

§ 1. Calculus of Variations

- Dirichlet principle
 Harmonic funcs. $\rightarrow \begin{cases} u|_{\partial\Omega} = g \\ \Delta u = 0 \text{ in } \Omega \end{cases}$
 are local minimizers of $\int_{\Omega} |\nabla u|^2 dx$

- Regularity $\Delta u = 0$ in Ω , $u \in C^\infty(\Omega)$

- Hilbert's 19th Problem (1900)

$$\mathcal{E}(u) = \int_{\Omega} F(\nabla u) dx, \quad u|_{\partial\Omega} = g$$

F smooth, uniformly convex function
 $\mathbb{R}^n \rightarrow \mathbb{R}$! $u \in H^1(\Omega)$

Q: any local minimizer is smooth?

$$u \in H^1(\Omega) \longrightarrow C^\infty(\Omega) ?$$

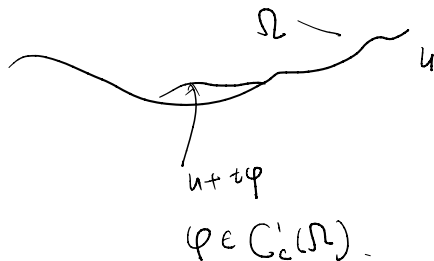
- Euler-Lagrange equation.

$$\mathcal{E}(u + t\varphi) \geq \mathcal{E}(u)$$

$$0 = \left. \frac{d}{dt} \right|_{t=0} \mathcal{E}(u + t\varphi)$$

$$= \left. \frac{d}{dt} \right|_{t=0} \int_{\Omega} F(\nabla u + t \nabla \varphi) dx$$

$$= \int_{\Omega} F_{\xi}(\nabla u) \cdot \nabla_{\xi} \varphi dx$$



$$= - \int_{\Omega} D_j F_j(\nabla u) \cdot \varphi \, dx$$

$$\Downarrow$$

$$D_j F_j(\nabla u) = 0$$

$$F: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$p \mapsto F(p)$$

$$F_{p_j} = \frac{\partial F}{\partial p_j}$$

$$\longrightarrow E-L$$

• If $u \in C^2$, $F_{ij}(\nabla u) \partial_{ij} u = 0$.

$$a_{ij}(x) = F_{ij}(\nabla u(x)) \quad , \quad a_{ij} u_{ij} = 0$$

• If $u \in C^{1,\alpha}$ $\Rightarrow \nabla u \in C^\alpha \Rightarrow a_{ij}(x) \in C^\alpha$

KEY

$$\Rightarrow u \in C^{2,\alpha} \Rightarrow \nabla u \in C^{1,\alpha} \Rightarrow a_{ij}(x) \in C^{1,\alpha}$$

Schauder

$$\Rightarrow D^2 u \in C^{1,\alpha} \Rightarrow u \in C^{3,\alpha} \xrightarrow{\text{bootstrap}} u \in C^\infty$$

$$\vec{e} \in S^{n-1}, \quad u_e = \partial_e u = v \quad \in C^\alpha? \quad \text{PDE?}$$

$$u \in H^1(\Omega)$$

$$v \in L^2(\Omega)$$

$$D_j (F_{ij}(\nabla u) \partial_i u_e) = 0$$

$$D_j (a_{ij}(x) \partial_i v) = 0$$

$$\lambda I_n \leq \begin{pmatrix} a_{ij}(x) \end{pmatrix} \leq \Lambda I_n$$

$$\operatorname{div} (A(x) \cdot \nabla v) = 0$$

2nd order, uniformly elliptic. PDE

in divergence form.

- $a_{ij}(x)$
- ① measurable L^1
 - ② bounded.
 - ③ uniformly elliptic.
- $\Downarrow$
- $v \in C^\alpha$

not continuous

Schauder theory
doesn't work.

Claim: Freezing coefficient $a_{ij}(0) = \delta_{ij}$

$$a_{ij} \partial_j u = f, \quad \Delta u = f + (a_{ij}(0) - a_{ij}(x)) \partial_j u$$

$$\|u\|_{C^{2,\alpha}} \leq C(\|f\|_{C^\alpha} + \varepsilon \|u\|_{C^{2,\alpha}})$$

$$\Rightarrow \|u\|_{C^{2,\alpha}} \leq C\|f\|_{C^\alpha} \quad \leftarrow \text{depending on regularity of } a_{ij}$$

Thm [De Giorgi 1957]

$$D_i(a_{ij}(x) \partial_j v) = 0 \quad \text{in } B_1, \quad a_{ij} \begin{matrix} \leq 0 \\ \geq 0 \\ \leq 0 \end{matrix} \quad \text{then } v \in C^\alpha(B_{1/2})$$

$$\text{and } \|v\|_{C^\alpha(B_{1/2})} \leq C\|v\|_{L^2(B_1)}$$

Proof: Sobolev inequality + energy identity □

For non-divergence equation

Thm [Krylov - Safonov 1979]

$$a_{ij}(x) \partial_j u(x) \text{ in } B_1, \text{ then } u \in C^\alpha(B_{1/2})$$

• Motivation Fully Nonlinear equation.

$$F(D^2 u) = 0 \quad \text{uniformly Elliptic}$$

$$F_{ij}(D^2 u) \partial_j u = 0$$

Krylov - Safonov
[1979]

$$a_{ij}(x) = F_{ij}(D^2 u), \quad a_{ij}(x) D_j u = 0 \Rightarrow u \in C^\alpha$$

$$\text{KEY: } u \in C^{2,\alpha} \Rightarrow a_{ij}(x) \in C^\alpha$$

$$\Rightarrow C^{1,\alpha}$$

$$\text{Schauder} \Rightarrow u \in C^{3,\alpha} \Rightarrow a_{ij} \in C^{1,\alpha}$$

$$\text{bootstrap} \Rightarrow u \in C^\infty$$

Evans - Krylov [1982]

① Sobolev inequality

Let W be supported in B_1 for some $p > 2$.

$$\|W\|_{L^p(B_1)} \lesssim \|\nabla W\|_{L^2(B_1)}$$

Proof: $W(0) = \int_{-\infty}^0 \frac{d}{dt} W(0+t\vec{v}) dt, \quad \vec{v} = S^{n-1}$

$$= - \int_0^\infty \vec{v} \cdot \nabla W(t\vec{v}) dt.$$

$$\therefore W(0) = - \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_0^\infty \vec{v} \cdot \nabla W(t\vec{v}) dt d\vec{v}$$

$$\begin{aligned} t\vec{v} &= y \\ dt d\vec{v} \cdot t^{n-1} &= dy \end{aligned} \quad = - \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_0^\infty \frac{t\vec{v} \cdot \nabla W(t\vec{v})}{t \cdot t^{n-1}} t^{n-1} dt d\vec{v}$$

$$= - \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \int_{\mathbb{R}^n} \frac{y \cdot \nabla W(y)}{|y|^n} dy$$

$$\Rightarrow \forall x \in \mathbb{R}^n \quad W(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla W(y) \cdot (x-y)}{|x-y|^n} dy$$

When W is supported in B_1 , then $\forall x \in B_1$,

$$W(x_0) = \frac{1}{\omega_{n-1}} \int_{B_1} \frac{\nabla W(y) \cdot (x_0 - y)}{|x_0 - y|^n} dy$$

$$\int |W|^p \approx \int_{B_1} \left(\int_{B_1} \frac{\nabla W(y) \cdot (x_0 - y)}{|x_0 - y|^n} dy \right)^p dx_0$$

