) - \varepsilon S \sigma_k(t)$

Then $f_\varepsilon(0) = f(0) - \varepsilon S \sigma_k(0)$

$$f'_\varepsilon(0) = f'(0) - \varepsilon C \sigma_k(0) \leq -\varepsilon < 0.$$

For ε small, $f_\varepsilon(t) < 0$.

It remains to show $f_\varepsilon(t) \leq 0$, $\forall t \in [0, l]$

Let $t_0 = \inf \{ t \in (0, l] \mid f_\varepsilon(t) = 0 \}$

(1) When $k \leq 0$, suppose $f_\varepsilon|_{[0, t_0]}$ attains its minimizer at t_1 .

$$f''_\varepsilon(t_1) + k f_\varepsilon(t_1) = f''(t_1) + k f(t_1) - \varepsilon (S \sigma_k''(t_1) + k S \sigma_k(t_1)) \leq 0$$

By Remark (2) on barrier func. $f_\varepsilon(t_1) < f_\varepsilon(t) < 0$ in barrier sense
 $-k f_\varepsilon(t_1) \geq 0 \Rightarrow f_\varepsilon(t_1) \geq 0$ Contradiction!

(2) When $K > 0$, we may assume $t_0 < l$. ($t_0 = l$, already true)

$$\text{Fix } \delta > 0 \quad [0, t_0] \subset [-\delta, \frac{\pi}{\sqrt{K+\delta}} - \delta]$$

$$\phi(t) = -\sin(\sqrt{K+\delta}(t+\delta))$$

$$\Rightarrow \phi''(t) + (K+\delta)\phi(t) = 0 \quad , \quad \phi < 0 \text{ on } [0, t_0]$$

Suppose $\frac{f_2}{\phi} \Big|_{[0, t_0]}$ attains its maximal at t_1

Let $g_{\epsilon, \delta'}$ be a barrier func. of f_2 at t_1 .

$$\text{i.e. } g_{\epsilon, \delta'}(t_1) = f_2(t_1), \quad g_{\epsilon, \delta'}(t) \geq f_2(t) \text{ near } t_1 \quad (1)$$

$$g_{\epsilon, \delta'}''(t_1) \leq -K f_2(t_1) + \delta' \quad (2)$$

$$\phi < 0 \Rightarrow \frac{g_{\epsilon, \delta'}}{\phi} \text{ also attains its maximal at } t_1.$$

$$\text{So } 0 = \left(\frac{g_{\epsilon, \delta'}}{\phi} \right)' \Big|_{t_1} = \frac{g_{\epsilon, \delta'}'(t_1) \phi(t_1) - g_{\epsilon, \delta'}(t_1) \phi'(t_1)}{\phi^2(t_1)} \Big|_{t_1}$$

$$0 \geq \left(\frac{g_{\epsilon, \delta'}}{\phi} \right)'' \Big|_{t_1} = \frac{g_{\epsilon, \delta'}''(t_1) + (K+\delta) g_{\epsilon, \delta'}(t_1)}{\phi(t_1)}$$

By the fun. of $\phi(t_1)$

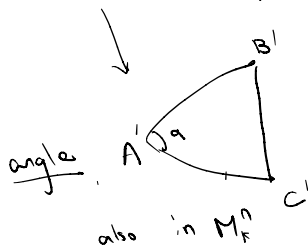
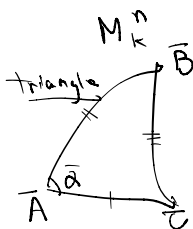
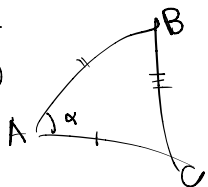
$$\begin{aligned} &\geq \frac{\delta' + \delta g_{\epsilon, \delta'}(t_1)}{\phi(t_1)} \\ (1) \quad (2) \end{aligned}$$

$$\text{Hence, } \delta > 0 \quad \delta' + \delta g_{\epsilon, \delta'}(t_1) \geq 0$$

$$\delta' \rightarrow 0. \quad g_{\epsilon, \delta}(t_1) \geq 0. \quad g_{\epsilon}(t_1) \geq 0.$$

Proof of Toponogov

Step 1. 1) $\Rightarrow$ 2)



$$\text{By (1)} \quad \bar{d}(B', C') \geq d(B, C) = d(\bar{B}, \bar{C}).$$

$$\Rightarrow \alpha \geq \bar{\alpha}$$

Step 2: Prove 1)

$$l_0 = l_{AB}, \quad l_1 = l_{CA}$$

l_0 minimizing

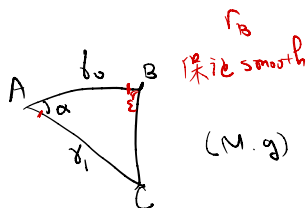
$$\varepsilon > 0, \quad \rho_\varepsilon(t) = d(l_0(l_0 - \varepsilon), l_1(t)), \quad t \in [0, l_1]$$

$$l_0(0) = A, \quad l_0(l_0) = B, \quad \exists \delta > 0, \quad \rho_\varepsilon(t) \text{ is smooth}$$

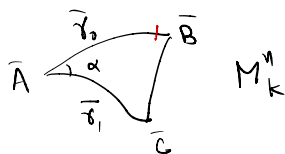
$$\forall t \in [0, \delta]$$

$$\rho_\varepsilon(0) = l_0 - \varepsilon.$$

$$\rho_\varepsilon'(0) = -\langle \dot{l}_0(0), \dot{l}_1(0) \rangle = -\cos \alpha.$$



