

Lecture 1, Heat Kernel

(M^n, g) , complete, Rie. manifold.

$$\text{Ric} \geq -(n-1)Kg, \quad K \geq 0$$

Δ : Laplacian of g in coord. $x = (x^1, \dots, x^n)$

$$g = g_{ij} dx^i \otimes dx^j$$

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial}{\partial x^j} \right)$$

Heat equation

$$\frac{\partial f}{\partial t} = \Delta f \quad (*)$$

$\uparrow$
 L^2 extension,

self-dual operator

$P_t f$: solution of $(*)$, starting at f .

$\Downarrow$

$$\text{Kernel} : (P_t f)(x) = \int_M \underbrace{H(x, y; t)}_{\text{Heat kernel}} f(y) dy$$

$t \rightarrow 0 \rightarrow f(x)$

Existence $\exists!$ minimal kernel $H(x, y; t)$, s.t.

① smooth: $H(x, y; t)$ smooth on $M \times M \times (0, \infty)$

② symmetry: $H(x, y; t) = H(y, x; t)$

③ positivity: $H > 0$

④ semi-group: $H(x, y, t+s) = \int_M H(x, z, t) H(z, y, s) dz$

⑤ Fundamental Solution: $H(x, y, t) \xrightarrow[t \rightarrow 0]{} \delta_y(x)$

⑥ $\int_M H(x, y, t) dy \leq 1$

If $\int_M H(x, y, t) dy = 1, \forall t$, then (M, g) is called stochastically complete.

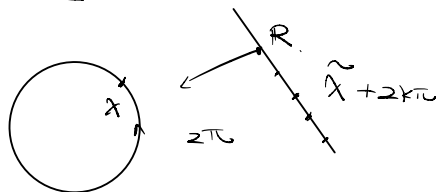
Grigor'yan: If $\int_1^\infty \frac{r dr}{\log \text{vol}(B_r(x))} = \infty$, then g is stoch. complete.

$$B_r(x) = \{y \in M \mid d(x, y) < r\} \quad \underline{\text{volume}}$$

Example $\text{Ric} \geq -(n-1)Kg \Rightarrow \text{Vol}(B_r(x)) \leq \text{Vol}(B_r) \sim \sinh r$

Example $\mathbb{R}^n, H(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$

Ex $S^1 = \mathbb{R} / 2\pi\mathbb{Z}$



$$h(x, t) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$$

$$\pi: \mathbb{R} \rightarrow S^1$$

$$\tilde{x} \mapsto x$$

$$H(x, y, t) = \sum_{k \in \mathbb{Z}} H(\tilde{x}, \tilde{y} + 2k\pi, t)$$

$\mathbb{R} \cong \mathbb{H}^1$
Heat kernel $= \sum_{k \in \mathbb{Z}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(\tilde{x} - \tilde{y} - 2k\pi)^2}{4t}}$

$$\bullet t \ll 1,$$

$$0 \leq x \leq \pi$$

$$\begin{aligned} \text{put } H(x, y; t) &= H(0, x, t) \\ H(x, t) &= H(\pi - x, t) \end{aligned}$$

$$\begin{aligned} H(x, t) &= \sum_{k \in \mathbb{Z}} (4\pi t)^{-\frac{1}{2}} e^{-\frac{(x-2k\pi)^2}{4t}} \\ &= \frac{(4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}}{\underbrace{\left(1 + e^{-\frac{\pi^2 - \pi x}{t}} + e^{-\frac{\pi^2 + \pi x}{t}} + \sum_{|k| \geq 2} e^{\dots} \right)}} \end{aligned}$$

$$x = \pi \quad (*) = 1 + o(1)$$

$$x \neq \pi \quad (*) = o(1)$$

$$\begin{aligned} \bullet t \gg 1, \quad k \geq 2, \quad e^{-\frac{k^2\pi^2 - k\pi x}{t}} &\lesssim e^{-\frac{k^2\pi^2 - k\pi^2}{t}} \\ \sum_{|k| \geq 2} e^{\square} &\lesssim \sum_{|k| \geq 1} e^{-\frac{k^2\pi^2}{t}} \lesssim e^{-\frac{(k-1)^2\pi^2}{t}} \end{aligned}$$

$$H(0, t) = (4\pi t)^{-\frac{1}{2}} \left(1 + \sum_{k \neq 0} e^{-\frac{k^2\pi^2}{t}} \right)$$

$$\left| e^{\frac{x^2}{4t}} H(x, t) - H(0, t) \right| \leq C \cdot t^{-\frac{1}{2}}, \quad t \gg 1$$

$$\int_{\mathbb{S}^1} H(x, t) dx = 1$$

$$\Rightarrow \left| H(x, t) - \frac{1}{2\pi} \right| \leq C \cdot t^{-\frac{1}{2}}, \quad t \gg 1.$$

Example Hyperbolic space, $\mathbb{H}^n$, $\text{sect} \equiv -1$.

$n=2$, McKean formula

$$H_2(x, y; t) = H_2(r, t) = \sqrt{2} e^{-\frac{t}{4}} (4\pi t)^{-\frac{3}{2}} \int_r^\infty \frac{s e^{-\frac{s^2}{4t}} ds}{\cosh s - \cosh r}$$

$r = d(x, y)$

$n=3$ $H_3(r, t) = \underbrace{(4\pi t)^{-\frac{3}{2}} \cdot e^{-\frac{r^2}{4t}}}_{\text{主项}} \cdot \underbrace{e^{-t} \cdot \frac{r}{\sinh r}}_{\text{修正项}}$

Recurrence formula

$$H_{n+2}(r, t) = -\frac{1}{2\pi} e^{-nt} \left(\frac{1}{\sinh r} \cdot \frac{\partial}{\partial r} \right) H_n(r, t)$$

Asymptotic $H_n(r, t) \sim t^{-\frac{n}{2}} (1+t+r)^{\frac{n-3}{2}} (1+r) e^{-\frac{r^2}{4t} - \frac{(n-1)^2}{4}t - \frac{(n-1)r}{2}}$

Davies - Mandouvalos 88

Grigor'yan - Noguichi 98

• Li-Yau Heat Kernel Estimate 1986

Thm (Li-Yau, differential Harnack)

Suppose $\partial B_{2r}(x) \neq \emptyset$ (球不能是整个流形, 因为 r 可能很小)

Let $f(x, t) > 0$, be a solution

$$\frac{\partial f}{\partial t} = \Delta f \quad \text{on} \quad B_{2r}(x) \times [0, T]$$

Then for $\forall 1 < a \leq 2$

$$|\nabla \log f|^2 - a \frac{\partial \log f}{\partial t} \leq \frac{n}{2t} a^2 + \frac{C(n)}{a-1} (r^{-2} + K)$$

on $B_r(x) \times (0, T]$

Corollary: f is a positive solution over $M \times (0, T]$

$$\text{Then } |\nabla \log f|^2 - a \frac{\partial \log f}{\partial t} \leq \frac{n}{2t} a^2 + \frac{C(n)}{a-1} K$$

In particular, if $\text{Ric} \geq 0$, then

$$|\nabla \log f|^2 - \frac{\partial \log f}{\partial t} \leq \frac{n}{2t} \quad (a \rightarrow 1)$$

Corollary (Harnack inequality of heat eqn) $\forall x_1, x_2 \in B_{r/2}$
 $0 < t_1 < t_2 \leq T$

$$f(x_1, t_1) \leq f(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\frac{n}{2}} \exp \left\{ \frac{a d^2(x_1, x_2)}{4(t_2 - t_1)} + \frac{C(n)}{a-1} (r^{-2} + K) (t_2 - t_1) \right\}$$

If, in particular, $\text{Ric} \geq 0$, f is a solution over $M \times (0, T]$

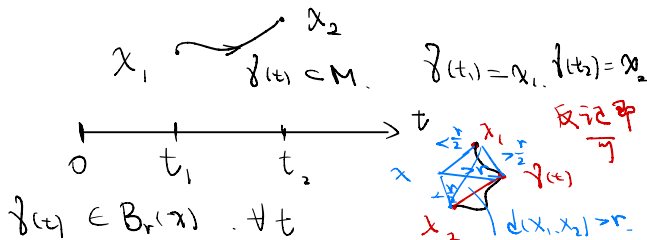
$$\text{then } f(x_1, t_1) \leq f(x_2, t_2) \left(\frac{t_2}{t_1} \right)^{\frac{n}{2}} \exp \left\{ \frac{d^2(x_1, x_2)}{4(t_2 - t_1)} \right\},$$

