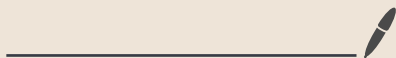


最优转运和曲率维数



1. Gradient flow: De Giorgi

Abstract setting / Wasserstein space

2. Optimal transport.

Tool, target

3. Ricci curvature / CD, RCD

OT stability, Wasserstein barycenter

Convexity (凸性?)

$\mathbb{R}$, f is "convex"

(1) $f \in C^2(\mathbb{R})$, $f'' \geq 0$

"geodesic convex"

(2) $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ e.g. $t = \frac{1}{2}$

(3) $f(\lambda_1 x_1 + \dots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \dots + \lambda_n f(x_n)$ "barycenter convex"

In metric space (X, d) , geodesic. we say $f: X \rightarrow \mathbb{R}$ is k -convex,

for $k \in \mathbb{R}$

$\overset{x}{\curvearrowright} y$
 $\forall x, y, \exists \gamma: [0, 1] \rightarrow X$, shortest.

if $f(\gamma_t) + \frac{k}{2} t(1-t) d^2(\gamma_0, \gamma_1) \leq tf(\gamma_1) + (1-t)f(\gamma_0)$

$\left\{ \begin{array}{l} \text{(weak)} \exists \gamma \\ \text{(strong)} \forall \gamma \end{array} \right.$

Remark: e.g. $\mathbb{R}^n, \|\cdot\|$. 不同范数的 weak / strong.

"weak" stable, Ricci limit $(\mathbb{R}^n, d_t) \xrightarrow{\text{lim}} \mathbb{R}^n$ on $\mathbb{R}^n$

"strong" $\rightarrow$ (太 strong 了!)

in many cases

uniqueness.

(Synthetic 框架下, "strong" 往往对应唯一性)

Riemannian manifold 上为函数的存在性 (不平凡)

↓
提升一下

Gradient flow

convex

(X, d) $\gamma_t : [0, 1] \rightarrow X$, ϕ is $\checkmark$ 向量集合 (很多空间上无意义)

$\mathbb{R}^n$ 中 $\rightarrow$ $\dot{\gamma}_t \in -\partial\phi(\gamma_t) = \{v : \phi(x) \leq \phi(y) + \phi(y-x)\}$
 $= -\nabla\phi(\gamma_t)$

$$\phi(\gamma_s) - \phi(\gamma_t) = \int_s^t \frac{1}{2} |\partial\phi|^2 + \frac{1}{2} |\dot{\gamma}_r|^2 dr$$

梯度流下降最快

local Lipschitz

$$\begin{pmatrix} \left| \frac{\partial}{\partial r} \phi(\gamma_r) \right| \\ = |\nabla\phi(\gamma_r) \cdot \dot{\gamma}_r| \\ \leq \frac{1}{2} |\partial\phi|^2 + \frac{1}{2} |\dot{\gamma}_r|^2 \end{pmatrix}$$

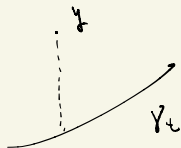
De Giorgi : 不同形式梯度流
 energy dissipation

(EDI) $\phi : X \rightarrow \mathbb{R}$, $|\partial\phi|(x) = \lim_{y \rightarrow x} \frac{|\phi(y) - \phi(x)|}{d(y, x)}$

Lemma: (γ_t) is absolutely continuous $\Leftrightarrow d(\gamma_t, \gamma_s) \leq \int_s^t g(r) dr$, $g \in L^1$
 $|\dot{\gamma}|$ is the minimal g .