(M, g)



M_K^n

$$\text{By Global Comparison} \quad (S_K \circ \rho_\varepsilon)''(t) \leq 1 - K S_K \circ \rho_\varepsilon(t)$$

in Barrier sense.

$$\text{Similarly on } \angle \bar{B}\bar{A}\bar{C}, \text{ we define } \bar{\rho}_\varepsilon(t) = d(\bar{l}_0(l_0 - \varepsilon), \bar{l}_1(t)).$$

$$\bar{\rho}_\varepsilon(0) = l_0 - \varepsilon, \quad \bar{\rho}_\varepsilon'(0) = -\cos \alpha.$$

$$(S_K \circ \bar{\rho}_\varepsilon)''(t) = 1 - K S_K \circ \bar{\rho}_\varepsilon(t)$$

$$\text{Let } f(t) = S_K \circ \rho_\varepsilon - S_K \circ \bar{\rho}_\varepsilon, \quad f''(t) + K f(t) \leq 0 \text{ in Barrier sense.}$$

$$f'(0) = 0, \quad f(0) = 0 \Rightarrow \rho_\varepsilon(t) \leq \bar{\rho}_\varepsilon(t) \quad \varepsilon \rightarrow 0. \Rightarrow \rho(t) \leq \bar{\rho}(t) \quad \square$$

Lecture 9

Thm 6.14 (M.g) Ric. mfd, $\gamma: [0,1] \rightarrow M$ minimal geodesic

$\gamma(0)=p$, $\text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) \geq (n-1)K$. Let $\bar{\gamma}$ be the related geodesic in M_K^n . Then $\Delta r_p(\gamma(t)) \leq \bar{\Delta} \bar{r}_{\bar{p}}(\bar{\gamma}(t)) \quad \forall t \in [0,1]$

"=" iff $\forall t \in [0,1]$, $\text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) = (n-1)K$.

For any $J(t) = J_{ac}^\perp(\gamma)$, $J(t) = \frac{S\eta_K(t)}{S\eta_K(a)} e(t)$.

$e(t)$ is a parallel vector field.

Proof: Take $b < a$, $\{e_i(t)\}$, $\{\bar{e}_i(t)\}$ be parallel o.n.B along $\gamma, \bar{\gamma}$

$$e_1(t) = \dot{\gamma}(t), \quad \bar{e}_1(t) = \dot{\bar{\gamma}}(t)$$

For $i \geq 2$. Let $X_i(t) \in J_{ac}^\perp(\gamma)$ with $X_i(0) = 0$, $X_i(b) = e_i(b)$

$\bar{X}_i(t) \in J_{ac}^\perp(\bar{\gamma})$ with $\bar{X}_i(0) = 0$, $\bar{X}_i(b) = \bar{e}_i(b)$

By the def of Δ

$$\Delta r_p(\gamma) = \sum_{i=1}^n (\nabla^2 r_p)_\gamma(e_i(b), e_i(b)) = \sum_{i=2}^n I(X_i, X_i)$$

Since $i=1$, $I(\dot{\gamma}(t), \dot{\gamma}(t)) = 0$

In M_K^n , $\bar{X}_i(t) = \frac{S\eta_K(t)}{S\eta_K(b)} \bar{e}_i(t)$

$$\tilde{X}_i(t) = \frac{S\eta_K(t)}{S\eta_K(b)} \cdot e_i(t)$$

$$\Rightarrow I(X_i, X_i) \leq I(\tilde{X}_i, \tilde{X}_i)$$

$$\sum_{i=2}^n I(\tilde{X}_i, \tilde{X}_i) = \sum_{i=2}^n \int_0^b (\|\nabla_{\tilde{Y}} \tilde{X}_i\|^2 - R(\tilde{Y}, \tilde{X}_i, \tilde{Y}, \tilde{X}_i)) dt$$

Here $\tilde{Y} \perp e_i(t) = \sum_{i=2}^n \int_0^b \left[\left(\frac{sn'_k(t)}{sn_k(b)} \right)^2 - \left(\frac{sn_k(t)}{sn_k(b)} \right)^2 K(\tilde{Y} \wedge e_i) \right] dt$

$$= \int_0^b \left[(n-1) \left(\frac{sn'_k(t)}{sn_k(b)} \right)^2 - \left(\frac{sn_k(t)}{sn_k(b)} \right)^2 R_{\tilde{Y}}(\tilde{Y}(t), \tilde{Y}(t)) \right] dt.$$

$$\leq \int_0^b \left[(n-1) \left(\frac{sn'_k(t)}{sn_k(b)} \right)^2 - \left(\frac{sn_k(t)}{sn_k(b)} \right)^2 (n-1) K \right] dt.$$

$$= \sum_{i=2}^n \int_0^b (\|\nabla_{\tilde{Y}} \bar{X}_i\|^2 + R(\tilde{Y}, \bar{X}_i, \tilde{Y}, \bar{X}_i)) dt.$$

$$= \sum_{i=2}^n I(\bar{X}_i, \bar{X}_i)$$

"=" $\Leftrightarrow R_{\tilde{Y}}(\tilde{Y}(t), \tilde{Y}(t)) = (n-1)K, \quad X_i = \tilde{X}_i, \quad 2 \leq i \leq n. \quad \square$

$$\mathbb{R}^3 \quad \Sigma: \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases} \quad (u, v) \in \Delta \subset \mathbb{R}^2.$$



$$\text{Area}(\Sigma) = \int_{\Sigma} dS = \int_{\Delta} |\vec{r}_u \wedge \vec{r}_v| du dv.$$

$|\vec{r}_u \wedge \vec{r}_v|$ is area of the parallelogram.