Kernel $G \approx \frac{1}{|x_0 - y|^{n-1}} \sim \text{almost } \frac{n}{n-1} - \text{integral.}$ " $G(x_0, y)$ "

$$\begin{aligned} \therefore \int |W|^p &\lesssim \int_{B_1} \left(\int_{B_1} |\nabla W| |G| \right)^p \\ &\lesssim \int_{B_1} \|\nabla W\|_{L^2}^p \|G\|_{L^2}^p \\ &\lesssim \|\nabla W\|_{L^2}^p \underbrace{\int_{B_1} \|G\|_{L^2}^p}_{\int \left(\frac{1}{r^{2n-2}} r^n \right)^{\frac{p}{2}}} \end{aligned}$$

Actually $r^{n + \frac{p(2-n)}{2}}$ bounded? $r \in [0,1]$

$$n + \frac{p(2-n)}{2} > 0$$

need. $2 < p < \frac{2n}{n-2}$

② Energy identity

if v is a sub solution. $Lv \geq 0, v \geq 0$

and a cut-off function $\varphi \in C_0^\infty(B_1)$

then
$$\int_{B_1} |\nabla(\varphi v)|^2 dx \leq C \sup |\nabla \varphi|^2 \int_{B_1 \cap \text{supp } \varphi} v^2$$

proof : $D_i(a_{ij} \partial_j v) = 0$

$\varphi^2 v \geq 0$

$0 \leq \int \varphi^2 v D_i(a_{ij} \partial_j v)$

$\int D_i(\varphi^2 v) \cdot a_{ij} \partial_j v \leq 0.$

$$\begin{aligned} a_{ij}(\varphi v)_i (\varphi v)_j &= v^2 a_{ij} \varphi_i \varphi_j \\ &\quad + \underline{a_{ij} \varphi^2 v_i v_j} \\ &\quad + \underline{2 a_{ij} \varphi_i v_j \varphi v} \end{aligned}$$

$$\int \underline{2 \varphi \varphi_i v \cdot a_{ij} v_j + \varphi^2 v_i a_{ij} v_j} \leq 0$$

i.e. $\int a_{ij}(\varphi v)_i (\varphi v)_j \leq \int v^2 a_{ij} \varphi_i \varphi_j$

$2 \int |\nabla(\varphi v)|^2 \leq$

$\leq \Lambda \sup |\nabla \varphi|^2$

$\int_{B_1 \cap \text{supp } \varphi} v^2$

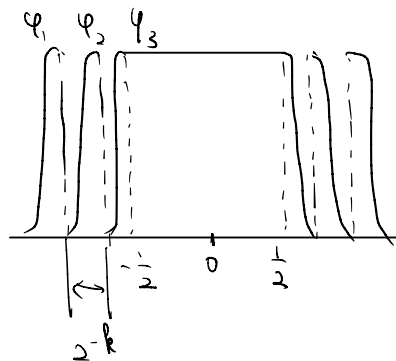
□

Proof of Thm De Giorgi

Part I. $L^2 \rightarrow L^\infty$

1) cut-off

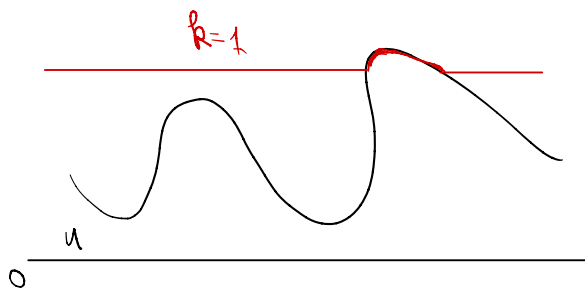
$$\varphi_k = \begin{cases} 1, & B_{\frac{1}{2}+2^{-k}} \\ |\nabla \varphi_k| \leq C \cdot 2^k \\ 0 & \text{outside } B_{\frac{1}{2}+2^{-k+1}} \end{cases}$$



Note $\varphi_{k-1} \equiv 1$ in the support of φ_k

$$u_k = (u - (1 - 2^{-k}))^+$$

$$k \rightarrow \infty, u_k = (u - 1)^+$$



Aim As $k \rightarrow \infty$ $\varphi_k u_k \rightarrow 0$.

Idea: to control $\int |\varphi_{k+1} u_{k+1}|^2$ by $\int |\varphi_k u_k|^2$

$$\rightarrow \int |\nabla(\varphi_k u_k)|^2$$

By Sobolev $\left(\int |u_k \varphi_k|^p \right)^{\frac{2}{p}} \lesssim \int |\nabla(\varphi_k u_k)|^2$

$$\int |u_k \varphi_k|^2 \leq \left(\int (u_k \varphi_k)^{2\bar{p}} \right)^{\frac{1}{\bar{p}}} \left(\int 1^{\bar{q}} \right)^{\frac{1}{\bar{q}}}$$

$$\bar{p} = \frac{p}{2} = \left(\int (u_k \varphi_k)^p \right)^{\frac{2}{p}} \left(\int 1 \right)^{\frac{p}{p-2}} \quad \Sigma = 1 - \frac{2}{p}$$

$$\leq \left(\int (\varphi_k u_k)^p \right)^{\frac{2}{p}} \left| \{ \varphi_k u_k > 0 \} \right|^2$$

Denote $A_k = \int (\varphi_k u_k)^2$

$$E_k = \int |\nabla(\varphi_k u_k)|^2$$

$$A_k \leq E_k \left| \{ \varphi_k u_k > 0 \} \right|^2 \leq \dots$$

$$\begin{aligned} E_k &= \int |\nabla \varphi_k u_k|^2 \lesssim \sup |\nabla \varphi_k|^2 \int u_k^2 \\ &\lesssim 2^{2k} \int_{\text{supp}(\varphi_k)} u_k^2 \\ &\quad \text{supp}(\varphi_k) \end{aligned}$$

Note on supp of φ_k

$$\Rightarrow \varphi_{k-1} \equiv 1. \quad u_k \leq u_{k-1}$$

$$\therefore E_k \lesssim 2^{2k} \int_{\text{supp}(\varphi_{k-1})} u_{k-1}^2 \lesssim 2^{2k} A_{k-1}$$

Hence $A_k \lesssim 2^{2k} \left| \{ \varphi_k u_k > 0 \} \right|^2 A_{k-1}$

§ 2. De Giorgi

Thm Let u be a solution of $D_i(a_{ij}(x) \partial_j u) = 0$ in $B_{1/2}$

a_{ij} satisfying

- ① measurable
- ② bounded
- ③ uniformly elliptic.

$$0 < \lambda Id \leq a_{ij}(x) \leq \Lambda Id.$$

Then $u \in C^\alpha(B_{1/2})$ $0 < \alpha < 1$.

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \|u\|_{L^2(B_1)}$$

Proof · Part I $L^2 \rightarrow L^\infty$ | Tools ① Sobolev inequality
 Part II $L^\infty \rightarrow C^\alpha$ | ② Energy estimate

Lemma 1 if $\|u^+\|_{L^2(B_1)}$ is small enough $< \delta$
 then $\sup_{B_{1/2}} u^+ \leq 1$.