Evolution variation

(EVI) ϕ is k -convex,
 $\forall y$, we have



"very strong" $\rightarrow$ implies
 ① strong convex
 ② Riemannian structure

Evolution variation
 $\frac{1}{2} \frac{d}{dt} d^2(\gamma_t, y) + \frac{1}{2} \frac{d}{dt} d^2(\gamma_t, y) \leq \phi(y) - \phi(\gamma_t)$
 EVI, $k=0$ Riemannian $\mathbb{R}^n$ $\gamma_t \cdot (\gamma_t - y)$

Assume $\gamma_t, \tilde{\gamma}_t$ are two gradient flows

$$k d^2(\gamma_t, \tilde{\gamma}_t) + \frac{1}{2} \frac{d}{dt} (d^2(\gamma_t, \tilde{\gamma}_t)) \leq 0.$$

By Aronwall lemma, $d(\gamma_t, \tilde{\gamma}_t) \leq e^{-kt} d(\gamma_0, \tilde{\gamma}_0)$.

imply Riemannian structure

"Wasserstein Space"

given (X, d) . Monge Problem 1781 $\inf_T \int C(x, T(x)) d\mu$

Given $\begin{cases} \mu \in \mathcal{P}(X) \leftarrow \text{概率测度} \\ \nu \in \mathcal{P}(X) \end{cases} \quad T_{\#}\mu = \nu$

$$C(x, y) \in C(X \times X)$$

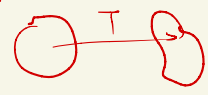
T may not exist

e.g. $\mu = \delta_c$ (dirac-measure)

$$\nu = \frac{\delta_1 + \delta_2}{2}$$

problem

push forward

$$\begin{aligned} T: X &\rightarrow X \\ T_{\#}\mu(A) &= \mu(T^{-1}(A)) \end{aligned}$$


Kantorovich problem

$$\inf_{\pi \in \Pi(\mu, \nu)} \int C(x, y) d\pi$$

$\in \mathcal{P}(X \times X)$

let P_1, P_2 be projection

$$P_{1\#}\pi = \mu, P_{2\#}\pi = \nu.$$

$$\text{let } C(x, y) = d^2(x, y).$$

$$W_2(\mu, \nu) = \sqrt{\inf_{\pi \in \Pi} \int d^2(x, y) d\pi} \quad \text{有三角不等式,}$$

$\text{且可加} + \infty$.

is extended metric space.

Equivalent class $\mathcal{P}_2(X) = \{\mu \in \mathcal{P}(X), \exists x_0, \int d^2(x, x_0) d\mu < +\infty\}$.

$\therefore (\mathcal{P}_2, W_2)$ is a metric space.

Moreover, if (X, d) is complete, geodesic, separable

$(\mathcal{P}_2, W_2)$ is $\sim$ $\sim$ $\sim$ (respectively)

Q: $(X, d) = (\mathbb{R}^n, \|\cdot\|)$, $(P_2(\mathbb{R}^n), W_2)$ is Alexandrov space with non-negative curvature. (right: non-neg. c. $\Rightarrow$ non-neg. c.)
But other curvature, $\nRightarrow$ other curvature. we don't know.

Theorem (Geodesics in (P_2, W_2)) **Ambrosio-Aigl users-guide.**

For any $(\mu_t) \in P_2(X)$, geodesic $\Leftrightarrow \exists \Pi \in P(\text{Geod}(X))$ s.t.

$(e_t)_\# \Pi = \mu_t$, $e_t(r) = r_t$ & $(e_0, e_1)_\# \Pi$ is OT in Kantorovich problem

In particular, $\|\dot{\mu}_t\|^2 = \int |\dot{r}|^2(r_t) d\Pi$



example $(P_2(\mathbb{R}^n), W_2)$

Monge problem?

$$W_2^2(\mu, \nu) = \sup_{\substack{(\varphi, \varphi^c) \\ \varphi(x) + \varphi^c(y) \leq d^2(x, y)}} \int \varphi d\mu + \int \varphi^c d\nu \quad (\text{Kantorovich duality}) \quad \underline{1989 Yann Brenier}$$

$$\varphi(x) + \varphi^c(y) \leq |x|^2 + |y|^2 - 2x \cdot y \quad \text{if } \varphi \text{ is differentiable.}$$

notice this term.

$$\varphi'(x) = -2y \quad \Leftrightarrow \quad \text{"}\mu\text{-a.e. differentiable"}$$

$\underbrace{C-C}_{\text{convex}}$ hypersurface, graph "convex-convex" function
(Monge Problem is solvable in $\mathbb{R}^n$)

Remark: We need a reference measure. (uniqueness? rigidity?)

