

Motivation: from sphere thm in cpt. manifold.

$$(M^n, g) = \text{cpt. mnfd.}$$

$$\underline{n=2}: R > 0 \Rightarrow M \cong S^2 \text{ (Gauss-Bonnet)}$$

$$\underline{n=3}: \text{Ric} > 0 \Rightarrow \tilde{M} \cong S^3 \text{ (Hamilton)}$$

$$n \geq 4: \text{differentiable sphere thm (Brendle-Schoen, Bihl-Wilking).}$$

($n=3, K > 0$)

$$\bullet R_m > 0 \text{ (as End } S^{\text{?}}_M) \xRightarrow{\text{BW}} \tilde{M} \cong S^n$$

$$\bullet \frac{1}{4}\text{-pinched } K_{\max} < 4K_{\min} \xRightarrow{\text{BS}}$$

All by RF

BS: Based on intermediate concept

: Isotropic curvature.

I think (1) Micallef-Moore studied

$$f: S^2 \longrightarrow M \text{ (minimal)}$$

Def: R_m is said to have ≥ 0 isotropic curvature.

if $K^{\mathbb{C}}(\Sigma) \geq 0$ if totally isotropic $2^{\mathbb{C}}$ -plane
(Require $n \geq 4$).

Def: R_m is said to be WPIC₁ if $M \times \mathbb{R}$ is of WPIC.

... WPIC₂ if $M \times \mathbb{R}^2$ is WPIC.

$$n=3: \text{WPIC}_1 \Leftrightarrow \text{Ric} \geq 0$$

$$\text{WPIC}_2 \Leftrightarrow K \geq 0$$

$$n \geq 4: \frac{1}{4}\text{-pinched} \Rightarrow \text{WPIC}_2 \Rightarrow \text{WPIC}_1 \Rightarrow \text{WPIC}$$

BS WPIC is preserved under RF

"non-cpt mfd."

- $n=2$: $R > 0 \Rightarrow M \cong \mathbb{R}^2$
- $n=3$: $\text{Ric} > 0 \Rightarrow M \cong \mathbb{R}^3$ (Schön-Yau).
- $n \geq 4$ Saul theorem $K > 0 \Rightarrow M \cong \mathbb{R}^n$

Q: What if $R_m > 0$ in Stronger sense??

$n=3$: Hamilton's conj: $(M^3, g) \mid \text{Ric} \geq \varepsilon R_g$ is either cpt or flat.

Thm (Perrone L. - Schulz - Simon-Topping, c.f. Lott, Chau-Zhu)

If (M, g) is complete s.t. $R_m - \varepsilon R \cdot \text{Id} \leq C \text{pt}$,

then $M = \text{cpt} / \text{flat}$. (non-cpt version of Barden's sphere thm)

Focus on $n=3 \rightarrow$ use RF + Cheeger-Colding.

Fake pf: Pretend $(M, g) = \text{mfd}$ w/ $AVR > 0$
 $\lim_{r \rightarrow \infty} \frac{V(x, r)}{W r^n}$

$C(x) := \lim_{R_i \rightarrow \infty} (M, R_i^2 g, X)$ for some $R_i \rightarrow \infty$, by Cheeger-Colding.

• Pretend: $X^2 = \text{smooth}$ (fake proof)

$\Downarrow$

$$g_{C(x)} = dr^2 + r^2 g_X$$

• $\text{Ric} \geq \varepsilon R$ is preserved (fake)

• $\text{Ric}(\partial_r, \partial_r) = 0 \Rightarrow R \stackrel{\text{3D}}{=} 0 \Rightarrow R_m = 0$.

$$X = S^2$$

$\Rightarrow C(x) = \mathbb{R}^3 \xrightarrow{\text{Colding}} M = \mathbb{R}^3$ (thanks to $\text{Ric} \geq 0$)

Issue: No regularity !!

- collapsing (?)

Special of Special case.

$(M, g) =$ gradient expanding soliton w/ $\text{Ric} \geq \varepsilon R$

(self-similar sol to RF)

$$2\text{Ric} + g = 2\nabla^2 u = \underbrace{\mathcal{L}_u g}_{\text{Lie derivative.}}$$

$$\text{Define } \begin{cases} \partial_t \phi_t = -t^{-1} du \\ \phi_1 = \text{id} \end{cases}$$

$$\text{then } \hat{g}_{(t)} = t \phi_t^* g, \text{ then } \partial_t \hat{g} = -2\hat{\text{Ric}}, \hat{g}(1) = g.$$

$$\text{pull back } \star \quad 2t\hat{\text{Ric}} + \hat{g} = 2\hat{\nabla}^2 \hat{u}$$

$$t \rightarrow 0^+ \quad \boxed{\hat{g}_0 = 2\hat{\nabla}^2 u_0} = \text{metric cone equation}$$

GERS = smoothed version of metric cone.