Back to proof. $\varphi_k u_k > 0$.

$$\varphi_k > 0 \Rightarrow \varphi_{k+1} \equiv 1$$

$$u_k > 0 \Rightarrow u_{k+1} \geq 2^{-k}$$

$$\begin{aligned} |\{\varphi_k u_k > 0\}| &\leq |\{(\varphi_{k+1} u_{k+1})^2 \geq 2^{-2k}\}| \\ &\leq 2^{2k} \int (\varphi_{k+1} u_{k+1})^2 = 2^{2k} A_{k+1} \end{aligned}$$

$$\Rightarrow A_k \lesssim 2^{2k(1+\varepsilon)} A_{k-1}^{1+\varepsilon}$$

Claim if $\int A_0 < \infty$

then $A_k \rightarrow 0$ as $k \rightarrow \infty$

Proof $A_k \lesssim C^k A_{k-1}^{1+\varepsilon} \quad k = 2^{2(1+\varepsilon)}$

$$\Rightarrow A_k \lesssim C^{k+(k-1)(1+\varepsilon)+(k-2)(1+\varepsilon)^2+\dots} A_0^{(1+\varepsilon)^k}$$

$$\lesssim C^{(1+\varepsilon)^k} A_0^{(1+\varepsilon)^k}$$

$$\lesssim \left((C A_0)^{1+\varepsilon} \right)^k$$

$$k \rightarrow \infty. \quad A_k \rightarrow 0.$$

Remark (Moser's proof) [1960]

$$\int |\nabla(\varphi v)|^2 \leq C \int v^2 |\nabla \varphi|^2$$

Moser: $\varphi^2 v^p \quad (p \geq 1)$

$$\Rightarrow \int |\nabla(v^{\frac{p+1}{2}} \varphi^2)|^2 \leq C \int v^{p+1} |\nabla \varphi|^2$$

$r_k = \frac{1}{2} + 2^{-k} \quad \oplus \quad \text{Sobolev}$

$$\Rightarrow \frac{1}{2} \quad p > 1 \quad \|v\|_{L^{2p^*}(B_{r_k})} \leq C \|v\|_{L^2(B_1)}$$

Lemma 2 Let $\text{Osc}_D u = \sup_D u - \inf_D u$

if u is a solution of $Lu=0$ in B_1 ,

and $\text{Osc}_{B_1} u$ is 2 (i.e. $|u| \leq 1$)

then $\text{Osc}_{B_{1/2}} u = \sigma \cdot 2$ for $\sigma < 1$ depending

on λ, I .

i.e. $\exists \sigma(\lambda, \lambda, n) < 1$ s.t. $\text{Osc}_{B_{1/2}} u \leq \sigma \cdot \text{Osc}_{B_1} u$.

★ Suppose $u^+|_{B_{3/4}} = 0$ except for a set of measure $\frac{\delta}{2} \Rightarrow u^+|_{B_{1/2}} \leq \frac{1}{2}$ done!

$$u^+|_{B_{3/4}} = 0 \geq 50\%$$

"half of time"

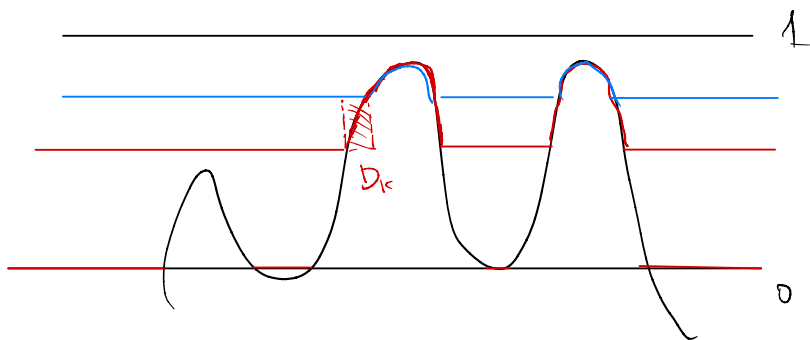
$\Downarrow$

"most of time"

$$u_1 = \frac{(u_0 - \frac{1}{2})^+}{2^{-1}}$$

$$\vdots$$

$$u_k = \frac{(u_0 - (1 - 2^{-k}))^+}{2^{-k}}$$



$$A_k = \{u_k = 0\} \cap B_{3/4}$$

$$C_k = \{u_k \geq \frac{1}{2}\} \cap B_{3/4}$$

$$D_k = \{0 < u_k < \frac{1}{2}\} \cap B_{3/4}$$

$$|A_k| > 1 - \frac{\delta}{2}$$

$$D_k = A_{k+1} - A_k$$

if $\exists k_0$ $k < k_0$.

Claim: $|D_k| \geq |C_k|^{2-\frac{2}{n}}$

s.t. $|C_k| < \frac{\delta}{2}$ done!

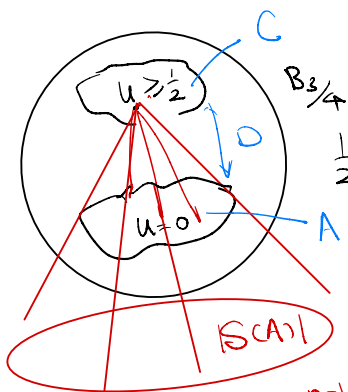
if not $|C_k| > \frac{\delta}{2}$

$$|D_k| \geq \left(\frac{\delta}{2}\right)^{2-\frac{2}{n}}$$

$$|A_k| = \sum_{k=0}^{k_0} |D_k| \gtrsim k_0 \left(\frac{\delta}{2}\right)^{2-\frac{2}{n}} > |B_{\frac{\delta}{4}}|$$

Energy identity $\Rightarrow u \in H^1(B_{\frac{\delta}{4}})$

$$x_0 \in \{u \geq \frac{1}{2}\}$$



$$\frac{1}{2} \leq u(x_0) = \int_{\text{segment}} u_r dr$$

$$= \frac{1}{|S(A)|} \int \frac{\nabla w \cdot (x_0 - y)}{|x_0 - y|^n} dy$$

cone.
projection to S^{n-1}
 $S(A) \subset S^{n-1}$

$$|A|_{H^{n-1}} \leq C |S(A)|_{H^{n-1}}$$

$$\Rightarrow |A| \lesssim \int_{\text{cone } A} \frac{\nabla w \cdot (x_0 - y)}{|x_0 - y|^n} dy$$

$$\lesssim \int_D \frac{|\nabla w(y)|}{|x_0 - y|^{n-1}} dy$$

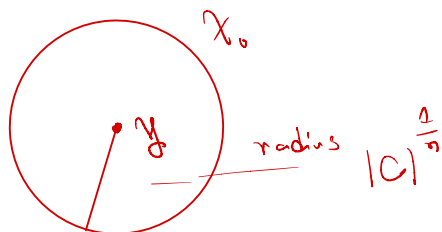
integrate x_0 in C .

$$|A||C| \leq \int_D dy \int_C \frac{|\nabla w(y)|}{|x_0 - y|^{n-1}} dx_0$$

$$\underbrace{|A||C|}_{\frac{1}{2}} \leq \int_D |\nabla w| dy \int_C \frac{1}{|x_0 - y|^{n-1}} dx_0$$

$$\lesssim |D|^{\frac{1}{2}} \cdot |C|^{\frac{1}{2}}$$

$$|D_K| \lesssim |C_K|^{2 - \frac{2}{n}}$$



$$\begin{aligned} & \int_C \frac{1}{|x_0 - y|^{n-1}} dx_0 \\ &= \int_0^{K^{\frac{1}{n}}} \frac{dr}{r^{n-1}} \\ &= \int_0^{K^{\frac{1}{n}}} \frac{1}{r^{n-1}} dr \\ &\lesssim |C|^{\frac{1}{n}} \end{aligned}$$

$dx_0 = r^{n-1} dr d\theta$

Gor $\text{OSC}_{B_{\frac{1}{2}}} u \leq \sigma \text{OSC}_{B_1} u, \quad \sigma < 1.$

$\searrow$

$$\text{OSC}_{B_{2^{-k}}} u \leq \sigma^k \text{OSC}_{B_1} u$$

$$\Rightarrow u \in C^\alpha(B_{1/2}), \quad \alpha \sim \sigma$$

Choose x near 0.

$$\begin{aligned} |x| = 2^{-k} \quad |u(x)| &\leq \sigma^k = \sigma^{-\log_2 |x|} \\ &= |x|^{-\frac{1}{\log_2 \sigma}} \end{aligned}$$

$\downarrow$
 α

Cor, Liouville-Thm

Let u be a solution of $D_j(a_{ij} \partial_i u) = 0$ in $\mathbb{R}^n$.