(Cheeger's work (1999))

Gradient flows

① V is convex on $\mathbb{R}^n$.

$$P_2(X) \ni \mu \mapsto \int V d\mu \quad \text{Denote } V(\mu)$$

(a) V is convex,

$$\int V d\mu_t = \int V(\gamma_t) d\pi$$

$$\leq \int [tV(\gamma_1) + (1-t)V(\gamma_0)] d\pi$$

$$= tV(\mu_1) + (1-t)V(\mu_0).$$

(b) μ_t : continuity equation

$$\exists \eta \text{ s.t. } \frac{d}{dt} \int \varphi d\mu_t = \int \langle \nabla \varphi, \eta_t \rangle d\mu_t \quad \eta - \text{test func.}$$

则称 η_t 为 μ_t 的速度场

η_t ?

Thm (Benamou - Brenier)

$$W_2^2(\mu, \nu) = \inf_{\substack{(\mu_t) \in AC(P_2, W_2) \\ \mu_0 = \mu, \mu_1 = \nu}} \int_0^1 \int |\eta_t|^2 dt. \quad (\text{BB formula})$$

Exercise. V 's gradient flow

(Otto's Calculus)

$$(W_2, P_2), \text{ porous medium equation } \partial_t f(x, t) = \Delta f^\alpha, \quad \alpha > 1.$$

$$\text{Functional: } \mu \longrightarrow \int \rho^\alpha dx$$

$$\mu = \rho dx$$

(inner energy. Santambrogio)

$$(\mu_t) \subseteq (P_2(X), W_2)$$

inner product

(0 to) Riemannian Structure.

$$\int \langle v_t, w_t \rangle d\mu_t, \quad v_t \text{ from continuous equation.}$$

example.

$$\mu_t \text{ is the geodesic } [0,1], \quad W_2(\mu_0, \mu_1) = \int_0^1 |\dot{\mu}_t| dt$$

参考动度

Assume (a) $\mu_t \ll m$ (关于 m 绝对连续? 有一组密度函数)

(b) (Monge problem) strong version, 一端 absolutely continuous,

M.P. solvable: (X, d, m) , weak version, 另一端 $\checkmark$.

$$\mu_0 \ll m, \exists T: X \rightarrow X$$

$$\text{OT, s.t. } \mu_1 = T_{\#} \mu_0. \rightarrow T \text{ 即为 M.P. 的解}$$

$$\text{on } \mathbb{R}^n \quad \mu_t = (\text{Id} + t(T - \text{Id}))_{\#} \mu_0 \quad (\text{欧拉线}). \quad (\text{Breitner}).$$

$$\text{on Riem. mfd.} \quad \mu_t = [\exp(t \nabla \varphi)]_{\#} \mu_0. \quad (\text{McCann})$$

Generally, let (φ, φ^c) be Kantorovich potentials.

$$v_t = -\nabla Q_{1-t}(-\varphi^c)$$

where $\{Q_t\}$ is Hamilton-Jacobi

$$Q_t \varphi(x) = \inf_y \left\{ \frac{d^2(x,y)}{t} + \varphi(y) \right\}$$

② Heat flow (equation)

一族动度的曲线

参数 t 的一族函数

$$\mathbb{R}^n. (\mu_t) \subseteq P_2 \quad (\text{Heat flow})$$

$$\mu_t = \rho_t \, dm \quad \text{Heat equation}$$

$$\Delta \rho_t = \frac{d}{dt} \rho_t \quad \text{Neumann 边界条件}$$

test func.

$$\varphi \in C^\infty, \quad \frac{d}{dt} \int \varphi \, d\mu_t = - \int \langle \nabla \varphi, \nabla \rho_t \rangle \, dm$$

$$= - \int \langle \nabla \varphi, \nabla \ln \rho_t \rangle \, d\mu_t$$

$\mathbb{R}^n$ 不是 Lebesgue measure

$$\text{Velocity field: } -\nabla \ln \rho_t$$

$$|\dot{\mu}_t|^2 = \int \frac{|\nabla \rho_t|^2}{\rho_t} \, dm$$

Finer information

W_2 不是 φ 的区间
Fisher vector

Gradient flow

On (P_2, W_2) , consider some examples of functionals.