Higher-dim. case.

$$(u, \varphi) \quad (M, g) \quad \text{Vol}(U) = \int_U d\text{Vol}$$

$$(x^1, \dots, x^n) \leftrightarrow (\partial_1, \dots, \partial_n)$$

$$= \int_{\varphi(U)} \underbrace{|V(\partial_1 \wedge \partial_2 \wedge \dots \wedge \partial_n)|}_{\substack{\text{"} \\ |\det A| = \sqrt{\det(g_{ij})}}} \circ \varphi^{-1} dx^1 dx^2 \dots dx^n.$$

Natural requires

$$V|_p(e_1 \wedge e_2 \wedge \dots \wedge e_n) = 1,$$

$\{e_1, \dots, e_n\}$ is O.N.B of $T_p M$ w.r.t. g_p

Suppose $\partial_i = a_i^j \overset{\text{depend on } p}{e_j}$

$$\text{Then } V_p(\partial_1 \wedge \dots \wedge \partial_n) = \det(a_i^j).$$

$$\begin{aligned} \text{Note } g_{ij} &= g(\partial_i, \partial_j) = g(a_i^k e_k, a_j^l e_l) \\ &= a_i^k a_j^l = (A A^T)_{ij}, \quad A = (a_i^j) \end{aligned}$$

$$\det(g_{ij}) = \det A \det A^T = (\det A)^2$$

$$\Rightarrow |\det A| = \sqrt{\det(g_{ij})}$$

$$\boxed{\text{Vol}(U) = \int_{\varphi(U)} \sqrt{G} \circ \varphi^{-1} dx^1 dx^2 \dots dx^n}$$

$$\{(U_\alpha, \varphi_\alpha)\} \text{ cover } M, \quad \{\rho_\alpha\}, \quad \sum_\alpha \rho_\alpha = 1$$

$$\text{Vol}(M) = \int_{U_\alpha} d\text{Vol} = \int_{\varphi_\alpha(U_\alpha)} (\rho_\alpha \sqrt{G}) \circ \varphi_\alpha^{-1} dx_\alpha^1 \dots dx_\alpha^n$$

Identify $T_p M$ with $\mathbb{R}^n$, $\{x^1, x^2, \dots, x^n\}$

$$\exp_p(\Sigma_p) = (M \setminus \text{cut}(p))$$

$$r = \sqrt{\sum_i (x^i)^2}, \quad \oplus = \frac{1}{r}(x^1, \dots, x^n) \in S_p M, \quad \|\oplus\| = 1$$

$$dx^1 dx^2 \dots dx^n = r^{n-1} dr d\oplus \quad r \text{ 不超出 } \Sigma_p.$$

$$Vol(B(p, r)) = \int_{B_r(p)} \boxed{\sqrt{g}(\exp_p(r \oplus))} r^{n-1} dr d\oplus$$

Better rep.?

$$\mu_p(r, \oplus)$$

Note: $\sqrt{g} = \sqrt{\det(g_{ij})}$,
depend on coordinate.

Prop 6.15 For $\oplus \in S_p M$, let $\gamma(t) = \exp_p(t \oplus)$. Suppose

$\{v_2, \dots, v_n\}$ is a basis of $(\dot{\gamma}(0))^\perp = \{v \in T_p M \mid \langle v, \dot{\gamma}(0) \rangle = 0\}$

Let $v_j(t) \in T_{\gamma(t)}$, s.t. $\nabla_{\dot{\gamma}(0)} v_j(t) \big|_{t=0} = v_j$, $2 \leq j \leq n$.

If $r \oplus \in \Sigma_p$, then

$$\mu_p(r, \oplus) = \frac{|\det(v_2(r), \dots, v_n(r))|}{|\det(v_2, \dots, v_n)|}$$

$$\begin{aligned} \text{where } |\det(v_2(r), \dots, v_n(r))| &= \sqrt{\det \langle v_i(r), v_j(r) \rangle_{2 \leq i, j \leq n}} \\ &= |V(v_2(r) \wedge \dots \wedge v_n(r))| \end{aligned}$$

Proof: Let $v_1 = \oplus$, then $\{v_1, v_2, \dots, v_n\}$ is a basis of $T_p M$

For any $y \in T_p M$, $y = y^1 v_1 + \dots + y^n v_n$

$$\{y^1, \dots, y^n\}, \quad \frac{\partial}{\partial y^i} = v_i$$

$$|\det(v_1, \dots, v_n)| dy^1 dy^2 \dots dy^n = dx^1 dx^2 \dots dx^n$$

$$\langle v_i, v_j \rangle = 0, \quad i \geq 2$$

$$= r^{n-1} dr d\oplus$$

$$\langle v_1, v_1 \rangle = 1 \quad |\det(v_1, \dots, v_n)| = |\det(v_2, \dots, v_n)|$$

$$\Rightarrow dy^1 dy^2 \dots dy^n = \frac{r^{n-1}}{|\det(v_2, \dots, v_n)|} dr d\oplus$$

In coordinate $\{y^1, \dots, y^n\}$

$\downarrow \frac{\partial}{\partial y^j}$

Note $V_j(t) = (\exp_p)_* \Big|_{t\oplus} (t V_j) = t (\exp_p)_* \Big|_{t\oplus} (V_j)$

Recall $\alpha(s, t) = \exp_p(t(u+s\xi))$, $\xi = V_j$

$V(t) = \frac{\partial \alpha}{\partial s} \Big|_{(t,0)} \Rightarrow V(t)$ is a Jacobi field.