$\forall x_1, x_2 \in M$

pointwisely, $t^{\frac{n}{2}} \cdot f(x, t) \nearrow$ in t for $\forall x \in M$. $0 < t_1 < t_2 \leq T$

proof of Harnack:

Take a minimal geodesic



$\gamma(t)$, $t_1 \leq t \leq t_2$, from x_1 to x_2

$$|\gamma'| = \frac{d(x_1, x_2)}{t_2 - t_1}$$

$$\begin{aligned} \frac{d}{dt} f(\gamma(t)) &= df_\gamma(\gamma'(t)) \\ &= \langle \nabla f, \gamma'(t) \rangle \end{aligned}$$

$$\log f(x_1, t_1) - \log f(x_2, t_2) = - \int_{t_1}^{t_2} \frac{d}{dt} \log(f(\gamma(t), t)) dt$$

$$= - \int_{t_1}^{t_2} \left[\langle \nabla \log f, \gamma'(t) \rangle + \frac{\partial \log f}{\partial t} \right] dt$$

$$\leq \int_{t_1}^{t_2} \left[|\nabla \log f| \cdot \frac{d}{t_2 - t_1} - \frac{\partial \log f}{\partial t} \right] dt$$

$$\leq \int_{t_1}^{t_2} \left[\frac{1}{a} |\nabla \log f|^2 - \frac{\partial \log f}{\partial t} + \frac{a d^2}{4(t_2 - t_1)^2} \right] dt$$

$$\leq \int_{t_1}^{t_2} \left[\frac{n a}{2t} + \frac{C(n)}{a(a-1)} (r^{-2} + K) + \frac{a d^2}{4(t_2 - t_1)^2} \right] dt$$

$$= \frac{n}{2t} a \log \frac{t_2}{t_1} + \frac{C(n)}{a(a-1)} (r^{-2} + K) (t_2 - t_1) + \frac{a d^2}{4(t_2 - t_1)} \quad \square$$

proof of diff Harnack Let $u = \log f$. $F = t(|\nabla u|^2 - a u')$

$$u' = \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|^2$$

$$\frac{\partial}{\partial t} |\nabla u|^2 = \Delta |\nabla u|^2 - 2|\nabla \nabla u|^2 + 2\langle \nabla u, \nabla |\nabla u|^2 \rangle - 2\text{Ric}(\nabla u, \nabla u)$$

$$\frac{\partial}{\partial t} \Delta u = \Delta \Delta u + 2|\nabla \nabla u|^2 + 2\langle \nabla u, \nabla \Delta u \rangle + 2\text{Ric}(\nabla u, \nabla u)$$

$$\frac{\partial}{\partial t} u' = \Delta u' + 2\langle \nabla u, \nabla u' \rangle$$

Apply Maximum Principle to F

$$\frac{\partial F}{\partial t} = \Delta F + 2\langle \nabla u, \nabla F \rangle + t^{-1}F - t(2|\nabla \nabla u|^2 + 2\text{Ric}(\nabla u, \nabla u))$$

$$|\nabla \nabla u|^2 \geq \frac{1}{n} (\Delta u)^2 \rightsquigarrow \Delta u \geq F?$$

$$t^{-1}F = |\nabla u|^2 - a(\Delta u + |\nabla u|^2)$$

$$= -a\Delta u + (1-a)|\nabla u|^2$$

$$\Delta u = -a^{-1}t^{-1}F - \frac{a-1}{a}|\nabla u|^2$$

$$|\nabla \nabla u|^2 + \text{Ric}(\nabla u, \nabla u)$$

$$\geq \frac{1}{n}(\Delta u)^2 - (n-1)K|\nabla u|^2$$

$$\geq \frac{a^2 t^{-2}}{n} F^2 + \frac{2a^{-2}(a-1)t^{-1}}{n} |\nabla u|^2 F - \frac{n(n-1)^2 a^2}{4(a-1)^2} K^2$$

$$\begin{aligned} \frac{\partial F}{\partial t} &\leq \Delta F + 2\langle \nabla u, \nabla F \rangle + t^{-1}F - \frac{2a^{-2}t^{-1}}{n} F^2 \\ &\quad - 4 \frac{a^{-2}(a-1)t^{-1}}{n} |\nabla u|^2 F + \frac{n(n-1)^2 a^2}{4(a-1)^2} K^2 t \end{aligned}$$

Take a cut-off func. $\phi_0^\infty(B_{2r}(x))$, $0 \leq \phi \leq 1$

Let $Q = \phi F$ Q satisfies:

$$\begin{aligned} \frac{\partial Q}{\partial t} &\leq \Delta Q - \Delta \phi F - 2\langle \nabla \phi, \nabla F \rangle + 2\phi \langle \nabla u, \nabla F \rangle + t^{-1}Q \\ &\quad + \phi \left[-\frac{2a^{-2}t^{-1}}{n} F^2 + \dots \right] \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{\partial Q}{\partial t} &\leq \Delta Q - 2\phi^{-1} \langle \nabla \phi, \nabla Q \rangle + 2\langle \nabla u, \nabla Q \rangle \\ &\quad - \frac{2a^{-2}t^{-1}}{n} \phi^{-1} F \end{aligned}$$

$$E = Q^2 - \frac{na^2 t}{2} \phi \left[-\phi^{-1} \Delta \phi Q + 2\phi^{-2} |\nabla \phi|^2 Q + \frac{na^2}{4(a-1)} \phi^{-2} |\nabla \phi|^2 Q \right]$$

$$+ t^{-1} Q + \frac{n(n-1)^2 a^2}{2(a-1)^2} K^2 \phi t \int$$

Choose $\phi(y) = \eta\left(\frac{d(x,y)}{r}\right)$ where $\eta \in C_0^\infty(\mathbb{R})$

$$\eta(s) = 1, s \leq 1; \quad \eta(s) = 0, s \geq 2, \quad -C\eta^{1/2} \leq \eta' \leq 0$$

$$E \geq \frac{Q^2 - \left(\frac{n}{2} a^2 + C(n) K a^2 t + C(n) \frac{a^4}{a-1} r^{-2} t \right) Q - \frac{C(n) a^4}{(a-1)^2} K^2 t^2}{G}$$

G

$$G = (Q-A)(Q-B), \quad A \leq B \quad \text{solution of } G=0.$$

$$\begin{aligned} \frac{\partial Q}{\partial t} &\leq \Delta Q - 2\phi^{-1} \langle \nabla \phi, \nabla Q \rangle + 2 \langle \nabla u, \nabla Q \rangle \\ &\quad - \frac{2a^{-2}t^{-1}}{n} \phi^{-1} (Q-A)(Q-B) \end{aligned}$$

$$\begin{aligned} \text{WTS } Q \leq B &= \frac{1}{2} \left(\frac{n}{2} a^2 + C(n) K a^2 t + C(n) \frac{a^4}{a-1} r^{-2} t + \sqrt{\Delta} \right) \\ &\leq \frac{1}{2} \left(n a^2 + C K t + \frac{C}{a-1} r^{-2} t + \frac{C}{a-1} K t \right) \\ &\leq \frac{1}{2} \left(n a^2 + \frac{C}{a-1} (r^{-2} + K) t \right) \end{aligned}$$

①

Claim $Q \leq \textcircled{1}$

$\cdot x_0$

$$Q \leq \overline{D(x) < D(x_0)} t_0$$

$$Q(\cdot, 0) \equiv 0 < D(x)$$

$\rightarrow 0$

first time

Assume $Q(x_0, t_0) = D(t_0)$ at some (x_0, t_0) , $0 < t_0 \leq T$
 $\Rightarrow$ support of ϕ . $\phi(x_0) \neq 0$.