P_2 上不是 function

define ① $\mu \rightarrow \int V \, d\mu$ "potential energy"

$$F: \quad \textcircled{2} \quad \text{Ent}(\mu) = \begin{cases} \int \rho \ln \rho \, dm & \mu = \rho_m \\ +\infty & \text{other} \end{cases}$$

$$\rho \, dm \rightarrow \int U(\rho) \, dm$$

" ∇F " 1st order variation

$$\frac{F(\mu + \varepsilon V) - F(\mu)}{\varepsilon}$$

$$\textcircled{1} \quad \nabla V$$

$$\textcircled{2} \quad \nabla U'(\rho)$$

Note that when $U(x) = x \ln x$, $U' = 1 + \ln x$ 熵的

$$\nabla U(p) = \nabla \ln p$$

可将抛物方程写成 ∇ 梯度流?

In conclusion,

Thm: heat flow is the gradient flow of $\text{Ent}(\cdot)$.

$\mathbb{R}^n$ Otto

Riem, Alex, CD, ...

In particular, if EVI gradient flow exist, heat flow exists for any

probability measure. $H_t(\mu)$, $H_t(\delta_x) \rightarrow$ heat kernel.

(如果设有从密度函数出发的测度可用一族去逼近, 但需 W_2 完备性)
有的话, 可把 heat eqn 解做 p_t

CD (curvature-dimension)

Def (Displacement convexity) McCann.

$F: (P_2, W_2)$ 上 functional. We say F is displacement convex.

$$\text{if } F(\mu_t) \leq t F(\mu_1) + (1-t) F(\mu_0)$$

Intuitively (Villani's first Book, Otto-Villani)

good book

Hopf-Lax

那书

Villani: Bakry-Emery

$$\frac{\partial}{\partial t} \text{Ent}(\mu_t) = \int \langle \nabla \ln f_t, \nabla_t \rangle d\mu_t$$

$$\text{Hopf-Lax: } Q_t f = \inf_y \left\{ \frac{d^2(y)}{t} + f(y) \right\}$$

$$\text{Hamilton-Jacobi-equation. } \frac{d}{dt} Q_t f + |\text{lip } Q_t f|^2 = 0.$$

$$\frac{\partial^2}{\partial t^2} \text{Ent}(\mu_t) = \int T_2 d\mu_t$$

Bakhar formula. left part.

(自然包含一些曲率信息)

CD condition

(Von Renesse-Sturm)

$\Rightarrow$ $CD(K, N)$

Def $F = \text{Entropy}(\text{Ent})$,
($CD(K, \infty)$)

Then $\text{Ricci} \geq K \Leftrightarrow CD(K, \infty)$ on Riem mfd.

(M, g) 仍有等价性 (Bog Xin Han).
Riem $\uparrow$ any $\uparrow$

$e^{-\frac{\text{Ent}}{N}}$ is concave
($CD(0, N)$)

Remark: log-concavity. Riemannian mfd 上结果在?

$\text{Ricci} \geq 0$, i.e., $[\text{Vol}(A_t)]^{\frac{1}{N}} \geq t [\text{Vol}(A_1)]^{\frac{1}{N}}$

$A_t = \{z \mid \exists x \in A_0, y \in A_1 \text{ s.t. } z = (1-t)x + ty\}$

$\Leftrightarrow z$ is t -intermidium pt

$\text{Vol}(A_t) \geq \text{Vol}(A_1)^t \text{Vol}(A_0)^{1-t}$

Remark: Villani et al.

$(\mathbb{R}^n, \|\cdot\|)$ is $CD(0, n)$
 $\downarrow$
+ norm

CD Setting

Gradient flows $\chi(t) = \int |\nabla f|^2 dm$ "Cheeger energy" 有一个 functional 总可考虑其 gradient flow

Thm (AGS). The L^2 -gradient flow of $\chi(\cdot)$ is the g.f. of $\text{Ent}(\cdot)$

Riemannian $CD(RCD)$

1. TFAE (1) $CD + \chi(\cdot)$ is quadratic.

保证几何
分析工具的使用

(2) Heat flow is linear

(3) Laplacian is unique

(3) Gradient flow of $\text{Ent}(\cdot)$ is an EVI gradient flow.