Maybe $V(t) = (\exp_p)_* \Big|_{t\xi} (t\xi)$?

$\frac{\partial}{\partial y^1} = \delta(r)$, $\frac{\partial}{\partial y^r} = \frac{1}{r} V_\delta(r)$.

$g_{ij} = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle = \frac{1}{r^2} \langle V_i(r), V_j(r) \rangle$.

$\Rightarrow G = \det(g_{ij})_{i,j \geq 1} = \det(g_{ij})_{i,j \geq 2}$
 $= r^{-2n+2} \det(V_2(r), \dots, V_n(r))$

$\Rightarrow dVol = \sqrt{G} dy^1 dy^2 \dots dy^n = \sqrt{G} \frac{r^{n-1}}{|\det(V_2, \dots, V_n)|} dr d\oplus$

$= \frac{|\det(V_2(r), \dots, V_n(r))|}{|\det(V_2, \dots, V_n)|} dr d\oplus$

$\mu_p(r, \oplus)$.

Cor 6.16 For $\oplus \in S_p M$, $r\oplus \in \Sigma_p$. Then

$\frac{\mu'_p(r, \oplus)}{\mu_p(r, \oplus)} = (\Delta r_p)(\exp_p(t\oplus))$

Where $\mu'_p(r, \odot) = \frac{d\mu_p(r, \odot)}{dr}$

Proof: Let $\gamma(t) = \exp_p(t, \odot)$, $t \in [0, r]$

Choose $\{e_i(t)\}$ parallel along γ , $e_1(t) = \dot{\gamma}(t)$

Let $V_j(t) \in \text{Jac}(\gamma)$, $V_j(0) = 0$, $V_j(r) = e_j(r)$. 为什么可取 e_j 呢?

At $q = \gamma(r)$, $\Delta r_p|_q = (\nabla^2 r_p)|_q (e_i(r), e_i(r)) = \sum_{i=2}^n I(V_i, V_i)$

Let $A(t) = (a_{ij}(t))_{(n-1) \times (n-1)}$, $a_{ij}(t) = \langle V_i(t), V_j(t) \rangle$

Then $A(r) = I$. Let $D(t) = \det A(t)$.

Note $D'(t) = D(t) \text{tr}(A^{-1} \cdot A')$

By Prop 6.15, $A(t) = t^{-2n+2} D(t)$
 $\parallel$
 $\det(g_{ij}(t))$.

$\Rightarrow \mu'_p(t, \odot) = \sqrt{A(t)} \cdot t^{n-1} = \sqrt{D(t)}$.

$\frac{\mu'_p}{\mu_p} = \frac{1}{2} \frac{D'(t)}{D(t)} = \frac{1}{2} \text{tr}(A^{-1} A')$

$\frac{\mu'_p}{\mu_p} \Big|_{t=r} = \frac{1}{2} \text{tr}(A') \Big|_{t=r} = \sum_{i=2}^n \langle V_i(t), \nabla_{\dot{\gamma}(t)} V_i(t) \rangle \Big|_{t=r}$
 $= \sum_{i=2}^n I(V_i(t), V_i(t))$

□

In M_K^n , by Prop 6.15

$$\mu_K(r) = \begin{cases} \left(\frac{\sin(\sqrt{K}r)}{\sqrt{K}} \right)^{n-1}, & K > 0 \\ r^{n-1}, & K = 0 \\ \left(\frac{\sinh(\sqrt{K}r)}{\sqrt{K}} \right)^{n-1}, & K < 0 \end{cases}$$

$\left[\begin{array}{c} \text{warped product} \\ \downarrow \\ M_K^n \end{array} \right]$ symmetric.

$\mu_K(r)$ is independent of Θ .

Prop 6.17 (M, g) Complete, $\text{Ric} \geq (n-1)K$, $\forall \Theta \in S_p M$.

$$r(\Theta) \in \Sigma_p.$$

$$1) \frac{\mu_p(r, \Theta)}{\mu_K(r)} \downarrow \quad 2) \lim_{r \rightarrow 0^+} \frac{\mu_p(r, \Theta)}{\mu_K(r)} = 1.$$

Further, we have $\mu_p(r, \Theta) \leq \mu_K(r)$ for all $r > 0$.

Proof: (1) $\frac{d}{dt} \left(\log \frac{\mu_p(r, \Theta)}{\mu_K(t)} \right) \stackrel{\text{Cor 6.16}}{=} \Delta r_p - \overline{\Delta} \bar{r}_p \stackrel{\substack{\geq \\ M_K^n}}{\leq} 0$
 $\downarrow$
Laplacian Comparison.

(2) By Taylor Formula.

$$V_j(t) = t \nabla_{\dot{\gamma}(t)} V_j + O(t^2)$$

By Prop 6.15

$$\lim_{r \rightarrow 0^+} \frac{\mu_p(r, \Theta)}{r^{n-1}} = \lim_{t \rightarrow 0^+} \frac{t^{n-1} |\det(V_2(t), \dots, V_n(t))|}{|\det(\nabla_{\gamma(0)} v_2, \dots, \nabla_{\gamma(0)} v_n)|} \cdot \frac{1}{t^{n-1}} \rightarrow 1.$$

Thm 6.18. (Bishop - Gromov Comparison)

(M, g) compact, $\text{Ric} \geq (n-1)K$, $p \in M$, $S_K(r)$ and $B_K(r)$ be the geodesic sphere and geodesic ball in M_K^n . Then

$$\frac{\text{Area}(S(p, r))}{\text{Area}(S_K(r))} \downarrow_{r \rightarrow 0} 1 \quad \frac{\text{Vol}(B(p, r))}{\text{Vol}(B_K(r))} \downarrow_{r \rightarrow 0} 1$$

Remark: For $t \oplus \notin \Sigma_p$, define $\mu_p(t, \oplus) = 0$. 
 $\partial \Sigma_p = \text{cut}(p)$ (well-defined since it's zero measure)

In this setting Cor 6.16 still right if $r \oplus \notin \Sigma_p$.