Claim: ^{3D} GERS w/ $\text{Ric} \geq \varepsilon R \Rightarrow \text{flat}$.

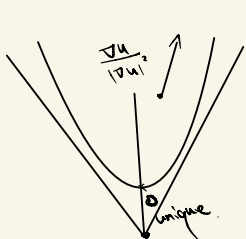
Lemma: $\nabla R + 2\text{Ric}(\nabla u, \cdot) = 0$.

$$\text{Pf: } (\nabla_i \nabla_j - \nabla_j \nabla_i) \nabla_k u = -R_{ijk\ell} u_\ell$$

$$\parallel \\ -\nabla_i R_{j\ell k} + \nabla_j R_{i\ell k} \quad \square$$

Let $\mathbb{F}_t$ be

$$\partial_t \mathbb{F}_t = \frac{\nabla u}{|\nabla u|^2} \quad \left(\underbrace{g \leq g + \text{Ric}}_{\sim} = \underbrace{2\nabla^2 u}_{\sim} \right)$$



$$\partial_t u(\Psi_t(x)) = 1$$

$$u = t, \quad \partial_t R(\Psi_t) = \langle \nabla R, \frac{\nabla u}{|\nabla u|^2} \rangle$$

$$\stackrel{(2ma)}{=} -2Ric \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right)$$

$$\leq -2\varepsilon R$$

$\Rightarrow |R_m|$ decay exponentially at ∞

$$\rightarrow M^3 = \mathbb{R}^3$$

e.g.: Greene-Peterson-Zhang (gpp thm)

• Cheeger (Blow-down = $C(x)$, $X \cong S^{n-1}/M$)
and $T = \{e, z\}$.

Q: ① why non-collapsing??

② why smoothable??

③ why smoothable is rigid??

Outline: Assume $M = \text{non-cpt}$. $Ric_0 \geq \varepsilon R_0 g_0$

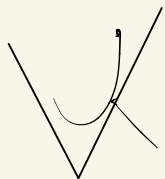
$$(|R_m - \varepsilon R|) \in C_{p,1} \quad (n \geq 4)$$

Step 1:

• L-Topping: then $\exists g(t)$ RF on $M \times [0, \infty)$

$$\text{s.t. } \begin{cases} g(\infty) = g_0, & Ric \geq \varepsilon R. \end{cases}$$

$$|R_m| \leq \alpha t^{-1} \quad \alpha(\varepsilon) > 0$$



expect = ERS

$$(M, g) \text{ GERS w/ } |R_m| \leq C.$$

$$\hat{g}(t), \quad |\hat{R}_m| \leq \frac{C}{t}$$



Step 2 (old approach)

Loth: If $g_0 \neq \text{flat}$, then $Ric > 0$ at $x_0 \in M$

$w \mid L. \text{ topping} \Rightarrow Ric > 0 \text{ on } M \text{ (WLOG)}$

$\Downarrow sy$

OR appeal to Liu $\Rightarrow M \cong \mathbb{R}^3$

then existence of RF smoothing $g_t, w \mid 0 < Ric_t < \frac{\alpha}{t}$

$\Rightarrow AVR_t \equiv AVR_0 > 0$ (Cheeger-Gromov-Fetaya)

Step 3 (assuming 1+2)

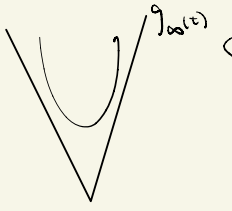
Appeal to CC, $C(x) = \varphi GH \text{ limit of } (M, R_i^2 g_0, x_0) \text{ as } R_i \rightarrow +\infty$

($g(t)$ = original RF)

Consider $g_{(t)} = R_i^2 g(R_i^2 t) \text{ w } \left\{ \begin{array}{l} 0 \leq Ric(g_{(t)}) \leq \frac{\alpha}{t} \\ AVR_t \equiv v_0 > 0 \end{array} \right.$

$\downarrow v = +\infty$ in (Cheeger-Gromov smooth optness)

$g_\infty(t) = RF \text{ on } M_\infty \times (0, +\infty)$

 $\left\{ \begin{array}{l} Ric_\infty \geq \Sigma R_\infty \geq 0 \\ |Rm_\infty| \leq \alpha t^{-1} \\ AVR_\infty = AVR_0 > 0 \end{array} \right.$

① $dg_{\infty(t)} \leq dg_{\infty(s)} \quad \forall t > s > 0 \text{ (Ric} \geq 0)$

② $dg_{\infty(t)} + C\sqrt{t} \uparrow$ (from $R_\infty \leq \alpha t^{-1}$)

$\Downarrow$

(by Hamilton, using 2nd variation)

$dg_{\infty(t)} \xrightarrow{GH} C(x) \text{ as } t \rightarrow 0^+$

Thm (Chan-L.-Pecceley, c.f. Deruelle-Schulze-Simon)

In this case, $g_{\alpha(t)}$ must be GERS
(DSS: is almost)

Rmk: Related to existence of Soliton

- 1) Bryant: $X = S^{n-1}$
- 2) AD: $X = \text{some smooth object (ODE)}$
- 3) Baulier-Chan: $X^2 = \text{smooth w} \mid \text{Scal}(X) \geq n(n-1)$
- 4) Schulze-Simon: $X = \text{smoothable w} \mid \text{Ric} \geq n(n-1)$.