不大光滑 / 无限维也可使用

$\checkmark$ "P2: take 2."
 $\int e^{-cd^2(x, x_0)} dm < +\infty$

(体积增长条件)

$\sup_{\mu \in \mathcal{P}} \text{Ent}(\mu) < +\infty$

- 强有界 $\rightarrow$ 弱收敛子列.

Appl.

b is harmonic func.

Splitting

$\rightarrow$ prop: $|\nabla b| \equiv 1$

b is Buseman function.

gradient flow of $b = ?$

Riem 中要解方程 $\dot{\gamma} = -\nabla b(\gamma_0)$ 什么意思?

但非光滑不行. $\rightarrow$ 把方程换到 W_2 中 - 一个 functional.

与 Riem.
mfd 的
不同处

$\mu \rightarrow \int b d\mu = B(\mu)$, consider the gradient flow of B

func. 保持凸性

denote by (μ_t) the g.f of B .

在 π 的观测意义下

$\exists \pi \in \mathcal{P}(AC(\mathbb{R}, X))$, s.t. $\mu_t = (e_t)_\# \pi$, π a.e. $\gamma \in AC(\mathbb{R}, X)$

γ is the gradient flow of b

通过 π 把方程解出来了. 对应一个解

(存在性?) 唯一性与 Monge question's sol 有关.

RCD 下可证 Monge question 有解. (3) 强凸的.

entropy 沿测度流

Wasserman barycenter

$\mu_0, \mu_1 \in \mathcal{P}_2(X)$

$\mu_{\frac{1}{2}} \in \argmin \frac{1}{2} W_2^2(\mu, \mu_1) + \frac{1}{2} W_2^2(\mu, \mu_0)$

(Condition: $W_2(\mu_{\frac{1}{2}}, \mu_1) = W_2(\mu_{\frac{1}{2}}, \mu_0) = \frac{1}{2} W_2(\mu_1, \mu_0)$) 刻画中点

WB problem

Gauss (最早) Cantan (最早严格).

$\Omega \in \mathcal{P}(\mathcal{P}_2(X))$, $\inf_{\mu} \int W_2^2(\mu, \nu) d\Omega(\nu)$. If $\mu \in \argmin$, we say barycenter of Ω .

$\mathbb{R}^n$ 中. 即坐标取平均值 $\int x d\mu$ $\Omega \rightarrow \mathbb{R}$
 $X \rightarrow \mu$

类似于概率中求 E (期望).

W space 上的概率平均度

Why "barycenter"? 而不是 geodesic.

Cantan

$f: X \rightarrow Y$. Ask: $\int f d\mu$?
 $\mu \in \mathcal{P}(X)$

在曲率空间上. 存在且唯一.

eg. 欧氏 space.

$f_\# \mu$ 在 Y 中的 barycenter

(PDE) A priori est.

- 1) if exist, satisfy some bound
- 2) Approximate. (也有类似 bound).
- 3) compactness.

对应

In Wasserstein geom.

- 1) Jensen inequality. (BCD)
if μ is barycenter of Ω .

$Ent(\mu) \leq \int Ent(\nu) d\Omega(\nu)$

- 2) Approximate. (Almost Jensen).

- 3) for Ω satisfying $\int Ent(\nu) d\Omega(\nu) < +\infty$

$\exists$ subspace. weak convergence.
1. W-distance 下降
连续吗

Tessier PP: EVI grad flow. Let μ be a barycenter, μ_t be the EVI grad flow from μ .

$$\forall \mu \in \mathcal{P}_2(X), \quad \frac{1}{2} \frac{d}{dt} W_2^2(\mu_t, \nu) \leq \text{Ent}(\nu) - \text{Ent}(\mu_t) \leq \text{Ent}(\nu)$$

$$\int_0^t \text{Ent}(\mu_s) ds + \frac{1}{2} (W_2^2(\mu_t, \nu) - W_2^2(\mu_0, \nu)) \leq t \text{Ent}(\nu)$$

Int grad w.r.t. Ω directly t on both sides, $t \downarrow 0$.

无 μ $\therefore \int \phi d\mu = 0$ 即全空间概率积分为 1.