$$0 \leq \frac{\partial Q}{\partial t}(x_0, t_0) \leq \Delta Q(x_0, t_0) - \frac{2a^{-2}t^{-1}}{n} \underbrace{\phi^{-1}(Q(x_0, t_0) - A)}_{\substack{\leq 0 \\ \nabla Q(x_0, t_0) = 0}} \underbrace{(Q(x_0, t_0) - B)}_{\substack{\leq 0 \\ \nabla Q(x_0, t_0) = 0}}$$

Max. Prin. $\Rightarrow$ Claim

On $B_r(x) \times (0, T]$

$$\sup_{B_r(x)} F \leq \sup_{B_{2r}(x)} Q \leq D(t)$$

□

Davies double integrated estimate

Lemma, $\Omega \subset M$ domain $\bar{\Omega}$ compact.

$$f \in C^\infty(\Omega \times (0, T]) \times \text{Lip}(\bar{\Omega} \times (0, T])$$

$$\text{Satisfies } \frac{\partial f}{\partial t} = \Delta f \quad \text{in } \Omega \times (0, T]$$

$$f = 0 \quad \text{on } \partial\Omega \times (0, T]$$

$$\text{Suppose } h \in \text{Lip}(\bar{\Omega} \times (0, T]) , \quad \frac{\partial h}{\partial t} = -|\nabla h|^2$$

Ex $h = \frac{d^2(z, \cdot)}{4t}$

Thm $\int_{\Omega} e^{2h} f^2 \downarrow \text{ in } t$

Pf:

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega} e^{2h} f^2 &= \int 2h' e^{2h} f^2 + 2e^{2h} f \cdot f' \\
 &= \int -2e^{2h} f^2 |\nabla h|^2 + 2e^{2h} f \cdot \Delta f \\
 &= 2 \int_{\Omega} (-|\nabla h|^2 f^2 - \langle \nabla f, \nabla f \rangle \\
 &\quad - f \langle \nabla f, \nabla h \rangle) e^{2h} \\
 &= 2 \int_{\Omega} -|f \nabla h + \nabla f|^2 e^{2h} \leq 0. \quad \square
 \end{aligned}$$

Proposition (Davies) $B_i = B_{r_i}(x_i)$, $i=1,2$.

$$\int_{B_1} \int_{B_2} H(x, y, t) dx dy \leq \text{vol}(B_1)^{\frac{1}{2}} \cdot \text{vol}(B_2)^{\frac{1}{2}} \cdot e^{-\frac{d^2(B_1, B_2)}{4t}}$$

where $d(B_1, B_2) = \inf_{\substack{x \in B_1 \\ y \in B_2}} d(x, y)$

Pf:

$$f_i(x) = \int_{B_i} H(x, y, t) dy = \int_M H(x, y, t) \chi_{B_i}(y) dy$$

Let $g_i(x, t) = \frac{d^2(x, B_i)}{4(t+\epsilon)}$, $\epsilon > 0$

$$\int e^{2g_i} f_i^2 \quad \downarrow$$

$$\int_M e^{2g_i} f_i^2 \leq \int_{B_i} e^{\frac{d^2(x, B_i)}{2\epsilon}} dx = \text{vol}(B_i)$$

$$\int_M e^{\frac{d^2(B_1, B_2)}{8(t+\epsilon)}} f_1 f_2 \leq \left(\int_M e^{\frac{d^2(x, B_1)}{2(t+\epsilon)}} f_1^2 \right)^{\frac{1}{2}} \left(\int_M e^{2g_2} f_2^2 \right)^{\frac{1}{2}}$$

$$d^2(B_1, B_2) \leq 2 d^2(x, B_1) + 2 d^2(x, B_2)$$

$$\leq \text{vol}(B_1)^{\frac{1}{2}} \text{vol}(B_2)^{\frac{1}{2}}$$

$$\underline{\text{LHS}} = \int_M e^{\frac{d^2(B_1, B_2)}{8(t+\epsilon)}} \int_{B_1} H(x, y, t) dy \int_{B_2} H(x, z, t) dz dx$$

Fubini's
thm

$$= \int_{B_1} \int_{B_2} e^{\sim} \int_M H(x, y, t) H(x, z, t) dx$$

$$= e^{\frac{d^2(B_1, B_2)}{8(t+\epsilon)}} \int_{B_1} \int_{B_2} H(y, z, 2t) dy dz,$$

□

Lecture 2. Li-Yau Heat kernel estimate

(M^n, g) complete Riem. $\text{Ric} \geq -(n-1)Kg$, $K \geq 0$.

Thm (L-Y)

$$H(x, y, t) \leq C(n) \text{vol}(B_{\sqrt{t}}(x))^{-\frac{1}{2}} \text{vol}(B_{\sqrt{t}}(y))^{-\frac{1}{2}} \cdot \exp\left\{-\frac{d^2(x, y)}{4at} + C(n)Kt\right\}$$

where $1 < a \leq 2$, and $C(n, a) = C(n) e^{\frac{b}{a-1}}$

$$\& H(x, y, t) \geq C(n) \text{vol}(B_{\sqrt{t}}(x))^{-\frac{1}{2}} \text{vol}(B_{\sqrt{t}}(y))^{-\frac{1}{2}} \cdot \exp\left\{-\frac{a d^2(x, y)}{4t} - \frac{C(n)}{a-1}(1+Kt)\right\}$$

Upper bound

$t' = (1+\delta)t$ for some $\delta > 0$.

$$H(x, y, t) \leq H(x, y', (1+\delta)t) (1+\delta)^{\frac{n\alpha}{2}} \exp\left\{\frac{a d^2(y, y')}{4\delta t} + \frac{CK}{a-1} \delta t\right\}$$

$\forall 1 < a \leq 2$,

$$\Rightarrow y' \in B_{\sqrt{t}}(y)$$

$$\int_{B_{\sqrt{t}}(y)} H(x, y, t) \leq \frac{(1+\delta)^{\frac{n\alpha}{2}}}{\text{vol}(B_{\sqrt{t}}(y))} \int_{B_{\sqrt{t}}(y)} H(x, y', (1+\delta)t) \exp\left\{\frac{a}{4\delta} + \frac{CK}{a-1} \delta t\right\} dy'$$

$$\forall x' \in B_{\sqrt{t}}(x), \quad t'' = (1+2\delta)t$$

$$H(x, y', t) \leq H(x', y', (1+2\delta)t) \left(\frac{1+2\delta}{1+\delta}\right)^{\frac{n\alpha}{2}} \exp\left\{\frac{a d^2(x, x')}{4\delta t} + \frac{CK}{a-1} \delta t\right\}$$

Integral

$$H(x, y, t) \leq \text{vol}(B_{\sqrt{t}}(x))^{-1} \text{vol}(B_{\sqrt{t}}(y))^{-1} (1+2\delta)^{\frac{n\alpha}{2}} \int_{B_{\sqrt{t}}(y)} \int_{B_{\sqrt{t}}(x)} H(x', y', t'') \cdot \exp\left\{\frac{a}{2\delta} + \frac{2CK}{a-1} \delta t\right\} dx' dy'$$

$$\Rightarrow H(x, y, t) \leq \text{vol}(B_{\sqrt{t}}(x))^{-\frac{1}{2}} \text{vol}(B_{\sqrt{t}}(y))^{-\frac{1}{2}} (1+2\delta)^{\frac{n\alpha}{2}} \exp\left\{\frac{a}{2\delta} + \frac{2CK}{a-1} \delta t\right\}$$

$$\cdot d(x, y) > 2\sqrt{t}.$$

$$-\frac{d^2(B_{\sqrt{t}}(x), B_{\sqrt{t}}(y))}{4(1+2\delta)t} = -\frac{(d(x, y) - 2\sqrt{t})^2}{4(1+2\delta)t} \leq -\frac{d^2(x, y)}{4(1+4\delta)t} + \frac{1}{2\delta}$$

$$\frac{d(x, y)\sqrt{t}}{4(1+2\delta)t} \leq \frac{d^2}{(1+2\delta)t} + \frac{t}{4\delta(1+2\delta)t}, \text{ take } \delta = \frac{1}{1+2\delta} - \frac{1}{1+4\delta}.$$

$$\cdot d(x, y) \leq 2\sqrt{t}, \Rightarrow d^2(B_{\sqrt{t}}(x), B_{\sqrt{t}}(y)) = 0.$$

$$H(x, y, t) \leq \boxed{\cdot} \cdot \boxed{\cdot} (1+2\delta)^{\frac{n\alpha}{2}} \exp \left\{ \frac{1+\alpha}{2\delta} + \frac{C_K}{\alpha-1} \delta t - \frac{d^2(x, y)}{4(1+4\delta)t} \right\}.$$

$$\text{Take } (1+4\delta) = \alpha, \exp \left\{ -\frac{d^2}{4\alpha t} + \dots \right\} \quad \square$$