Proof:
$$\text{Area}(S(p, r)) = \int_{S^{n-1}} \mu_p(r, \oplus) d\oplus$$

$$\text{Vol}(B(p, r)) = \int_0^r \int_{S^{n-1}} \mu_p(r, \oplus) d\oplus dr,$$

$$\frac{\text{Area}(S(p, r))}{\text{Area}(S_K(r))} = \frac{\int_{S^{n-1}} \mu_p(r, \oplus) d\oplus}{\int_{S^{n-1}} \mu_K(r) d\oplus} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \frac{\mu_p(r, \oplus)}{\mu_K(r)} d\oplus \downarrow 1.$$

$$\begin{aligned} \frac{d}{dt} \left(\log \frac{\text{Vol}(B(p, r))}{\text{Vol}(B_k(r))} \right) &= \frac{\text{Area}(S(p, r))}{\text{Vol}(B(p, r))} - \frac{\text{Area}(S_k(r))}{\text{Vol}(B_k(r))} \\ &= \frac{\int_0^r (\text{Area}(S(p, r)) \cdot \text{Area}(S_k(t)) - \text{Area}(S_k(r)) \cdot \text{Area}(S(p, t))) dt}{\text{Vol}(B(p, r)) \text{Vol}(B_k(r))} \end{aligned}$$

(Hope) ≤ 0

T lemma: $f(t) \downarrow, t \in [0, \infty), S(t) > 0, \forall r \leq R$

$$\frac{\int_0^r f(t) S(t) dt}{\int_0^r S(t) dt} \geq \frac{\int_0^R f(t) S(t) dt}{\int_0^R S(t) dt}$$

Proof of Lemma

$$F(x) = \frac{\int_0^x f(t) S(t) dt}{\int_0^x S(t) dt}, \quad F'(x) = \frac{S(x)}{(\int_0^x S(t) dt)^2} \int_0^x [F(x) - f(t)] S(t) dt \leq 0.$$

Let $S(t) = S_k(t) > 0, f(t) = \frac{\mu_p(t, \Theta)}{S_k(t)} \downarrow$

$$\Rightarrow \frac{\int_0^r \mu_p(t, \Theta) dt}{\int_0^r S_k(t) dt} \geq \frac{\int_0^R \mu_p(t, \Theta) dt}{\int_0^R S_k(t) dt}$$

$$\Rightarrow \int_{\Theta} \frac{\int_0^r \mu_p(t, \Theta) dt}{\int_0^r S_k(t) dt} d\Theta \geq \int_{\Theta} \frac{\int_0^R \mu_p(t, \Theta) dt}{\int_0^R S_k(t) dt} d\Theta$$

$$\Rightarrow \frac{d}{dt} \left(\log \frac{\text{Vol}(B(p, r))}{\text{Vol}(B_k(r))} \right) \leq 0.$$

□

Remark: "=" holds $\mu_p(t, \Theta) = \mu_K(t)$, $\Delta r = \bar{\Delta} \bar{r}$

$\downarrow$

sectional cur.

$$\text{Ric}(\bar{r}(t), \bar{r}(t)) = (n-1)K,$$

$$\langle R(J_K, \bar{r}) \bar{r}, J_K \rangle = K \quad \leftarrow \quad (\text{Jacobi field}), J_K(t) = \frac{\sin_K(t)}{\sin_K(1)} e_1(t),$$

$\Downarrow$

2nd coordinate τ $g_{ij} = \bar{g}_{ij}$.

$\Rightarrow (S(p, r), g)$ isometric to $(B_K(r), g_K)$

Thm (Cheng).

(M, g) Complete, $\text{Ric} \geq (n-1)K$, $\text{diam} = \frac{\pi}{\sqrt{K}} \Rightarrow (M, g) \stackrel{\text{iso}}{\hookrightarrow} (\mathbb{S}^n, g_{\mathbb{S}^n})$.

proof. $\forall p \in M$. By Bishop-Gromov $\text{dep. } g = \frac{\pi}{\sqrt{K}}$.

$$\frac{\text{Vol}(B(p, \frac{\pi}{2\sqrt{K}})) \xrightarrow{B_K(r)}}{\underbrace{\text{Vol}(B(p, \frac{\pi}{\sqrt{K}}))}_{= M}} \geq \frac{\text{Vol}(B_K(\frac{\pi}{2\sqrt{K}}))}{\text{Vol}(B_K(\frac{\pi}{\sqrt{K}}))} = \frac{1}{2}$$

$$\text{Vol}(B(p, \frac{\pi}{2\sqrt{K}})) \geq \frac{1}{2} \text{Vol}(M)$$

$$\text{Vol}(B(p, \frac{\pi}{2\sqrt{K}})) + \text{Vol}(B(q, \frac{\pi}{2\sqrt{K}})) \leq \text{Vol}(M)$$

$$\Rightarrow \text{Vol}(B(p, \frac{\pi}{\sqrt{K}})) + \text{Vol}(B(q, \frac{\pi}{\sqrt{K}})) = \text{Vol}(M).$$

$\Rightarrow$ isometric.