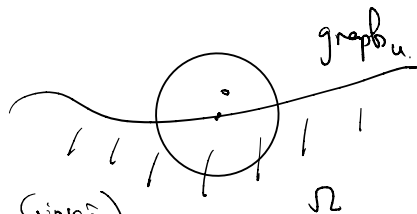
$$\& \|u\|_{L^\infty(\mathbb{R}^n)} \leq C.$$

Then u must be a constant.

§ 3.

$$\begin{cases} \min \int_{\Omega} F(\nabla u) dx \\ u = g \text{ on } \partial\Omega \end{cases}$$

$$F(\nabla u) = \sqrt{1 + |\nabla u|^2}$$



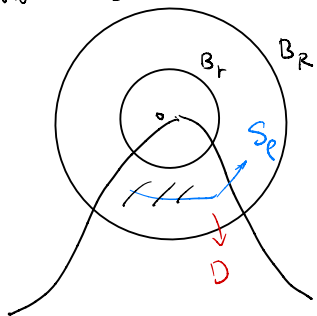
• Uniformly density lemma (by De Giorgi)

$$\text{Vol}(\Omega \cap B_r(x)) \geq \delta_0 > 0$$

Idea: If $\text{vol}(\Omega \cap B_r(x)) < \delta_0$, then $\Omega \cap B_{\frac{r}{2}}(x) = \emptyset$

$\exists x \in \Omega$ and

Two tools:



$$\textcircled{1} \text{ vol}(D) = \int_r^R \text{Area}(S_p) dr$$

$$\min \text{Area}(S_p) \leq \frac{\text{vol}(D)}{R-r}$$

$$\text{vol} \sim \int u^2$$

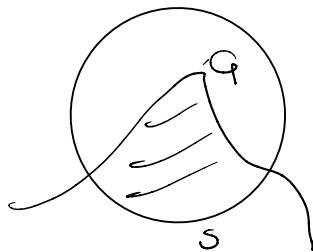
$$\text{area} \sim \int |\nabla u|^2$$

$$\int |\nabla u|^2 \leq \text{const.} \cdot \int u^2 \sim \text{Energy identity}$$

$\textcircled{2}$ Isoperimetric identity

$$\text{vol} \lesssim (\text{area}(G \cup S))^{\frac{n}{n-1}}$$

$$G \text{ min. surf} \lesssim (\text{area } S)^{\frac{n}{n-1}}$$



$$B_k = B_{\frac{1}{2} + \frac{1}{2^k}}$$

$$D_k = (B_k - B_{k+1}) \cap \Omega$$

$$S_k = \min_{\frac{1}{2} + \frac{1}{2^{k+1}} < \rho < \frac{1}{2} + \frac{1}{2^k}} \text{Area}(S_\rho)$$

$$\text{From } ① \quad S_k \leq \frac{\text{Vol}(D_k)}{2^k} = 2^k \text{Vol}(D_k)$$

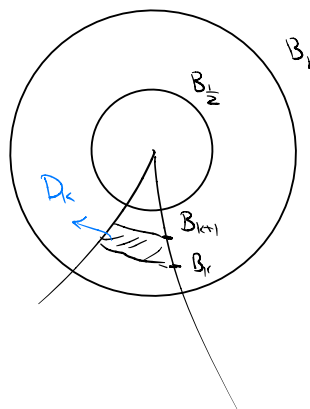
$$\text{From } ② \quad \text{Vol}(D_k) \leq C S_{k-1}^{\frac{n}{n-1}}$$

$$S_k \leq C \cdot 2^k S_{k-1}^{\frac{n}{n-1}} \leq C^k S_{k-1}^{n+1}$$

$\Rightarrow$ if $S_1 < S_0$, then $S_k \rightarrow 0$. By isoper. ineq.

$$\text{Vol}(\Omega \cap B_{\frac{1}{2}}) = 0$$

(De Giorgi: $L^2 \rightarrow L^\infty$ 几何版本).



$$L^\infty \rightarrow L^\alpha$$

Moser's Harnack inequality

{ local maximum principle
weak Harnack (Two steps)}

① (local m.p.)

Lemma. Let u be a solution of $D_j(a_{ij}u_i) \geq 0$

we have $\operatorname{ess\,sup}_{B_{1/2}} u \leq C \left(\int_{B_1} u^p \right)^{\frac{1}{p}} \quad \forall p > 0$

$$\text{pf: } \left(\int_{B_1} u^2 \right)^{\frac{1}{2}} = \left(\int_{B_1} u^p \cdot u^{2-p} \right)^{\frac{1}{2}} \leq \|u\|_{L^\infty}^{\frac{2-p}{2}} \left(\int_{B_1} u^p \right)^{\frac{1}{2}}$$

$$\left[ab \leq \frac{a^p}{p} + \frac{1}{p} \frac{b^q}{q} \right] \leq \frac{\|u\|_{L^\infty}^p}{p} + \frac{C}{p} \|u\|_{L^p}^p \cdot \frac{p}{2}$$

$$\therefore \|u\|_{L^\infty(B_{1/2})} \leq 2\|u\|_{L^p(B_1)} + C \left(\int_{B_1} u^p \right)^{\frac{1}{p}}$$

Iteration ...

□

② Weak Harnack.

$D_j(a_{ij}u_i) \leq 0$, $\operatorname{ess\,inf}_{B_{1/2}} u \geq C \left(\int_{B_1} u^{p_0} \right)^{\frac{1}{p_0}}$ for some $p_0 > 0$

pf. if u supersolution. $\rightarrow u \neq 0$, by consider $u + \varepsilon$

then $\frac{1}{u}$ subsolution

$$\uparrow \text{ then apply ①: } \frac{1}{\operatorname{ess\,inf}_{B_{1/2}} u} \leq C \left(\int_{B_1} \left(\frac{1}{u} \right)^{p_0} \right)^{\frac{1}{p_0}}$$

$$\operatorname{ess\,inf}_{B_{1/2}} u \geq C \left(\int_{B_1} u^{-p_0} \right)^{-\frac{1}{p_0}}$$

$$\geq C \left[\left(\int_{B_1} u^{-p} \right) \left(\int_{B_1} u^p \right) \right]^{-\frac{1}{p}} \left(\int_{B_1} u^p \right)^{\frac{1}{p}}$$

if $\left(\int_{B_1} u^{-p} \right) \left(\int_{B_1} u^p \right) \leq C$ then done!

John-Nirenberg (for some p_0 , true)

① + ② = ③ (Moser's Harnack ineq.) if u is a sol.

then $\operatorname{ess\,sup}_{B_{1/2}} u \leq C \operatorname{ess\,inf}_{B_{1/2}} u$.