Corollary: $\exists r(t) = C(n)\sqrt{t}(1+\sqrt{kt})$

$$\text{s.t.} \quad \int_{B_{r(t)}(x)} H(x, y, t) dy \geq \frac{1}{2}.$$

Pf: Heat Kernel upper bound + Bishop-Gromov volume comparison

$$\int_{M \setminus B_{r(t)}(x)} H(x, y, t) dy = \int_{r(t)}^{\infty} \int_{\partial B_\rho(x)} H(x, y, t) dy$$

$$\xrightarrow{\text{volume comparison}} \leq \int_{r(t)}^{\infty} \boxed{\cdot} \cdot \text{vol}(\partial B_\rho(x)) d\rho$$

$\square$

lower bound

$1 < a \leq 2, \delta > 0$, for some δ

$$H(x, z, t) \leq H(x, y, (1+\delta)t) (1+\delta)^{\frac{na}{2}} \exp \left\{ \frac{a d^2(y, z)}{4\delta t} + \frac{Ck}{a-1} \delta t \right\}$$

$$z \in B_{r(t)}^{(x)}$$

$$\frac{1}{2} \leq \int_{B_{r(t)}^{(x)}} H(x, z, t) dz \leq H(x, y, \underbrace{(1+\delta)t}_{\delta}) (1+\delta)^{\frac{na}{2}} \int_{B_{r(t)}^{(x)}} \exp \left\{ \frac{a d^2(y, z)}{4\delta t} \right\} dz + \frac{Ck\delta}{a-1} t$$

$$\frac{a d^2(y, z)}{4\delta t} \leq \frac{a (d(x, y) + r(t))^2}{4\delta t} \leq \frac{d^2}{4s} \cdot \frac{a(2+\delta)}{\delta} + C \frac{a(2+\delta)}{\delta} (1+kt)$$



For $1 < b \leq 2$

$$\text{Let } \frac{a(2+\delta)}{\delta} = b \quad \text{for some } \delta > 0$$

$$\text{Choose } a = \frac{2b}{1+b}, \quad \delta = \frac{2+\delta}{\delta} = \frac{1+b}{2}$$

$$\leq H(x, y, s)^{\frac{na}{2}} \exp \left\{ \frac{d^2(x, y)}{4s} b + \frac{Ck t}{b-1} \right\} \text{vol}(B_{r(t)}^{(x)})$$

$$\text{Vol}(B_{r_2}^{(x)}) \leq \text{Vol}(B_{r_1}^{(x)}) C(n) \left(\frac{r_2}{r_1}\right)^n e^{(n-1)K r_2}$$

$r_2 > r_1$

$$\Rightarrow H(x, y, s) \geq \frac{C(n) \text{vol}(B_{\sqrt{s}}^{(x)})^{-1} \exp \left\{ -\frac{b d^2(x, y)}{4s} - C(n) \frac{1+ks}{b-1} \right\}}{\text{vol}(B_{\sqrt{s}}^{(y)})^{-1}}$$

By symmetry

$$H(x, y, s) \geq \text{vol}^{-\frac{1}{2}} \text{vol}^{-\frac{1}{2}} \dots$$

□

$$\underline{\mathbb{R}^n}: H(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$$

$$\underline{H^n}: H(x, y, t) \sim t^{-\frac{n}{2}} \underbrace{(1+t+d)^{\frac{n-3}{2}} (1+d)}_{\sim (1+d)^{\frac{n-1}{2}}} e^{-\frac{d^2}{4t} - \frac{(n-1)^2}{4}t - \frac{n-1}{2}d}$$

反映出渐近性质 $t \ll 1, d \sim 1$

$$d \gg 1$$

$\underline{S^n}$: Molchanov 75

north pole.

$$\downarrow$$

$$H(N, S, t) t^{-\frac{n}{2}} \left(\frac{d(N, S)}{\sqrt{t}} \right)^{n-1} e^{-\frac{d^2(N, S)}{4t}}$$

$Ric \geq 0$ then

$$H(x, y, t) \leq C(n) \text{Vol}(B_{\sqrt{t}}(x))^{-\frac{1}{2}} \text{Vol}(B_{\sqrt{t}}(y))^{-\frac{1}{2}} \left(1 + \frac{d(x, y)}{\sqrt{t}} \right)^{n-1} e^{-\frac{d^2(x, y)}{4t}}$$

$$\& H(x, y, t) \geq c(n) \text{Vol}^{-\frac{1}{2}} \text{Vol}^{-\frac{1}{2}} \left(1 + \frac{d(x, y)}{\sqrt{t}} \right)^{-(n-1)} e^{-\frac{d^2(x, y)}{4t}}$$

L_i, L_u

Asymptotic behavior at $t = \infty$

$\text{Ric} \geq 0$, complete, non-compact. $\underline{B}_r \subset \mathbb{R}^n$, $|\underline{B}_r| = \text{vol}(\underline{B}_r)$

Fix any $p \in M$, $\frac{\text{vol}(\underline{B}_r(p))}{|\underline{B}_r|} \downarrow$ in r .

Suppose $\lim_{r \rightarrow \infty} \frac{\text{vol}(\underline{B}_r(p))}{|\underline{B}_r|} = \theta > 0$ ($\theta \leq 1$)

• $\lim_{r \rightarrow \infty} \frac{\text{vol}(\underline{B}_r(p))}{|\underline{B}_r|}$ indep of $p \in M$

$$\Rightarrow \lim_{r \rightarrow \infty} \frac{\text{vol}(\partial \underline{B}_r(p))}{\text{vol}(\partial \underline{B}_r)}$$

Lemma $\lim_{t \rightarrow \infty} (4\pi t)^{\frac{n}{2}} H(x, y, t) = \theta^{-1}$, $\forall x, y \in M$

Pf Recall Harnack: for $y = z$, $t < t'$

$$H(x, y, t) \leq H(x, y, t') \left(\frac{t'}{t}\right)^{\frac{n}{2}}, \quad \forall x, y \in M$$

$$t^{\frac{n}{2}} H(x, y, t) \uparrow \text{ in } t.$$

$$H(x, y, t) \leq C(n) \text{vol}(B_{\sqrt{t}}(x))^{-\frac{1}{2}} \text{vol}(B_{\sqrt{t}}(y))^{-\frac{1}{2}} \underbrace{e^{-\frac{d^2}{4\pi t}}}_{\leq 1}$$

$$\text{vol}(B_{\sqrt{t}}(x)) \geq \theta |\underline{B}_r| \geq \theta \cdot |\underline{B}_1| \cdot r^n$$

$$H(x, y, t) \leq C(n) \theta^{-1} t^{-\frac{n}{2}}$$

$$\Rightarrow \lim_{t \rightarrow \infty} (4\pi t)^{\frac{n}{2}} H(x, y, t) = \mu(x, y)$$

$$\cdot \mu(x, y) = ?$$

$$\begin{aligned} (4\pi t)^{\frac{n}{2}} H(x, z, t) &\leq \frac{(4\pi t')^{\frac{n}{2}} H(x, y, t)}{e^{\frac{d^2(z, y)}{4(t'-t)}}} \\ \mu(x, z) &\leq \mu(x, y) \quad \downarrow \text{Take } t' = 2t, \quad t \rightarrow \infty \\ &\quad \quad \quad 1 \end{aligned}$$