Idea:

metric cone: $(C(X), d, X + \text{cp.})$ satisfies

$$\nabla^2 \left(\frac{1}{2} d^2 \right) \stackrel{\approx}{=} g_{C(X)} \quad (\text{not well-defined})$$

But CC: If $(M, g, X) \approx C(X)$ in GH, under
then $\star$ holds (almost) in L^2_{loc}

lemma (cf. Cheeger-Colding, Cheeger-Jiang-Naber)

$$\exists \varepsilon_i, \delta_i \rightarrow 0, R_i \rightarrow +\infty, U_{i,0} \in C^\infty(B_{\delta_i}(x_0, R_i))$$

$$\text{s.t. } \begin{cases} \Delta_{t_i} U_i = \frac{n}{2} \\ \sup_{B_{\delta_i}(R_i)} \left| U_i - \frac{1}{4} d_{t_i}^2(x_0, \cdot) \right|^2 \leq \varepsilon_i \\ \textcircled{\star} \int_{B_{\delta_i}(x_0, R_i)} \left| \nabla^2 U_i - \frac{1}{2} g_{(t_i)} \right|^2 \leq \varepsilon_i \\ |\nabla U_i| \leq C(d_{g_{(t_i)}} + 1) \end{cases}$$

$$\text{Ric}_t \geq 0$$

$$0 \leq |R_m| \leq \alpha t$$

$$\text{GH limit} = C(X)$$

(Reverse Engineering) smooth $\star$ Some cutoff

$$U_i(t) = \int_M K(x, t; y, t_i) (\varphi_i, U_{i,0})(y) dy_{t_i}$$

$$\text{i.e. } \begin{cases} (\partial_t - \Delta_g) U_i = 0 & \text{on } M \times [t_i, \infty) \\ U_i(t_i) = \varphi_i U_{i,0} \end{cases}$$

Main goal

$$U_i(t) \longrightarrow U_\infty(t)$$

$$\begin{cases} (\partial_t - \Delta_{g_\infty}) U_\infty = 0 & \text{on } (0, +\infty) \\ U_\infty \stackrel{C^0}{=} \frac{1}{4} d_0^2(X_0, 0) (= C(x)) \end{cases}$$

Based on Ric ≥ 0

where

$$\star \quad 2t \text{ Ric}_\infty + g_\infty = 2 \nabla_{g_\infty}^2 U_\infty$$

Based on studying Lichnerowicz heat operator

lemma (evolution of $S := 2t \text{ Ric} + g - 2 \nabla^2 U$)

(direct computation)

$$(\partial_t - \Delta) S_{ij} = \underbrace{2 \text{ Ric}_{ik} g_{kj} S_{ik}}_{\text{careful treatment}} - \underbrace{(R_i S_{kj} + R_j S_{ik})}_{\text{O.N.}}$$

Uhlenbeck task, consider $\{E_i(t)\}_{i=1}^n$ s.t. $\partial_t E_i = R_i^k E_k$, $\nabla E_i = 0$

$$① \partial_t g(E_i, E_j) = 0 \quad ② \square (S(E_i, E_j))$$

"Smooth case" Q: $S_i(0) \stackrel{L^2}{\approx} 0$ as $i \rightarrow \infty \Rightarrow S_\infty(t) \equiv 0$.

easy if curvature is good!

$$\square |S|^2 \leq \underbrace{C_n |R_n|}_{\frac{C}{t}} |S|^2$$

$$\Rightarrow |S(t)| \leq e^{Ct} |S|_{0,\infty}^2 \quad \times$$

Lemma: Under $\text{Ric} \geq 0$ (in 3D)

$$\left(\begin{array}{l} 2\text{-non neg } K \text{ (in general)} \\ (R_{1212} + R_{1331} \geq 0) \end{array} \right)$$

then $(S \otimes g)_\pm = \varphi$ satisfies

$$\boxed{(\partial_t - \Delta - R) \varphi \leq 0}$$

$$\left(\begin{array}{l} S_{ij} = \lambda_i \delta_{ij} \\ \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \end{array} \right)$$

$$\left\{ \begin{array}{l} (S \otimes g)_+ = (\lambda_1 + \lambda_2)_+ \\ (S \otimes g)_- = (\lambda_n + \lambda_{n-1})_- \end{array} \right.$$