• Moser's Harnack ineq. $\Rightarrow C^\alpha$.

Pf: $M(r) = \operatorname{ess\,sup}_{B_r} u$, $m(r) = \operatorname{ess\,inf}_{B_r} u$,
 $w(r) = M_r - m_r$ ($\operatorname{osc}_{B_r} u$).

(i) $V = M(r) - u$, apply Harnack to V

$$M(1) - m(\tfrac{1}{2}) \leq C(M(1) - M(\tfrac{1}{2})) \quad (1)$$

(ii) $V = u - m(r)$. ~~~~~

$$M(\tfrac{1}{2}) - m(1) \leq C(m(\tfrac{1}{2}) - m(1)) \quad (2)$$

$$(1) + (2) \quad w(1) + w(\tfrac{1}{2}) \leq C(w(1) - w(\tfrac{1}{2}))$$

$$\therefore w(\tfrac{1}{2}) \leq \underbrace{\frac{C-1}{C+1}}_{< 1} w(1)$$

§ 4. Krylov - Safonov

Thm [1979] Let $u \geq 0$ be a solution of $a_{ij} D_{ij} u = 0$

in $B_{1/2}$, w/ $\{a_{ij}\}$ uniformly elliptic.

Then $\sup_{B_{1/2}} u \leq C \inf_{B_1} u \leftarrow$ weak Harnack

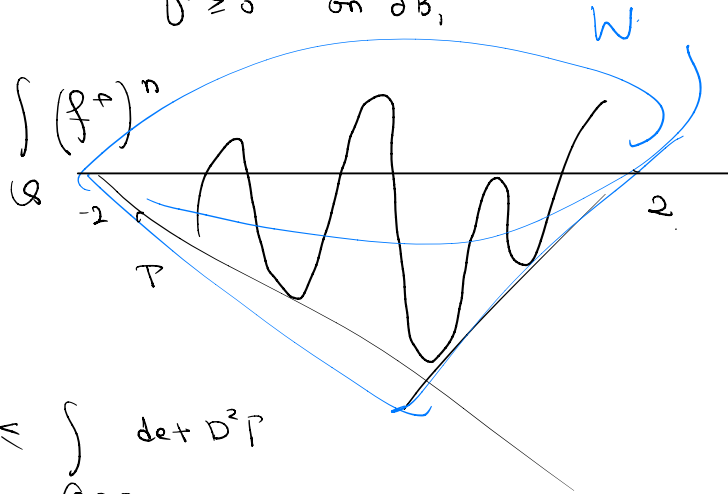
Preliminaries

local $m-p$ $a_{ij} D_{ij} v = f$ in $B_1(0)$

① A-B-P Lemma.

$v \geq 0$ on ∂B_1

$$(\sup v^-)^n \leq \int (f^+)^n$$



Proof: $\text{Vol}(\nabla \Gamma(B_1))$

upper bound

$$\text{Vol}(\nabla \Gamma(B_1)) \leq \int_{Q \cap B_1} \det D^2 \Gamma$$

$$\leq \int_{Q \cap B_1} \det D^2 v$$

$$\lesssim \int_{Q \cap B_1} (a_{ij} D_{ij} v)^n$$

$$\lesssim \int_{Q \cap B_1} (f^+)^n$$

Lower bound

$$|\nabla Q| \leq \frac{\sup v^-}{4} \Rightarrow \text{Vol}(\nabla \Gamma(B_1)) \geq \frac{(\sup v^-)^n}{4^n}$$

$$\nabla W(B_1) \subset \nabla \Gamma(B_1)$$

$$|\nabla W(B_1)| \leq |\nabla \Gamma(B_1)|$$

② C-Z (A-B lemma)

$$A \subset B \subset Q_1$$

$$(i) |A| \leq \delta$$

$$(ii) \frac{|A \cap Q|}{|Q|} > \delta, \text{ then } Q^* \subset B$$

$$\text{Then } |A| \leq \delta |B|$$

$$Q_0, Q_1, Q_2 \dots \text{ disjoint}$$

$$\cup Q_i \supset A$$

$$Q_0^*, Q_1^*, Q_2^* \dots$$

Q_i^*, Q_j^* either identity or disjoint.

$$\text{Take } Q_i^*, Q_j^* \text{ disjoint}$$

$$\cup Q_i^* = A$$

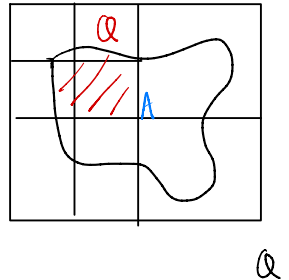
$$\frac{|A \cap Q_i|}{|Q_i|} > \delta \stackrel{(ii)}{\Rightarrow} Q_i^* \subset B$$

$$\frac{|A \cap Q_i^*|}{|Q_i^*|} \leq \delta.$$

$$|A| = |A \cap \cup Q_i^*|$$

$$= \sum_i |A \cap Q_i^*|$$

$$\leq \delta \sum_i |Q_i^*| = \delta |\cup Q_i^*| \leq \delta |B|.$$



Lemma 1 (Weak Harnack)

Let $a_{ij} D_{ij} v \leq 0$, $v \geq 0$ in B_1 , $v|_{\partial B_1} = 1$
 then $v \in L^2_{\text{weak}}(B_{1/2})$ in the weak sense.

$$\text{i.e. } |\{v > \lambda\}| \leq \lambda^{-2}$$

Idea: $\tilde{v} = v + 2(|x|^2 - 1)$

$$\tilde{v}|_{\partial B_1} = -1, \quad \tilde{v}|_{\partial B_1} \geq 0$$

$$L \tilde{v} = \underline{L} v + 2L(|x|^2 - 1)$$

$$\leq C(|a_{ij}|)$$

$$\leq \int (\sup v^-)^n \leq \int_{\text{contact set}} C \leq C |\text{contact set}|$$

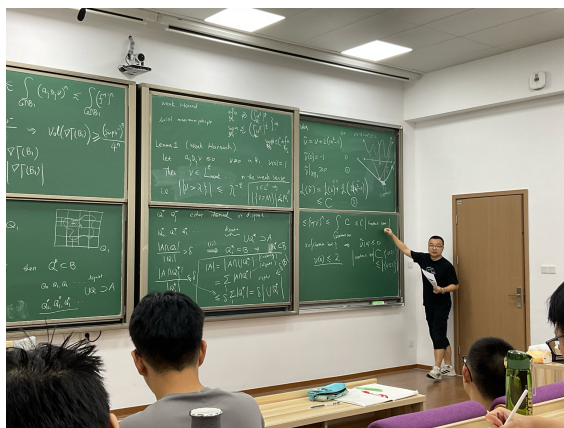
$$x \in \{\text{contact set}\}$$

$$\Rightarrow \tilde{v}(x) \leq 0$$

$$\Rightarrow v(x) \leq 2$$

$$\{\text{contact set}\} \subset \{v \leq 2\}$$

$$|\{\text{contact set}\}| \leq |\{v \leq 2\}|$$



$$1 \leq C |\{v \leq 2\}| \Rightarrow |\{v \leq 2\}| \geq \theta > 0$$

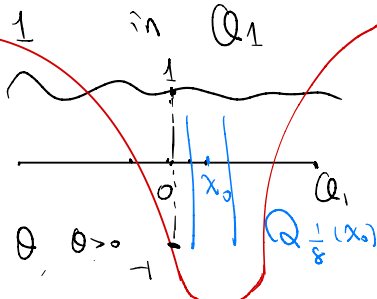
$$\Rightarrow |\{v > 2\}| \leq 1 - \theta.$$