Similarly $\mu(z, x) \leq \mu(y, x) \quad \forall x, y, z,$

$\Rightarrow \mu(x, y)$ indep. of x, y

$\mu(x, y) \leq \mu(x, z) \leq \mu(w, z), \quad \forall x, y, z, w.$

$1^\circ \mu \leq \theta^{-1} \quad \forall x, y, t < t'$

$$(4\pi t)^{\frac{n}{2}} H(x, x, t) e^{-\frac{d^2(x, y)}{4(t'-t)}} \leq (4\pi t')^{\frac{n}{2}} H(x, y, t).$$

integral for y .

$$\begin{aligned} (4\pi t)^{\frac{n}{2}} H(x, x, t) \int_M e^{-\frac{d^2(x, y)}{4(t'-t)}} dy &\leq (4\pi t')^{\frac{n}{2}} \\ &\quad \quad \quad \int_0^\infty \int_{\partial B_r(x)} e^{-\frac{r^2}{4(t'-t)}} dy dr \\ &= \int_0^\infty e^{-\frac{r^2}{4(t'-t)}} \text{vol}(\partial B_r(x)) dr \\ &\geq \int_0^\infty e^{-\frac{r^2}{4(t'-t)}} \theta |\partial B_r| dr = \theta [4\pi(t'-t)]^{\frac{n}{2}} \end{aligned}$$

$$1 = \int_{\mathbb{R}^n} [4\pi(t'-t)]^{\frac{n}{2}} e^{-\frac{|x|^2}{4(t'-t)}} dx$$

$$= \int_0^\infty [4\pi(t-t')]^{\frac{n}{2}} e^{-\frac{r^2}{4(t-t')}} |\partial B_r| dx$$

$$\Rightarrow (4\pi t)^{\frac{n}{2}} H(x, x, t) \theta [4\pi(t'-t)]^{\frac{n}{2}} \leq (4\pi t')^{\frac{n}{2}}$$

$$\therefore \mu \leq \theta^{-1}$$

仅含 t' - t

• $\mu \geq \theta^{-1}$ Cheeger - Yau comparison

$$H(x, y, t) \geq (4\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x, y)}{4t}}$$

距离一致的点

$$H(x, y, t) \quad d(x, y) = |x - y| = d(x, y)$$

For any $r > 0$,

$x, y \in \mathbb{R}^n$, H = Heat kernel on $\mathbb{R}^n$

$$\int_{B_r(x)} H(x, y, t) dy \geq \int_{\underline{B}_r(x)} \underline{H}(x, y, t) d\underline{y} \quad (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-x|^2}{4t}}$$

$$(4\pi t)^{\frac{n}{2}} \int_{B_r(x)} H(x, y, t) dy \geq (4\pi t)^{\frac{n}{2}} \int_{\underline{B}_r(x)} \underline{H}(x, y, t) d\underline{y}$$

$$= \int_{\underline{B}_r(x)} e^{-\frac{|x-x|^2}{4t}} d\underline{y}$$

$t \rightarrow \infty$

$$\mu \cdot \text{vol}(B_r(x)) \geq |\underline{B}_r(x)| \quad \forall r,$$

$$r \rightarrow \infty \Rightarrow \mu \geq \theta^{-1}$$

□

C-Y Comparison

Fix $p \in M$, $r > 0$. Let $u(x, t) = \int_{B_r(p)} H(x, y, t) dy$

$$\frac{\partial}{\partial t} u = \Delta u, \text{ on } M \times (0, T] \quad u > 0 \quad = \int_M H(x, y, t) \int_{B_r(p)} dy$$

$$\frac{\partial}{\partial t} \underline{u} = \Delta \underline{u} \quad \text{on } \mathbb{R}^n \times (0, \infty) \quad \underline{u} > 0, \quad \underline{u}(x, t) = \int_{B_r(p)} H(x, y, t) dy$$
$$= \underline{u}(p, t), \quad p = |x - p|$$

Want: $u(x, t) \geq \underline{u}(x, t)$ for x with $|x - p| = d(x, p)$

Construct $\tilde{u}(x, t) = \underline{u}(p, t)$, $p = d(x, p)$
on $M \times (0, \infty)$

$$\tilde{u} \text{ satisfies: } \frac{\partial \tilde{u}}{\partial t} = \frac{\partial \underline{u}}{\partial t} = \Delta \underline{u}(p, t) = \frac{\partial^2 \underline{u}}{\partial p^2} + \frac{\partial \underline{u}}{\partial p} \Delta p$$

$$\Delta \tilde{u}(x, t) = \operatorname{div}(\nabla \tilde{u}(x, t))$$
$$= \operatorname{div}\left(\frac{\partial \underline{u}}{\partial p} \cdot \nabla p\right)$$
$$= \frac{\partial^2 \underline{u}}{\partial p^2} \underbrace{|\nabla p|^2}_1 + \boxed{\frac{\partial \underline{u}}{\partial p}} \Delta p \quad (\geq \Delta \underline{u})$$

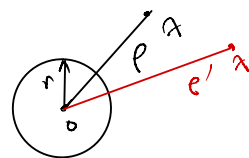
≤ 0 why?

$\Delta p \leq \Delta p$ (Laplace comparison thm)

In fact, $\frac{\partial \underline{u}}{\partial p} \leq 0 \quad \forall (p, t) \in (0, \infty) \times (0, \infty)$.

$$\underline{u}(p, t) = \int_{B_r(0)} (4\pi t)^{-\frac{n}{2}} e^{-|p-x|^2/4t} dy \quad |x|=p.$$

关于 $p \downarrow$.



$$p < e'$$

热核走远 (以 α 为中心)
在 $B_r(0)$ 上积分自然少

$$\frac{\partial}{\partial t} \tilde{u} = \Delta \tilde{u} \leq \Delta u$$

$$\frac{\partial}{\partial t} (\tilde{u} - u) \leq \Delta (\tilde{u} - u)$$

$$M.P. \Rightarrow \tilde{u} - u \leq 0. \quad \forall t.$$

□

$$\underline{\text{Thm (Peter, Li)}} \quad |\pi_1(M)| \leq \theta^{-1}$$

Def: H : heat kernel on M

$\tilde{H}$: ----- $\tilde{M}$

$\pi: \tilde{M} \rightarrow M$
universal
covering space.

$$H(x, y, t) = \sum_{\sigma \in \pi_1(M)} H(\tilde{x}, \sigma \cdot \tilde{y}, t)$$

$$(4\pi t)^{\frac{n}{2}} H(x, y, t) = (4\pi t)^{\frac{n}{2}} \sum_{\sigma \in \pi_1(M)} H(\tilde{x}, \sigma \cdot \tilde{y}, t)$$

$$t \rightarrow \infty, \text{ we have } \theta^{-1} = \tilde{\theta}^{-1} \cdot |\pi_1(M)|$$

$$\tilde{\theta} = \lim_{t \rightarrow \infty} \frac{\text{vol}(B_r(\tilde{x}))}{|B_r|} \leq 1$$

□

Lecture 3

Final Remark on heat kernel

$$(M^n, g), Ric \geq 0, \quad \theta = \lim_{r \rightarrow \infty} \frac{Vol(B_r(p))}{|B_r|} > 0$$

$$\text{Then, } \lim_{t \rightarrow \infty} (4\pi t)^{\frac{n}{2}} H(x, y, t) = \theta^{-1}, \quad \forall x, y \in M$$

$$\lim_{t \rightarrow \infty} Vol(B_{\sqrt{t}}(x)) H(x, y, t) = (4\pi)^{-\frac{n}{2}} |B_1|$$

$$\parallel$$

$$\frac{Vol(B_{\sqrt{t}}(x))}{|B_r|} \cdot |B_1| t^{\frac{n}{2}} H(x, y, t)$$

Xu, Guoyi 13 Large time behavior of the heat kernel

Tangent cone at infinity

$$(M, g_i) \xrightarrow[t_i \rightarrow \infty]{} (M_\infty, d_\infty, x_\infty, \mu_\infty)$$

g_i
 $\Sigma_i \rightarrow 0$
 $\theta = 0$

$\frac{1}{2} \rightarrow \frac{1}{4}$
 $d \rightarrow \frac{1}{4}$
 $\downarrow$
 0 体积无穷大性质