Lecture 10. Ricci flow.

Hamilton introduce the Ricci flow 1982

$$\frac{\partial g(t)}{\partial t} = -2 \operatorname{Ric}_{g(t)}$$

$S^3 \sim$

Thm 1 (Hamilton) 3D. compact mfd. with $\operatorname{Ric} > 0$ is 3D space form
constant sectional curv.

Key ideas: ① $\operatorname{Ric} - \varepsilon Sg \geq 0$ is preserv. along Ricci flow.

Blow up.

$$t \rightarrow T, S(t) \rightarrow +\infty, |R_{ijkl}| \rightarrow +\infty$$
$$\textcircled{2} \frac{|\operatorname{Ric} - \frac{1}{3} Sg|}{S^2} \leq \frac{C}{S^\alpha}$$

Thm 2 (Hamilton) 4D compact + curvature operator $> 0 \Rightarrow$ 4D space form.

Thm 3 (Bohm-Wilking) $n \geq 5$, compact + curv. operator $> 0 \Rightarrow$ Space form
(~~non~~ ≥ 0 eigenvalue sum) 2-operator > 0

Thm 4 (Schoen-Brendle) $\frac{1}{4}$ -pinched sectional curv. compact

($\frac{1}{4}$ -Differential Sphere conjecture) w PIC₂

$\downarrow$
sphere

定向曲率 > 0

(Compact Mfd with PIC)

4D Hamilton, Chen-Zhu 4D PIC 拓扑分类

Brendle $n \geq 13$ PIC.

?
uniformly PIC + WPIC₂

Chen (Tian's student) $n \geq 10$.

5-9 still open, with PIC.

- Ricci flow exists for a short time (Hamilton)

Similar result hold for noncompact manifolds with bounded (W. Shi) curvature.

By Ricci flow $\Rightarrow B(p, r) \cdot |Rm| \leq C,$
 $\Rightarrow B(p, \frac{r}{2}) \cdot |\nabla^k Rm| \leq \frac{C}{t^k}$

Example: $(S^n, g(t)) \quad g(t) = (1 - 2(n-1)t) g_0 \quad g_0, K = 1,$
exists $(-\infty, \frac{1}{2(n-1)})$

$(\mathbb{R}^n, g(t)) \quad g(t) = g_E \quad (-\infty, +\infty) \text{ Long time exists.}$

If the Ricci flow blow up at finite time, then

$$\lim_{t \rightarrow T} |Rm|_{g(t)} = +\infty$$

(Conj. $\lim_{t \rightarrow T} S(t) = +\infty$? Open $n \geq 4$).

$t \rightarrow T, \quad (p_i, t_i), \quad t_i \rightarrow T$

Cheeger - Gromov Convergence

$$(M_i, g_i(t), p_i) \xrightarrow{C-G} (M_\infty, g_\infty(t), p_\infty) \text{ if}$$

① $\exists U_i \in M_i \quad \forall \text{ compact } K \subset M_\infty, \exists i(K), \forall i \geq i(K),$

② $\exists \phi_i: U_i \rightarrow \phi_i(U_i) \subset M_\infty \quad K \subset \phi_i(U_i)$

$$\phi_i^* g_i \xrightarrow{C^\infty} g_\infty \text{ in } K \quad (K \text{ compact subset of } M_\infty)$$



Remark: $\begin{cases} |\nabla^k R_m| \leq C(k, K) \quad \forall x \in K \\ \text{inj}(p_i) \geq V \iff \text{Vol}(B(p_i, 1)) \geq V \end{cases}$

For any (p_i, t_i) when $|R_m|(p_i, t_i) \rightarrow +\infty$, Let $K_i = |R_m|(p_i, t_i)$

$$\text{If } (M, K_i g(K_i^{-1} t), p_i) \rightarrow (M_\infty, g_\infty^{(t)}, p_\infty), \quad t \in (-K_i t_i, 0]$$

$$\quad \quad \quad \downarrow$$

$$\quad \quad \quad g_i(t) \quad \quad \quad t \in (-\infty, 0]$$

$(M_\infty, g_\infty^{(t)}, p_\infty)$ is ancient flow
is called singularity model

Key: Classify singularity model.

By Perelman's noncollapsing Thm $\Rightarrow (M_\infty, g_\infty^{(t)})$ is
 κ -noncollapsed.

(Def of κ -noncollapsed).

Remark: If $|R_m|_{g(t)} \leq \frac{1}{r^2}$, $B_{g(t)}(p, r) \Rightarrow \text{Vol}_{g(t)}(B(p, r)) \geq \kappa \cdot r^n$

κ -noncollapsed $\Leftrightarrow (M, g(t))$ is μ -function $\geq C$

Hence, $(M_\infty, g_\infty^{(t)})$ is a κ -noncollapsed ancient flow
 κ -solution.

Def 5. If $\exists f \in C^\infty(M)$, $\lambda \in \mathbb{R}$ (const)

$$\text{Ric} + \lambda g = \text{Hess } f$$

$$(\text{Ric} + \nabla \nabla f = \frac{1}{2} \lambda g).$$

then (M, g, f) is called a gradient Ricci soliton.