Lemma 0

• $\sup_{D_j} v \leq 0, \quad v \geq 0, \quad v(0) = 1$ in Q_1

$\forall x_0 \in Q_{\frac{1}{8}}$

then $|\{v > M\} \cap Q_{\frac{1}{8}}(x_0)| \leq 1 - \theta, \quad \theta > 0$



$\sup_{D_j} v \leq C$

Proof: $\tilde{v} = v + f \leftarrow \text{construct}$

s.t. $\tilde{v}(0) = -1 \quad L[\tilde{v}] = L[v] + L[f] \leq 0$

$\tilde{v}|_{\partial B_1} \geq 0 \leq C \chi_{Q_{\frac{1}{8}}(x_0)}$

Applying A-B-P to $\tilde{v}$

$$1 \leq \left(\sup \tilde{v}^- \right)^n \leq \int_C \leq C |\Phi \cap B_{\frac{1}{8}}(x_0)|$$

contact set Φ

$x \in \Phi$, then $\tilde{v} \leq 0 \Rightarrow v(x) + f(x) \leq 0$

$v(x) \leq M$

$\Rightarrow |\{v \leq M\} \cap Q_{\frac{1}{8}}(x_0)| \geq \theta > 0$

Proof of Lem 1 (weak Harnack)

$$\bullet A_k = \{v > M^k\} \cap Q_{\frac{1}{2}} =: A \quad \text{decay estimate on } |A_k|$$

$$A_{k-1} =: B$$

Apply A-B lemma.

$$\text{if } Q$$

$$\frac{|A \cap Q|}{|Q|} \geq \underbrace{1-\theta}_S, \text{ then } Q^* \subset B$$

$$\frac{|A_k \cap Q|}{|Q|} \geq 1-\theta. \quad \text{aim } Q^* \subset A_{k-1}$$

$$\uparrow$$

$$\forall x \in Q^*, v(x) > M^{k-1}$$

$$\text{if not, } \exists x_0 \in Q^*, v(x_0) \leq M^{k-1}.$$

$$\tilde{v} = \frac{v}{M^{k-1}}, \quad \tilde{v}(x_0) \leq 1$$

Apply Lem 0 to $\tilde{v}$

$$\frac{|\{\tilde{v} > 1\} \cap Q|}{|Q|} \leq 1-\theta$$

$$\Downarrow$$

$$\frac{|\{v > M^k\} \cap Q|}{|Q|} < 1-\theta \quad \text{矛盾}$$

By A-B Lemma. $\Rightarrow |A_k| \leq (1-\theta)^k |A_0|$

$$\left| \left\{ v > \frac{M^k}{\lambda} \right\} \cap Q_{\frac{1}{2}} \right| \leq C(1-\theta)^k$$

choose $\varepsilon > 0$ small

$$\left| \left\{ v > \lambda \right\} \cap Q_{\frac{1}{2}} \right| \leq C M^{-k\varepsilon} \lesssim \lambda^\varepsilon \quad \checkmark$$

$$\begin{aligned} \int_{Q_{\frac{1}{2}}} v^2 &\leq M^2 \left| \left\{ v < M \right\} \right| + M^{2^2} \left| \left\{ M \leq v < M^2 \right\} \right| + \dots \\ &\leq M^2 + M^{2^2} (1-\theta) + M^{3^2} (1-\theta)^2 \\ &\leq \frac{M^2}{1-M^2(1-\theta)} \leq +\infty \end{aligned}$$

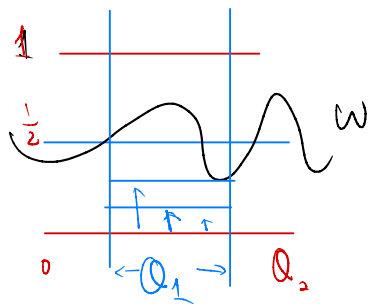
we get weak Harnack

$$\begin{aligned} v &\geq 0 \quad a_{ij} D_{ij} v \leq 0 \quad \text{in } Q_3 \\ \text{then } \left(\int_{Q_1} v^2 \right)^{\frac{1}{2}} &\leq C \inf_{Q_2} v \end{aligned}$$

Thm Let u be a solution of $a_{ij} D_{ij} u = 0$.
Then $u \in C^\alpha$.

$$\text{Proof.} \quad \text{aim} \quad \text{osc}_{Q_1} u \leq \theta \text{osc}_{Q_2} u \quad \theta < 1$$

Let $w = \frac{u - \inf_{Q_2} u}{\operatorname{osc}_{Q_2} u}$ then $Lw = 0$, $0 \leq w \leq 1$ in Q_2



Case 1. $|\{w \geq \frac{1}{2}\} \cap Q_1| \geq \frac{1}{2} |Q_1|$

apply weak Harnack to w .

$$\underbrace{\left(\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} |Q_1|\right)^{\frac{1}{2}}}_{\downarrow \frac{1+\varepsilon}{2}} \leq \left(\int_{Q_1} w^2\right)^{\frac{1}{2}} \leq C \inf_{Q_2} w \leq C \inf_{Q_1} w$$

$$\Rightarrow \inf_{Q_1} w \geq \left(\frac{1}{2}\right)^{\frac{1+\varepsilon}{2}} > 0.$$

Case 2 $|\{w \geq \frac{1}{2}\} \cap Q_1| < \frac{1}{2} |Q_1|$

Apply weak Harnack to $1-w$

$$|\{1-w \geq \frac{1}{2}\} \cap Q_1| > \frac{1}{2} |Q_1|$$

$$\left(\frac{1}{2}\right)^{\frac{1+\varepsilon}{2}} \leq \inf_{Q_1} (1-w)$$

$$\sup_{Q_1} w \leq 1 - \frac{1}{C} \left(\frac{1}{2}\right)^{\frac{1+\varepsilon}{2}} < 1.$$

□

Exercise

Krylov-Safonov's Liouville thm

if $Lu=0$ in $\mathbb{R}^n$ & $\|u\|_{L^\infty} \leq C$

then u must be a constant.

Lemma 2 (local maximum principle)

if w is a subsolution & $w \in L_{weak}^\varepsilon$ in Q_1 .

then $\sup_{Q_{1/2}} w \leq C$

Proof: if $w(x_0) = M_0$ 很大

then we can choose $Q(x_0)$ 不太大

δ 足够小, s.t. $\sup_{Q(x_0)} w > (1+\delta) M_0$

TBD

if not. claim that $|\{w < \frac{1}{2} m_0\} \cap Q_r| = |A|$

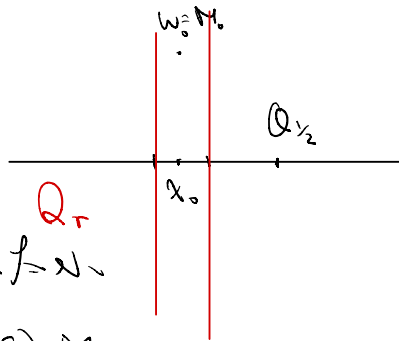
$|\{w > \frac{1}{2} m_0\} \cap Q_r| = |B|$

$$|A| < \frac{1}{2} |Q_r|$$

$$|B| < \frac{1}{2} |Q_r| \quad \text{矛盾}$$

By assumption $w \in L_{weak}^\varepsilon(Q_1)$.