$(M^n, g_i = t_i^{-1} g, x_i, \mu_i) \xrightarrow{G.H.} (M_\infty, d_\infty, x_\infty, \mu_\infty)$

$\downarrow$ 基点 $\swarrow$ normalized measure

$$\mu_i(E) = \frac{Vol_{g_i}(E)}{Vol_{g_i}(B_1(x_i))}$$

Cheeger-Colding $g_i: H_i(x, y, 1)$

$$\underline{Thm} (Xu) \quad \lim_{i \rightarrow \infty} Vol(B_{\sqrt{t_i}}(x_i)) H(x, y, t_i) = H(x_\infty, y_\infty, 1)$$

depend on M_∞ & t_i

Green function

fundamental solution to Poisson equ.

$$\Delta u = f, \quad u(x) = \int_M G(x, y) f(y) dy$$

M^n , non-compact.

$$G(x, y) = \int_0^\infty H(x, y, t) dt. \quad \boxed{\text{if RHS exists?}}$$

Li-Yau: $\text{Ric} \geq 0$, noncompact, then

$$C(n)^{-1} \int_{d^2}^\infty \text{vol}(B_{\sqrt{t}}(x))^{-1} dt \leq G(x, y) \leq C(n) \int_{d^2}^\infty \text{vol}(B_{\sqrt{t}}(x))^{-1} dt$$

pf: lower bound

$$\begin{aligned} \int_0^\infty H(x, y, t) dt &\geq \int_{d^2}^\infty C(n) \text{vol}(B_{\sqrt{t}}(x))^{-1} e^{-\frac{d^2(x, y)}{3t}} dt \\ &\geq C(n) \int_{d^2}^\infty \text{vol}(B_{\sqrt{t}}(x))^{-1} dt \end{aligned}$$

Upper bound

$$\int_0^\infty H(x, y, t) dt \leq \int_0^{d^2} + \int_{d^2}^\infty C(n) \text{vol}(B_{\sqrt{t}}(x))^{-1} e^{-\frac{d^2}{3t}} dt$$

$$\int_0^{d^2} \text{vol}(B_{\sqrt{t}}(x))^{-1} e^{-\frac{d^2}{3t}} dt \leq \int_{d^2}^\infty \text{vol}(B_{\sqrt{t}}(x))^{-1} \frac{\text{vol}(B_{\sqrt{t}}(x))}{\text{vol}(B_{\sqrt{t}}(x))} dt$$

$$\theta = \frac{d^4}{t}$$

$$\frac{d^4}{s^2} ds$$

$$\begin{aligned} & \cdot e^{-\frac{s}{5d^2}} \cdot \frac{d^4}{s^2} ds \\ & \leq \int_{d^2}^{\infty} \text{vol}(B_{\sqrt{s}}(x))^{-1} \left(\frac{s}{d^2} \right)^{n-2} e^{-\frac{s}{5d^2}} ds \leq \left(\frac{s}{t} \right)^{n/2} \\ & \leq C \int_{d^2}^{\infty} \text{vol}()^{-1} ds = \left(\frac{s^2}{d^4} \right)^{n/2} \end{aligned}$$

Assume $\text{Ric} \geq 0$ & $\theta = \lim_{r \rightarrow \infty} \frac{\text{vol}(B_r(x))}{|B_r|} > 0$

$n \geq 3$. then $\text{vol}(B_{\sqrt{t}}(x)) \geq \theta \cdot |B_1| t^{n/2}$

$$G(x, y) \sim \int_{d^2}^{\infty} \text{vol}(B_{\sqrt{t}}(x))^{-1} dt \sim \theta \int_{d^2}^{\infty} t^{-n/2} dt \sim \theta d^{2-n}$$

Compare $G(x, y)$ on $\mathbb{R}^n$

$$\int_0^{\infty} (4\pi t)^{-\frac{n}{2}} e^{-\frac{d^2}{4t}} dt = C(n) \cdot d^{2-n}$$

Colding-Minicozzi 97

Normalize $G(x, y)$, s.t.

$$\lim_{d \rightarrow 0} \frac{G(x, y)}{d^{2-n}} = 1$$

$$\lim_{d \rightarrow 0} \frac{\underline{G}(x, y)}{|x-y|^{2-n}} = 1 \quad (\text{on } \mathbb{R}^n)$$

Then $\lim_{\substack{y \rightarrow \infty \\ d(x, y) = |x-y|}} \frac{G(x, y)}{G(x, \underline{y})} = \theta^{-1}$

- $G(x, y) \geq d(x, y)^{2-n}$, $\forall x, y$
- $\Delta_y G(x, y) = 0 \quad \forall y \neq x$

G smooth on $M \times M \setminus \{x, x\}$

$$\begin{aligned}
 \Delta_y d^{2-n}(x, y) &= \operatorname{div}((2-n) d^{1-n} \nabla d) \\
 &= (2-n)(1-n) d^{-n} |\nabla d|^2 + (2-n) d^{1-n} \Delta d \\
 &\geq (2-n)(1-n) d^{-n} + (2-n) d^{1-n} \frac{n-1}{d} \\
 &= 0
 \end{aligned}$$

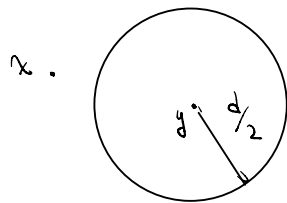
$$\Delta_y (d^{2-n} - G(x, y)) \geq 0 \Rightarrow d^{2-n} \leq G \quad \square$$

- Let $b(y) = G^{-\frac{1}{n-2}}(x_0, y)$ for fixed $x_0 \in M$
- $b(y) \leq d(x_0, y) \quad \forall y \in M$

Lemma $| \nabla b | \leq C(n)$

Pf. Cheng-Yau gradient estimate

$$\Delta G = 0$$



Thm (Colding) $| \nabla b | \leq 1$

Pf. $\sup_{\partial B_r(x_0)} | \nabla b |^2 \leq 1 + \varepsilon, \quad \forall r \leq r_0$

$\forall \varepsilon > 0, \quad \sup_{\partial B_{r_0}(x_0)} G(x_0, y) \leq \varepsilon$

$\exists r_0 < 1, \quad \forall \varepsilon < 1$

Define $u(y) = [|\nabla b|^2 - (1+\epsilon)] G - c^2 \epsilon$

On annulus $A = B_{r_0+1}(x_0) \setminus B_{r_0}(x_0)$

$$u \leq 0 \text{ on } \partial A \quad \& \quad \frac{\Delta u \geq 0}{\text{WT Proof}}$$

$$\Rightarrow u \leq 0 \text{ in } A$$

$$\Rightarrow [|\nabla b|^2 - (1+\epsilon)] G \leq c^2 \epsilon \text{ over } M.$$

$$\epsilon \rightarrow 0 \quad |\nabla b| \leq 1. \quad \square$$

Claim: $\Delta u = \frac{1}{2} b^{-2} \cdot G \cdot \left(\left| \nabla \nabla b^2 - \frac{\Delta b^2}{n} g \right|^2 + \text{Ric}(\nabla b^2, \nabla b^2) \right)$

Pf of the claim: HW. Euclidean space. both one zero. □

Colding - Minicozzi ²⁰¹³

Tangent cone uniqueness at ∞ , and a point.