($\lambda < 0$ shrinking, $\lambda = 0$ steady $\lambda > 0$ expanding)

$$\begin{cases} \frac{\partial g}{\partial t} = -2\text{Ric} \\ g^{(0)} = g \end{cases}$$

$$\phi_t \text{ generated } \frac{1}{1+2\lambda t} \nabla f$$

$$g^{(t)} = (1+2\lambda t) \phi_t^* g$$

$$(1+2\lambda t) > 0$$

$$\lambda > 0 \quad t \in \left(-\frac{1}{2\lambda}, +\infty\right)$$

$$\lambda = 0 \quad t \in (-\infty, +\infty)$$

$$\lambda < 0 \quad t \in \left(-\infty, -\frac{1}{2\lambda}\right).$$

Remark: (M, g) complete $\Rightarrow \Phi_t$ is well-defined for $t \in (-\infty, +\infty)$

flow's completeness is good!

3D:
compact: $(M_i, g_i^{(t_i)}, p_i) \xrightarrow{\phi_i} (M_\infty, g_\infty^{(t)}, p_\infty)$ compact

For i large, $\phi_i(M) = M_\infty$

(陈英杰)

Chen 3D ancient flow $\text{sec} \geq 0 \Rightarrow (M_\infty, g_\infty^{(t)}) \quad \text{sec} \geq 0$

$\Rightarrow \text{Ric}_{g_i^{(t_i)}} > 0. \Rightarrow M_i \cong \mathbb{S}^3$
By Hamilton

$\text{Ric} > 0.$

$$(M_\infty, g_\infty^{(t)}) \cong (\mathbb{S}^3, g_{\mathbb{S}^3}^{(t)}) / \sim$$

Noncompact: sec not strictly $\Rightarrow (\mathbb{S}^2 \times \mathbb{R}, g_{\mathbb{S}^2}(t) + dt^2) / \sim$
 $\mathbb{S}^2 \times \mathbb{R} / \mathbb{Z}_2$ only this quot.
 $\text{sec} > 0 \Rightarrow 3\text{D Noncompact K-solution} + \text{sec} > 0?$

(*)

For (*), Perelman's claim and conjecture.

Claim: 3D Steady Ricci Soliton is unique. $\leftarrow$ Brendle K-noncollapsed 2013.
 fly wing

Conjecture: 3D noncompact K-solution (Lai 21).

+ $\text{sec} > 0$ is a flow of Bryant Solution

Steady gradient Ricci

Brendle confirm the conj.

$\mathbb{I} \times_{\varphi} \mathbb{S}^{n-1}$

(Remark: K-solution, Perelman's over solution $\mathbb{S}^3$, Brendle-S-D)

Thm 6. For any 3D $(M_i, g_i(t), P_i)$ noncompact K-solution, $\text{sec} > 0$.

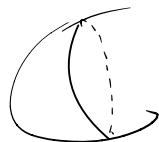
If $S(P_i) = 1$, then $(M_i, g_i(t), P_i) \xrightarrow{C-G} (M_\infty, g_\infty, P_\infty)$.

$S_\infty(P_\infty) = 1$.

$\text{sec} \geq 0$.

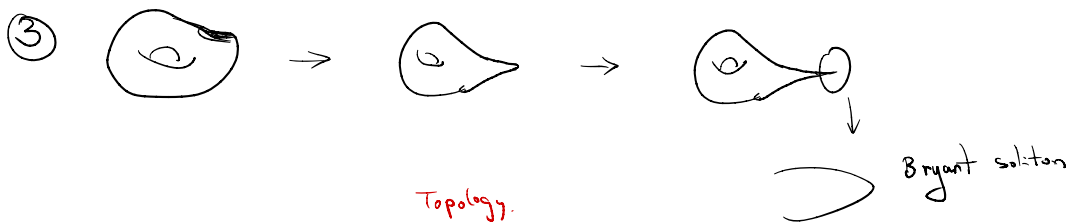
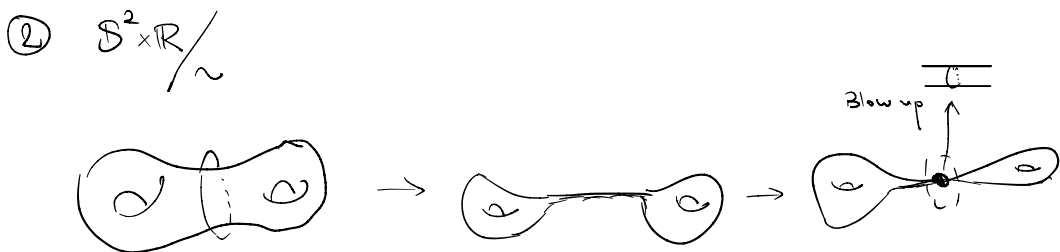
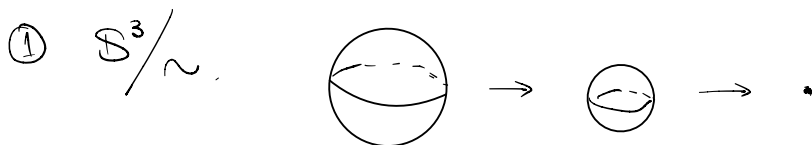
$S(x) d(x, x_0)^2 \rightarrow +\infty$ as $d(x, x_0) \rightarrow +\infty$

Thm 7. $(M_\infty, g_\infty(t), P_\infty) = (\mathbb{S}^2 \times \mathbb{R}, g_{\mathbb{S}^2}(t) + dt^2)$