$$\Rightarrow |\{w > \frac{1}{2}\} \cap Q_1| \leq M_0^{-\varepsilon}$$



Take ϵ small

$$\text{s.t. } |Q_r| = 2M_0^{-2}$$

$$\leq \frac{1}{2}|Q_r|$$

" δ "

by contradiction, $w < (1+\delta)M_0$ in Q_r

w , subsolution

$$\text{then } (1+\delta)M_0 - w \geq 0$$

$\downarrow$
super solution.

$$\tilde{w} = \frac{(1+\delta)M_0 - w}{\delta M_0}$$

$$\tilde{w}(x_0) = 1.$$

$$\Rightarrow \tilde{w} \text{ is } L_{\text{weak}}^2$$

in the set $\{w < \frac{1}{2}M_0\}$.

$$\Downarrow \\ \left\{ \tilde{w} > \frac{\frac{1}{2} + \delta}{\delta} \right\}.$$

$$\left| \left\{ \tilde{w} > \frac{\frac{1}{2} + \delta}{\delta} \right\} \cap Q_r \right| \leq \left(\frac{\delta}{\frac{1}{2} + \delta} \right)^2 |Q_r|$$

$$\parallel \\ |A \cap Q_r|$$

Choose δ small enough

$$\text{s.t. } \left(\frac{\delta}{\frac{1}{2} + \delta} \right)^2 < \frac{1}{2}$$

conclusion: $W \geq 0$ subsolution & $W \in L^\infty_{\text{weak}}$

and $W(x_0) = M_0$, then $\sup W \geq (1+\delta) M_0$.

$$\frac{\varepsilon}{n} = \widehat{\varepsilon} > 0 \quad Q_{M_0, \widehat{\varepsilon}}(x_0)$$

$$\begin{aligned} W(x_0) = M_0 \quad \exists x_1 \text{ s.t. } |x_1 - x_0| &\leq M_0^{-\frac{1}{\widehat{\varepsilon}}}, \quad W(x_1) \geq (1+\delta) M_0 \\ \exists x_2 \text{ s.t. } |x_2 - x_1| &\leq ((1+\delta) M_0)^{-\frac{1}{\widehat{\varepsilon}}}, \quad W(x_2) \geq (1+\delta)^2 M_0 \\ &\vdots \end{aligned}$$

$$W(x_k) \geq (1+\delta)^k M_0.$$

$\sum |x_k - x_{k-1}|$ bounded. $\therefore x_\infty$ 不会跑太远.

$$|x_\infty| < \frac{1}{\widehat{\varepsilon}}.$$

§ 5. Evans - Krylov.

Review. $a_{ij} D_{ij} u = 0$. no regularity of a_{ij}

$$\|(D^2 u)^+\| \sim \|(D^2 u)^-\| \quad \text{uniformly elliptic}$$

$$\lambda \text{Id} \leq (a_{ij}) \leq \Lambda \text{Id}$$

• If $a_{ij} D_{ij} u \leq 0$ supersolution $u \geq 0$ in B_1

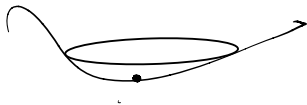
$$u(0) = 1 \Rightarrow \|u\|_{L^2_{\text{weak}}(B_{1/2})} \leq C \inf_{B_1} u$$



一点处值可控制 nbd. 积分平均

• If $a_{ij} D_{ij} u \geq 0$ subsolution $u \geq 0$ in B_1

$$u(0) = 1, \text{ then } L^2_{\text{weak}} \text{-norm controls } \sup_{B_1} u$$



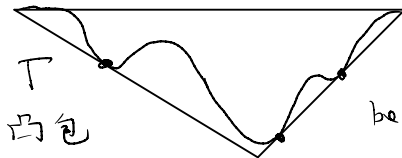
$$\left(\sup_{B_1} u \leq \left(\int_{B_1} u^2 \right)^{\frac{1}{2}} \cdot C \right)$$

• contact set

$$\text{"}\Sigma\text{"-theory } U \rightarrow L^\infty$$

$$DU \rightarrow L^\infty$$

$$\text{one-side } D^2 U \rightarrow L^\infty$$



belongs to contact set.

Reference: ① Evans 85 TAMS

② F. Lin 86 PAMS

Remark 1 (turn up-side down of weak Harnack)

• If u is a supersol. $u \geq 0$, $u(0) \leq 1$

$$\Rightarrow |\{u > M\}| \lesssim M^{-2}$$

• If u is a subsol. $u \leq 0$, $u(0) \geq -1$

$$|\{u < -M\}| \leq M^{-2}.$$

Conclusion If U is a supersol, $U \leq 0$, and

$$|\{U < -1\}| = \theta > 0, \text{ then } U(0) \leq -C(\theta).$$

(stay away from 0)

Remark 2 Consider uniformly elliptic PDE

$$F(D^2 u) = 0, \quad w/. \quad F(0) = 0$$

positive matrix

uniformly elliptic means. $F(M + N^+) = F(M) + O(\|N^+\|)$

$$F(M + N^-) = F(M) - O(\|N^-\|)$$

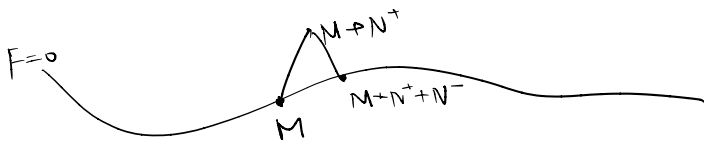
常規寫法. $F(M + N) - F(M)$

$$\leq \lambda \|N^+\| - \lambda \|N^-\|$$

Regard $F(M) = 0$ as a "surface" in the space of matrix $(\mathbb{R}^{n \times n})$

$$\text{if } F(M) = F(M + N^+ + N^-) = 0$$

then $\|N^+\| \sim \|N^-\|$
comparable.



$$F(M) = 0 \Rightarrow \|M^+\| \sim \|M^-\|.$$

Remark 3 (normalisation) $F(D^2 u) = 0$

$$\tilde{u} = \lambda u(\mu x) \quad \text{scaling.}$$

$$\text{If } \lambda = \frac{1}{\mu^2}, \quad D^2 \tilde{u} = D^2 u, \quad F(D^2 \tilde{u}) = 0.$$

$$\text{If not,} \quad D^2 \tilde{u} = \lambda \mu^2 D^2 u, \quad F\left(\frac{1}{\lambda \mu^2} M\right) = 0.$$

$C^{2,\alpha}$ -theory

In 2-D $\mathbb{R}^2$

Thm [Nirenberg 1953]

$$F(D^2 u) = 0 \quad \text{in } B_1 \subset \mathbb{R}^2 \quad \text{unif. ellip.}$$

$$\text{Then} \quad \|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\lambda, \mu) \|u\|_{L^\infty(B_1)}$$

$$\text{Proof: Let } v = \partial_e u, \quad a_{ij} = F_{ij}(D^2 u)$$

$$\sum_{i,j=1}^2 a_{ij} D_{ij} v = 0$$

$$a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2, \quad \xi_i = \xi_j = e_2, \quad \therefore a_{22} \geq \lambda_0$$

$$a_{11} \partial_{11} v + a_{12} \partial_{12} v + a_{21} \partial_{21} v + a_{22} \partial_{22} v = 0$$

$$\Rightarrow a_{11} \partial_{11} v + 2a_{12} \partial_{12} v + a_{22} \partial_{22} v = 0$$

$$\frac{a_{11}}{a_{22}} \partial_{11} v + 2 \frac{a_{12}}{a_{22}} \partial_{12} v + \partial_{22} v = 0$$

i.e. $a \partial_{11} V + b \partial_{12} V + \partial_{22} V = 0$

Let $W = \partial_1 V$

$\partial_1 (a \partial_1 W + b \partial_2 W) + \partial_{22} W = 0$

Define $A = \begin{pmatrix} a(x) & b(x) \\ 0 & 1 \end{pmatrix}$ DeGiorgi $\nabla W = \begin{pmatrix} \partial_1 W \\ \partial_2 W \end{pmatrix}$

$\operatorname{div}(A(x) \cdot \nabla W) = 0 \Rightarrow W \in C^\alpha \Rightarrow V \in C^{1,\alpha}$

W satisfies a divergence form $\Rightarrow U \in C^{2,\alpha} \quad \square$

In $\mathbb{R}^n$

Thm [Evans - Krylov 1982]

$F(D^2 u) = 0$ in B_1 . Let $u \in C^\infty$ be a sol.