Einstein, & tangent cone $C(x): X \times [0, \infty)$

X smooth $(n-1)$ -dim mfd $d_X x \, dr^2$

$$\text{Ric} \equiv 0$$

$$\text{Ric} \geq 0$$

(M^n, g) compact. then

H's mean value.

$$G(x, y) = \int_0^\infty (r(x, y, t) - \text{vol}(M)^{-1}) \, dt$$

Harnack

$$t' = (1+\epsilon)t$$

$$H(x, y, t) \leq H(x, x, t') \left(\frac{t'}{t}\right)^{\frac{n}{2}} \exp \left\{ \frac{d^2(x, y)}{4(t'-t)} \right\}$$

$$= H(x, x, t') (1+\epsilon)^{\frac{n}{2}} e^{\frac{d^2(x, y)}{4\epsilon t}} \quad \text{bounded}$$

$$\lim_{\substack{t \rightarrow \infty \\ t' \rightarrow \infty}} \frac{H(x, y, t)}{H(x, y, t')} = 1 \quad \text{for some } t' > t \rightarrow +\infty \quad \forall y$$

$$Q(x, y) = \int_0^\infty (H_0(x, y, t) - \text{Vol}^{-1}) dt \quad D = \text{diam}(M)$$

$$\underline{y \sim x} = \underbrace{\int_0^{d^2} (H_0(x, y, t) - \text{Vol}^{-1}) dt}_{\textcircled{1}} + \underbrace{\int_{d^2}^{D^2}}_{\textcircled{2}} + \underbrace{\int_{D^2}^\infty (H_0 - \text{Vol}^{-1}) dt}_{\textcircled{3}}$$

$$\textcircled{1} \sim -d^2 \text{vol}^{-1} + \int_0^{d^2} \text{vol}(B_{\sqrt{t}}(x))^{-1} e^{-\frac{d^2}{4t}} dt$$

$$\leq \int_0^{d^2} \text{vol}(B_{d^{(n)}})^{-1} \frac{\text{Vol}(B_{d^{(n)}})}{\text{Vol}(B_{\sqrt{t}}^{(n)})} e^{-\frac{d^2}{4t}} dt \quad \text{bounded.}$$

$$\sim \text{vol}(B_{d^{(n)}})^{-1} d^2 \leq \left(\frac{d}{\sqrt{t}}\right)^n \leq C$$

$$\textcircled{2} \sim \text{vol}(M)^{-1} D^2$$

$$\textcircled{3}. \text{ const}$$

Lecture 4.

Demailly 92 X^n , $\dim_{\mathbb{C}} X$, compact, complex manifold

T , closed current, positive (1,1), s.t.

$$T = \alpha + \sqrt{-1} \partial \bar{\partial} \psi, \quad \alpha: \text{real, smooth, (1,1)}$$

ψ : real

Assume: ① ω , a Hermitian metric, s.t. $\forall \theta, \xi \in T^{1,0} X$

$$(\mathbb{H}_{\omega} + u \otimes \text{Id}_{T_X})(\theta \otimes \xi, \theta \otimes \xi) \geq 0, \quad \langle \theta, \xi \rangle_{\omega} = 0$$

for some nonnegative, continuous (1,1)-form u

② $T \geq \gamma$, a continuous (1,1) form

Then: $\exists$ smooth ψ_{ϵ} , $\epsilon > 0$, s.t.

③ $\psi_{\epsilon} \downarrow \psi$, as $\epsilon \downarrow 0$

$$\textcircled{4} \quad T_{\epsilon} = \alpha + \sqrt{-1} \partial \bar{\partial} \psi_{\epsilon} \geq \gamma - \lambda_{\epsilon} u - \delta_{\epsilon} \omega$$

where $\textcircled{4.1} \quad \lambda_{\epsilon} \in C^{\infty}(X)$, $\lambda_{\epsilon} \downarrow \gamma(T)$ pointwisely

$\textcircled{4.2} \quad \delta_{\epsilon} \in \mathbb{R}$, $\delta_{\epsilon} \downarrow 0$ as $\epsilon \downarrow 0$.

(Z-Z)

Thm: X^n , Kähler, compact, ω_0 : Kähler metric

θ : smooth, (1,1)-form

ψ_0 : real, $\uparrow$ s.t. $\theta + \sqrt{-1} \partial \bar{\partial} \psi_0 \geq 0$.

upper semi-continuous

$\psi: L^1 \leq nAt$
($\psi_0 \leq 0$)

Assume, ① $\theta \leq A\omega$, $\Delta \theta \leq A\omega$.

$$\textcircled{2} \quad R(u, \bar{u}, v, \bar{v}) \geq -\Lambda, \quad \forall u, v \in T^{1,0}X, \quad |u|=|v|=1.$$

for some $A, \Lambda > 0$

Then for $\psi(x, t)$, subsolution of $\frac{\partial \psi}{\partial t} = \Delta \psi + nA$
with $\psi_{(0)} = \psi_0$

we have

$$\textcircled{3} \quad \Delta \psi \geq -nA, \quad \forall t$$

so. $\psi(x, t) \nearrow$ in t . for $\forall x \in X$.

$$\textcircled{4} \quad \lim_{t \rightarrow \infty} \frac{\psi(x, t)}{\log t} = \lim_{t \rightarrow \infty} t \Delta \psi(x, t) = \frac{1}{2} \nu(\psi_0, x)$$

$$\textcircled{5} \quad \theta + \sqrt{-1} \partial \bar{\partial} \psi(t) \geq -C(n) \cdot \Lambda \cdot \lambda(x, t) \cdot \omega_0, \quad \forall t < \Lambda^{-1}$$

$$\lambda(x, t) = \frac{\partial \psi}{\partial \log t} = t(\Delta \psi + nA)$$

$$\textcircled{6} \quad \text{For any } c > 0, \delta > 0, \quad \psi_{c, \delta}(x) = \inf_{t \in (0, \delta]} \left(\psi(x, t) - c \log \frac{t}{\delta} \right)$$

Kiselman-Legendre transformation of ψ , satisfies

$$\theta + \sqrt{-1} \partial \bar{\partial} \psi_{c, \delta} \geq -C(n) \cdot \Lambda \cdot \inf(c, \lambda(\delta, \cdot)) \cdot \omega_0$$

$\psi_{c, \delta}$ may be not bounded.

but lower bound is better

proof $\Delta \psi \geq -nA$

$\psi(t) \in C^\infty(X)$ for $t > 0$, since

$$\psi(x, t) = \int_X \psi_0(y) H(x, y, t) dy$$

$$|\psi(x, t)| \leq \int_X |\psi_0(y)| \cdot |H(x, y, t)| dy \leq \sup_M H \|\psi_0\|_{L^1}$$

$\bullet \text{ } 0 \leq t \leq t_0 \text{ } \psi + \Delta \psi_0 \leq nA + \Delta \psi_0 \text{ in distribution}$

For any $t_0 > 0$

$$\int_X f_0 \cdot \Delta \psi(t_0) \geq \int_X -nA f_0$$

For any $f_0 \in C^\infty(X)$, $f_0 \geq 0$.

$\forall f_0 \in C^\infty, f_0 \geq 0$

$-\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau} = \Delta f$ $f(\tau=0) = f_0$
 $f(t_0) = f_0$

$\tau(t) = t_0 - t$

$$\frac{d}{dt} \int_X f(t) \Delta \psi(t) = 0$$

$$\begin{aligned} \int_X f_0 \Delta \psi(t_0) &= \int_X \Delta f(t_0) \psi(t_0) = \int_X \Delta f(0) \psi_0 \\ &\geq -nA \int_X f(0) \\ &= -nA \int_X f_0 \end{aligned}$$