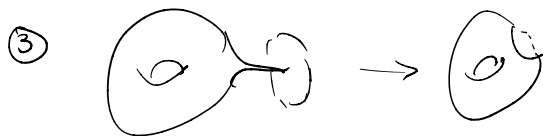
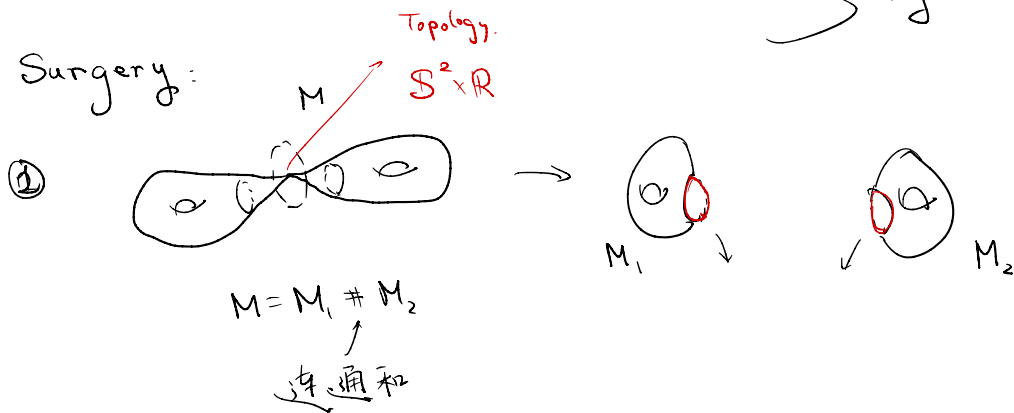


Toponogov Comparison

3 types of singularities



• Surgery:



may be

$$S^2 \times \mathbb{R} / \mathbb{Z}_2$$

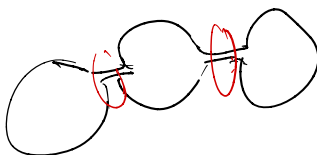
Topology don't change
cut Bryant Cap $\subseteq B(o, r)$

change on $S^2 \times \mathbb{R} / \mathbb{Z}_2$

iso. 拓扑不变性

κ -noncollapsed. need uniform

Surgery Time?



$$(M, g(t)) \rightarrow T.$$

conv. blow up everywhere.

$$\left\{ \begin{array}{l} \text{e.g. } S^3 \\ \text{e.g. } S^2 \times S^1 \\ \quad ds^2 \end{array} \right.$$

case 1 $S^3/\sim$ (local)

↓
good things!

2. $\mathbb{R} \times S^2 \cong \mathbb{R}^3$

3. Bryant solution by $\mathbb{R} \times S^2$.

① $S^2 \times S^1$

4. $\mathbb{R} \times S^2/\mathbb{Z}_2 \rightarrow \mathbb{RP}^3 \# \mathbb{RP}^3$

② $S^2 \times^{\sim} S^1$ (反接).

5. $\mathbb{R} \times S^2/\mathbb{Z}_2 + \text{Bry. } \mathbb{RP}^3$

finite topology.

Prime Decomposition of 3-dim manifold

$$M^3 \approx P_1 \# P_2 \# P_3 \# \dots \# P_n$$

$$P_i \text{ is prime if } P_i = M \# N$$

⇓

$$M \approx S^3 \text{ or } N \approx S^3.$$

• Aspherical : universal cover is contractible.

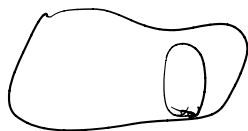
(spherical : ...).

M every P_i is not aspherical

If $\pi_1(M)$ is finite, then each p_i of M is not aspherical

Thm 8: If M contains no aspherical factor, then the Ricci flow with surgery extinct at finite time.

$$\pi_2(M) =$$



$$i \rightarrow i+1$$

$$|T_i - T_{i+1}| \geq \varepsilon.$$

Perelman: $A(\Sigma)$. $A(t) \searrow$ scale cur.

$$\frac{d}{dt} A(t) \leq -2\pi - \frac{1}{2} S_{\min}(t) A(t).$$

Colding-Minicozzi: another way.

宽度 $\rightarrow W(t)$. nonincreased when surgery. $\square$

$$\frac{dS}{dt} = \Delta_{g(t)} S + 2|\text{Ric}|^2 = \Delta S + \frac{2}{3} S^2 + \underbrace{2|\text{Ric}|^2}_{\text{trace-free part}}$$

By PDE BODE, we have

$$R_{\min}(t) \geq -\frac{3}{2} \frac{1}{t+C}$$

$$\bar{A}(t) = A(t) \frac{1}{t+C}$$

$$\Rightarrow \frac{d\bar{A}(t)}{dt} \leq -\frac{2\pi}{t+C} \quad \text{By integral}$$

Keep $A(t) \geq 0$. $T = t_{\max} \leq \square$

T

$(M, g(t)) \quad t \in (A, +\infty)$

$$|Rm| \leq \frac{C}{t}.$$

Thm 9 $M = \overset{\text{Coll.}}{M_{\text{thin}}} \cup \overset{\text{non Coll.}}{M_{\text{thick}}}$
 $\Downarrow$
Seifer space $\sim \rightarrow M_{\infty}$ admit metric $K \equiv -1$

4D - Singularity very difficult.

Shrinking FIK concept cone $\mathbb{CP}^1$ like bundle.

Bamler - C, - D, - R bounded, curv.

4D
Kahler Maz - Li, remove bounded curve.

Bamler open (he thinks $\exists |Rm|$ unbounded.).

Steady
also hardy

Appleton $U(2)$ $S^3/\sim$

Lai fly over $n \geq 4$ noncollapsing

Bamler : $\frac{11}{8}$ -conjecture

李直通 4维光滑流形拓扑分类

Notices AMS

He pointed out many problem when using Ricci flow.