Stability of OT.

Recall $\sup_{\phi(x)+\psi(y) \leq d^2(x,y)/2} \left\{ \int \phi(x) d\mu(x) + \int \psi(y) d\nu(y) \right\}$.

if $-c+c$ (但 g 有变化)

optimal ϕ is Kantorovich potential. (RCD)

$$\text{Map: } T(x) = \exp_x(-\nabla \phi)$$

(Alex)

有下界

$$\text{Goal: } \|\phi_\mu - \phi_\nu\|_{L^2(\mu)} \leq W_p^\alpha(\mu, \nu)$$

"Wasserstein space 嵌入 L^2 中的一种方式"
Hilbert space.

[Ambrosio-Cigli]

$$p=2, \alpha=\frac{1}{2}, \text{ almost.}$$

对应反向柯西不等式的最佳系数.

[Andoni-Naor-Neeman]

$$(\mathcal{P}_2(\mathbb{R}^d), W_2), \text{ bi-Hölder, } d \geq 3 \text{ (即不能嵌入 Hilbert space)}$$

Setting

(用 Roman-chain argument).

① supp ρ

John-domain S

arc length



fixed x_0 s.t. $\forall x \in S, \exists r$ from x_0 to x

s.t. $d(\gamma(r), S^c) \geq \eta r$.

Rem = OT 的

稳定性

Kitagawa (张若冰的师兄)

$$\textcircled{2} \alpha_1 m \leq \rho \leq \alpha_2 m$$

$$\textcircled{3} \mathbb{E}_\rho(\phi) = 0.$$

(法国人给出上述反例. Letrouit, Mériçot, Delalande)

截而曲率空间有估计下界。
(与距离函数有关)

"把距离用热核逼近替代"

Motivation: Varadhan's formula.

$$\textcircled{1} \lim_{t \rightarrow 0} -t \ln P_{t/2}(x, y) = \frac{1}{2} d^2(x, y)$$

heat kernel

$\textcircled{2}$ Schrödinger, Entropic OT.

与 Cole-Hopf transport 有关, 也与

Chen: d_1

Chen-Tian: Toric-Kähler.

further: Kähler-Einstein.

Berman/Daves

RIC, Ric $\geq$ K, Talagrand ineq.

conj: ...

Kähler geom. 中有对应, e.g. Dwyer's functional

Def: $\varphi \in \text{Lip}(X)$, $t > 0$. we define a functional.

$$\Phi_t[\varphi](x) = -t \ln \int e^{\varphi(y)/t} P_{t/2}(x, y) dm = -t \ln \left(\int e^{\frac{\varphi(y) - \frac{1}{2} d^2(x, y)}{t}} dm \right)$$

如何得到 Ric 信息?

$$-t \ln (H_{t/2}(e^{\varphi/2}))$$

$$K_t[\varphi] := \int \Phi_t[\varphi] d\mu \leftarrow \text{研究 } K_t \text{ 的凸性}$$

Lemma 1^{st} variation, $\nabla K_t[\varphi] = \int \mu_t^t d\varphi$

$$\nabla \Phi_t[\varphi] = \mu_x^t \quad \frac{d}{ds} \bigg|_{s=0} \nabla K_t[\varphi + s\psi]$$

$\uparrow$
变数
 μ_t 是概率测度

$\text{supp } \mu_t \subset Y \subset X$

测度

的支集

Thm $\forall \varphi, \psi \in \text{Lip}(Y)$ φ_*, ψ_* are proper Lipschitz extension.

$$\int (\Phi_t[\varphi_*](x) - K_t[\varphi_*]) - (\Phi_t[\psi_*](x) - K_t[\psi_*]) d\mu(x)$$

为了让它有意义 (空间无所谓) $\varphi_0 = \varphi_* + s(\psi_* - \varphi_*)$ 线性插值

$$\leq C \left| \int \mu_t^t[\varphi_*] - \mu_t^t[\psi_*] (\varphi_* - \psi_*) \right|^{\frac{1}{2}}$$

"fake proof"

$$L = \int_0^1 \int \langle \nabla K_t[\phi_s], v \rangle ds$$

$$(*) \leq K \left(- \int_0^1 \langle \nabla^2 K_t v, v \rangle ds \right)^{\frac{1}{2}}$$