Asymptotics as $t \rightarrow 0$

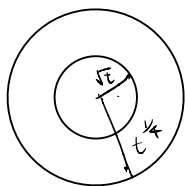
$$\nu(\psi_0, x) = \sup \left\{ c \geq 0 \mid \psi_0(y) \leq c \log |x-y| + O(1) \right\}_{y \rightarrow x}$$

$$= \lim_{r \rightarrow 0} \frac{\int_{\partial B_r(x)} \psi_0}{\log r} = \lim_{r \rightarrow 0} r^2 \int_{B_r(x)} \Delta \psi_0$$

$$\psi(x, t) = \int_X \psi_0(y) H(x, y, t) dy$$

$$H(x, y, t) = (4\pi t)^{-n} e^{-\frac{d^2(x, y)}{4t}} (1 + O(t)) \quad t \rightarrow 0$$

$$\psi_0(y) (4\pi t)^{-n} e^{-\frac{d^2(x, y)}{4t}} = \int_{B_{t^{1/4}}(x)} + \int_{X \setminus B_{t^{1/4}}(x)}$$



$$\int_{B_{t^{1/4}}(x)} \psi_0(y) (4\pi t)^{-n} e^{-\frac{d^2(x, y)}{4t}}$$

$$= \int_0^{t^{1/4}} \int_{\partial B_r(x)} \psi_0(y) (4\pi t)^{-n} e^{-\frac{r^2}{4t}} dy dr$$

$$\log r(\nu(x) \pm \varepsilon) = f \leq \nu(x) \pm \varepsilon$$

$\forall \varepsilon > 0, \exists r_0 < \varepsilon$

$$\left[\begin{aligned} \nu(x) &= \lim_{r \rightarrow 0} \frac{\int_{\partial B_r} \psi_0}{\log r} \\ &= \int_0^{t^{1/4}} (4\pi t)^{-n} e^{-\frac{r^2}{4t}} (\nu(x) \pm \varepsilon) \log r dr \end{aligned} \right]$$

$$|\partial B_r| = ?$$

$$\text{Vol}(\partial B_r)$$

$$\begin{aligned}
 \frac{r}{\sqrt{t}} = s &= (V(x) \pm \epsilon) (4\pi t)^{\frac{n}{4}} \int_0^{t^{-\frac{1}{4}}} e^{-\frac{s^2}{4}} (\log \sqrt{t} + \log s) ds \\
 &= (V(x) \pm \epsilon) (4\pi)^{-n} t^{-n+\frac{1}{2}} \left[\log \sqrt{t} \int_0^{t^{-\frac{1}{4}}} e^{-\frac{s^2}{4}} ds + \int_0^{t^{-\frac{1}{4}}} \log s \cdot e^{-\frac{s^2}{4}} ds \right] \\
 &\quad \text{Vol}(\partial B_r) \quad \text{Vol}(\partial B_r) \quad \text{Vol}(\partial B_r) \\
 &\quad \text{Vol}(\partial B_r) \quad \text{Vol}(\partial B_r) \quad \text{Vol}(\partial B_r)
 \end{aligned}$$

$$= (V \pm \epsilon) \cdot (1 \pm \epsilon) \cdot (1 + \epsilon) \log \sqrt{t}.$$

⑤ Maximum principle

$\xi = 0 + \sqrt{-1} \partial \bar{\partial} \psi$ let $\xi_1(x, t) \leq \xi_2 \leq \dots \leq \xi_n$ be
eigenvalues of ξ , pointwisely $\cdot x = x(t), \xi_1(x, t) = \inf \xi(t)$

$$\begin{aligned}
 \frac{\partial}{\partial t} \xi &= \frac{\partial}{\partial t} \partial \bar{\partial} \psi \\
 &= \partial \bar{\partial} \Delta \psi \\
 &= \Delta_L \partial \bar{\partial} \psi \\
 &= \Delta_L \xi - \Delta_L \theta \\
 &\geq \Delta_L \xi - A
 \end{aligned}$$

(eigenvalue) $\frac{\partial \xi_1}{\partial t} \geq \Delta_L \xi_1 - A$

$$\begin{aligned}
 &\xi: (1,1) \text{ form} \\
 &\xi = \xi_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \\
 &(\Delta_L \xi)_{i\bar{j}} = \Delta \xi_{i\bar{j}} + R_{k\bar{l}i\bar{j}} \xi_{k\bar{l}} \\
 &\quad - \frac{1}{2} (R_{i\bar{k}} \xi_{k\bar{j}} + R_{k\bar{j}i\bar{l}} \xi_{l\bar{k}})
 \end{aligned}$$

$$\geq \Delta \xi_1 - n \left(\text{tr}_{\omega} \xi - n \xi_1 \right) - A$$

$$\text{tr}_{\omega} \xi = \text{tr}_{\omega} \theta + \Delta \psi \leq nA + \Delta \psi$$

$$\therefore \frac{\partial \mathfrak{z}_1}{\partial t} \geq \Delta \mathfrak{z}_1 - \Lambda(\Delta \psi + nA - n\mathfrak{z}_1) - A \quad (*)$$

$$(\Delta_L \mathfrak{z})_{i\bar{i}} = \Delta \mathfrak{z}_{i\bar{i}} + \sum_k R_{k\bar{k}i\bar{i}} \mathfrak{z}_k - R_{i\bar{i}} \mathfrak{z}_1$$

$$\sum_k R_{k\bar{k}i\bar{i}} \mathfrak{z}_k = R_{i\bar{i}i\bar{i}} \mathfrak{z}_1 + \sum_{k>1} R_{k\bar{k}i\bar{i}} \mathfrak{z}_k$$

$$R_{i\bar{i}} = \sum_{k=1}^n R_{k\bar{k}i\bar{i}}$$

$$\begin{aligned} \Delta_L \mathfrak{z}_{i\bar{i}} &= \Delta \mathfrak{z}_{i\bar{i}} + \underbrace{\sum_{k>1} R_{k\bar{k}i\bar{i}}}_{\geq -\Lambda} \underbrace{(\mathfrak{z}_k - \mathfrak{z}_1)}_{\geq 0} \\ &\geq \Delta \mathfrak{z}_{i\bar{i}} - \Lambda \sum_{k>1} (\mathfrak{z}_k - \mathfrak{z}_1) \\ &= \Delta \mathfrak{z}_{i\bar{i}} - \Lambda (\text{tr}_{\omega_0} \mathfrak{z} - n\mathfrak{z}_1) \end{aligned}$$

$$(*) \geq \Delta \mathfrak{z}_1 - \Lambda(\Delta \psi + nA) + nA\mathfrak{z}_1 - A$$

$$\Rightarrow \underline{MP} \quad \boxed{e^{ct} \mathfrak{z}_1}$$

⑥

Kiselman minimal Principle.

satisfies $f(w, z) = f(w, z')$

if $z = x + \sqrt{-1}y$

$z' = x + \sqrt{-1}y'$

$\Omega \subset \mathbb{C}^n$ pseudoconvex $T \subset \mathbb{C}$

$\parallel$
 $\{z = x + \sqrt{-1}y \mid a < x < b\}$

$f \in \text{Psh}(\Omega \times T)$
 i.e. $\sqrt{-1} \partial \bar{\partial} f \geq 0$

s.t. $g(\omega) = \inf_{z \in T} f(\omega, z) = 0$, the $g \in \text{Psh}(\Omega)$

Convexity of ψ in t

Lemma $u > 0$ satisfies

$$\frac{\partial}{\partial t} u = \Delta u$$

then $t \Delta u \geq -2n u$. $t \leq \frac{n^{-1}}{b}$

pf: $\text{Ric} \geq -(n-1)\Lambda$

$$|\nabla \log u|^2 - a \frac{\partial}{\partial t} (\log u) \leq n \cdot \left(\frac{a^2}{a-1} \Lambda + \frac{a^2}{t} \right)$$

$$\downarrow$$

$$-a \frac{\Delta u}{u}$$

$$t \frac{\Delta u}{u} \geq -n \left(a + \frac{a}{a-1} \Lambda t \right) u$$

$$1 < a \leq 2$$

$$\eta = \frac{t^{2n}}{2n-1} (\Delta u + A) \quad \leftarrow \quad \Delta u(t) \geq -A$$

$$\frac{\partial}{\partial t} \eta \geq \frac{2n}{2n-1} \frac{t^{2n}}{t} (\Delta u + A) + \frac{t^{2n}}{2n-1} \underbrace{\Delta (\Delta u + A)}_{\Delta \eta}$$

$$\geq 0$$

$$\Delta \tilde{u} = \frac{\partial}{\partial t} \tilde{u}$$

Corollary

$u + A t$ conve $\left(\begin{matrix} \text{in } x \\ \text{pointwisely} \end{matrix} \right)$ $\xrightarrow{\text{By Lem. we have estimate}} \text{in } t$

$\text{in } t^{1-2n} = s(t)$

$$\frac{\partial}{\partial s} (u + A_t) = \frac{\Delta u + A}{s'} = \frac{t^{2\alpha}}{s} (\Delta u + A)$$

$$\frac{\partial^2}{\partial s^2} (u + A_t) \geq 0.$$