If F is uniformly elliptic and "concave" $\Rightarrow$ 有反例

Then $u \in C^{2,\alpha}(B_{1/2})$ and $\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C \|u\|_{L^\infty(B_1)}$

Concave: $F(D^2 u) = 0$

$F_{ij} D_{ij} u = 0$

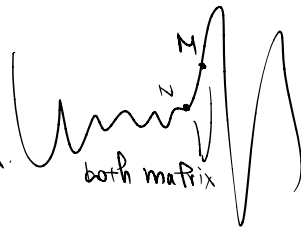
≤ 0
 $F_{ijkl} D_{ij} u D_{kl} u + F_{ij} D_{ij} u = 0$



$F_{ij} D_{ij} u \geq 0$

vector version

$\mathbb{R} \rightarrow \mathbb{R}^{n+1}$



$u \in C^{2,\alpha}, D^2 u \in C^\alpha$

$\operatorname{osc}_{B_{1/2}} D^2 u \leq \theta \operatorname{osc}_{B_1} D^2 u$

Proof $u \in C^{1,1}$, $|\partial^2 u(x)| \leq C$

$\mathcal{H}(B_1)$ is bounded

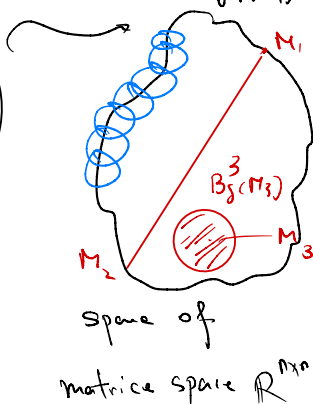
$$\text{diam}(\partial^2 u(B_1)) = 1$$

δ^{n^2}



$\mathcal{H}: x \rightarrow \partial^2 u(x)$

$\mathcal{H}(B_1)$



AIMs: We want to show that for some $\rho_0 > 0$, $\text{diam}(\partial^2 u(B_{\rho_0})) \leq \frac{1}{2}$

Step ①. use $\{B_\delta^3(M_j)\}$ to cover $\mathcal{H}(B_1) = \partial^2 u(B_1)$

need $\sim \delta^{-n^2}$ many balls

$$\Rightarrow \{ \mathcal{H}^{-1}(B_\delta^3(M_j)) \} \text{ cover } B_{1/2}$$

$\Rightarrow$ there is, at least one covers area of δ^{n^2}

$$\exists B_\delta^3(M_3), \text{ s.t. } |\mathcal{H}^{-1}(B_\delta^3(M_3))| > \pi \delta^{n^2}$$

Step ②. As for as image $\text{diam}(\mathcal{H}(B_1)) \geq \frac{1}{2}$

$$\exists M \in \mathcal{H} \text{ s.t. } \|M - M_3\| \geq \frac{1}{4}$$

Note $F(M) = 0$, $F(M_3) = 0$.

Rmk 2 $\|(M - M_3)^+\| \sim \|(M - M_3)^-\|$ comparable.

$$\|(M - M_3)^+\| \gtrsim \frac{1}{8} \Rightarrow u_{ee}(x) \geq u_{ee}(x_3) + \frac{1}{8}$$

$$\Rightarrow \exists e \in S^{n-1} \text{ s.t. } \sup_{B_1} u_{ee} \geq u_{ee}(x_3) + \frac{1}{2 \cdot 26}$$

Step ③. choose δ small enough s.t.

$$t := \sup_{B_1} u_{ee} \geq u_{ee}(x) + \frac{1}{2026} \quad \forall D^2 u(x) \in B_\delta^3(M_3)$$

i.e. $\forall x \in H^{-1}(B_\delta^3(M_3))$

$$t - u_{ee}(x) \geq 0.$$

supersolution

Apply weak Harnack to $t - u_{ee}(x)$ in B_1

$$\Rightarrow \inf_{B_{1/2}} (t - u_{ee}) \geq C \left(\int_{B_1} (t - u_{ee})^{q_p} \right)^{\frac{1}{q_p}}$$

$$\geq C \frac{1}{2026} \left| H^{-1}(B_\delta^3(M_3)) \right|^{\frac{1}{q_p}}$$

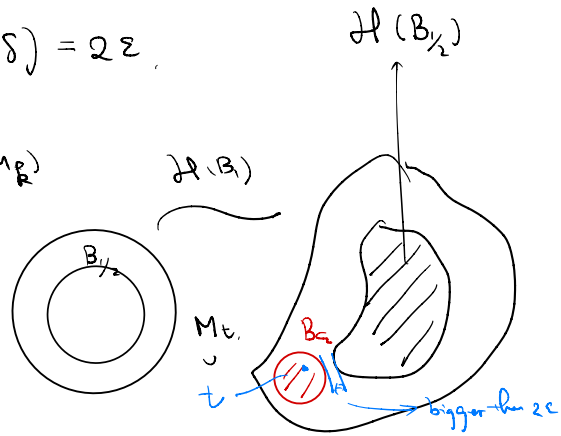
$$\geq \frac{C}{2026} (\pi \delta^{n^2})^{\frac{1}{q_p}}$$

$$= \theta(\delta) > 0.$$

$$\Rightarrow \text{On } B_{1/2} \quad t - u_{ee} \geq \theta(\delta) = 2\varepsilon.$$

①* replace $B_\delta^3(M_\delta)$ by $B_\varepsilon^k(M_k)$

need $\varepsilon^{-n \times n}$ many balls



$$\textcircled{4} \exists B_2 \text{ cover } t = \sup_{B_1} u_{\epsilon\epsilon}$$

in order to cover $f(B_{1/2})$

we can throw away B_2 ! $|B_2| \sim \epsilon^{n^2}$

\textcircled{5} Now, if $\text{diam}(f(B_{1/2})) \leq \frac{1}{2}$ done !

$$\text{If } \text{diam}(f(B_{1/2})) > \frac{1}{2},$$

we are in business for Step 2. in $B_{1/2}$

$$\text{Let } v(x) = 2^2 u\left(\frac{x}{2}\right). \quad B_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

$$D^2 v(x) = D^2 u(x/2)$$

$$D^2 v(B_{1/2}) = D^2 u(B_{1/4}).$$

over the threshold.

throw away B_2 .

$$|B_2| \sim \epsilon^{n^2}$$

$$\text{diam}(D^2 v(B_{1/2})) \leq \frac{1}{2}. \checkmark$$

$\exists k$! st. right.

$$\text{diam}(D^2 u(B_{2^{-k}})) \leq \frac{1}{2}$$

$\downarrow$
 B_{ϵ}