(*) from ①. 1-Poincare inequality

② Li-Yau heat kernel (use Ricci curv.).

$$|\nabla_x \log P_t(x, y)| \leq e^{-\frac{Kt}{2}} \frac{\Delta_x P_t(x, y)}{P_t(x, y)}$$

Lemma: $\Phi_t[\psi_x](x) = -t \ln \left(\int e^{\frac{\psi(y) - \frac{1}{2} d^2(x, y)}{t}} d\mu \right)$

$\downarrow \text{p.a.e.}$
 $\downarrow t \rightarrow 0.$

$$\psi^c(x)$$

$$K_t[\psi_x](x) \longrightarrow \int \psi^c(x) d\mu(x)$$

Lemma: $\forall \varphi$ of $\nu \rightarrow \rho$ the Kantor. potential.

$$\mu^t[\varphi_x] \xrightarrow{\text{weak}} \nu$$

$$\mu_x^t[\varphi_x] \longrightarrow \delta_{\Gamma(x)}$$

$$\nabla \Phi_t$$

Letting $t \rightarrow 0$, $\int |\varphi^c - \psi^c| d\rho \leq \int (\varphi - \psi) d(\mu - \nu)$

$$= \int (\varphi - \psi) d\mu - \int (\varphi - \psi) d\nu$$

$\forall \pi$, coupling of μ, ν .

$$= \int [(\varphi - \psi)(x) - (\varphi - \psi)(y)] d\pi(x, y)$$

$$\leq \text{Lip}(\varphi - \psi) \int d(x, y) d\pi.$$

□

而且, m_e -a.e. (x, v) geodesic 可无限延长.

(即几乎所有点不过非正则点)

Thm (Brue-Mondino-Semola, 2023)

RCD setting.

Alex. without bdy. no. m.m.bdy (?).

Def. (X, d, μ) , 令 $b_r(x) = \mu(B_r(x))$

对 r 求导
用 Taylor 展开.

$$v_r = \left(1 - \frac{b_r(x)}{b_n r^n}\right), \text{ Riem. mfd. } v_r = \frac{1}{b(n+2)} \text{ Scal } r^2$$

metric measure boundary (m.m.b.) is defined as the limit of

$$\left\{ v_r / r \right\}_{r \in (0,1)} \xrightarrow{r \downarrow 0}$$

刻画流形边界.

TM
SI?

$$\mathbb{R}^n \ni \int |\nabla \phi_\mu^{(n)} - \nabla \phi_\nu^{(n)}|^2 d\mu^{(n)} = \frac{1}{d_n} \iint_{S_X} \langle \nabla \phi_\mu^{(n)} - \nabla \phi_\nu^{(n)}, \nu \rangle^2 d\sigma_X(\nu) d\mu^{(n)}$$

$$f := \phi_\mu - \phi_\nu \text{ 在 } s \mapsto \phi_s(x, v) \text{ 下} = C \int_{S_X} \int_0^1 \langle \nabla f, \phi_s(x, v) \rangle b_s(x, v) ds.$$

ν 的方向导数. 沿射线 (S. Obata) domain 上
可能会跑出 X 再回来. 而积分 f 是 John r . $\therefore [0,1]$ 其实分成很多小段系数可能

then consider ① 1-dim 的反向 Poincaré ineq. (Marigot...). 爆炸.

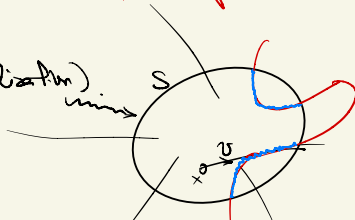
② 一条线上 (ϕ_s) 穿过边界 s 的次数.
在平均意义下不是有限的.

Sudakov 苏联人做了.
不少类似结果.

1-dim 上, s 在线上的等周不等式中的等周常数.

Isoperimetric ineq (localization)

Klartag / Nedde-deep.



"把高维视为一维".

$$m = m_g d_g \quad \int m_g d_g \leq 1 m.$$

nedge 分成很多 nedge. 且每条有 0 条件