* in RF, $\partial_t - \Delta$

$$\boxed{\partial_t - \Delta - R}$$

$$\partial_t \left(\int_M \varphi d\mu_t \right) \leq \int_M \overset{\text{Stoke}}{\cancel{\Delta \varphi}} + \overset{\circ}{\cancel{R \varphi}} - \overset{\circ}{\cancel{R \varphi}} d\mu$$

$$(d\mu_t)' = \frac{1}{2} \text{tr}_g (\partial_t g) d\mu = -R d\mu$$

that is why " $\partial_t - \Delta - R$ " is important !!

$$\begin{aligned} \text{Proof: } \square(S_{11} + S_{22}) &= 2R_{1kk1} \lambda_k + 2R_{2kk2} \lambda_k \\ &\leq 2 \sum_{k=3}^n (R_{1kk1} + R_{2kk2}) \left(\frac{\lambda_1 + \lambda_2}{2} \right) + 2R_{1221} (\lambda_1 + \lambda_2) \\ &\leq R (S_{11} + S_{22}) \end{aligned}$$

By the lemma:

$$\varphi(x, t) \leq \int_M \underbrace{G(x, t; y, t_i)}_{\substack{\text{Li-Yau estimate based} \\ \text{on } \int_M G \equiv \text{const}}} \underbrace{\varphi(y, t_i)}_{\substack{\circ \\ \int \cdot \text{ in } L^2}} dy_{t_i}$$

(Li-Yau estimate based on $\int_M G \equiv \text{const}$) $\xrightarrow{+ \text{cutoff}}$ cutoff error

of $U_{i,0}$
by growth rate

Recall: proved a special case of Hamilton conj.

Thm (Deruelle-Schulze-Simon, Chen-L-Peachey)

If (M^n, g_0) is complete non-cpt which admits RF = $g(t)$ smoothing

$$\text{from } g^{(0)} = g_0 \quad \text{s.t.} \quad \begin{cases} 0 \leq \varepsilon R \leq \text{Ric} \leq \varepsilon t^{-1} \\ \downarrow \text{(collapsing)} \\ \text{AVR}_t \equiv \text{AVR}_0 > 0 \end{cases}$$

$$\Rightarrow (M, g_0) \cong \mathbb{R}^3$$

Rmk: Now difficulties is to construct smoothing AND non-collapsing

Might consider SAME question in Kähler case. (conformally flat case)

(M^n, g) Kähler complete

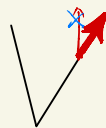
Def: $BK \geq \varepsilon R$

$$R(x, \bar{x}, Y, \bar{Y}) \geq \varepsilon R(|x|^2 |Y|^2 + |x \cdot Y|^2) \quad \forall x, Y \in TM$$

Q: $BK \geq \varepsilon R \Rightarrow \text{flat OR cpt}??$

$\text{Ric} \geq \varepsilon R \Rightarrow \text{Ricci flat}$

Rmk: Ricci flow dynamic in complex is very different!



But B well-understood in Maximal vol. case
($\text{AVR} > 0$)

Thm (Liu)

Complete non-cpt Kähler w/ $\{BK \geq \varepsilon R > 0, \text{AVR} > 0\}$ is $\mathbb{R}^{2n} = \mathbb{C}^n$

Pf Liu proved under $\text{AVR} > 0$, $BK \geq 0$, $\exists C > 0$ s.t. $\int_M \text{Ric}^n \leq C$

$$(\sum \lambda_i)^n \leq \lambda_1 \cdots \lambda_n \quad \text{bisectional.} \quad \int_M \text{Ric}^n \leq C \quad \parallel \quad \det \text{Ric} \quad (\text{Moph Z})$$

Under pinching, $\Rightarrow \int_M R^n \leq C$

Chaege $\Rightarrow$ any metric cone $C(x) = \text{flat outside tips (?)}$

Lin. L-Tam $C(x) = c p^k$ $\uparrow$ based on RF

$mfd = X \cong S^{2n-1}$
 $\xrightarrow{\text{aff}} X \cong S^{2n-1}$

□

Wish: discuss existence of RF smoothing

Setting $(M, g_0) = \text{completed mfd.}$

$$RF \begin{cases} \partial_t g = -2 Ric \\ g|_0 = g_0 \end{cases} = \text{weakly problem}$$

$\downarrow$
No direct existence even for cpt mfd.

★: unlike Kähler case.

$\therefore$ holc coordinate is good gauge.

(Monge-Ampère II)

Now usually appeal to DeTurck (fixed gauge)

Consider gauged version PDF

⊗ $\begin{cases} \partial_t \hat{g} = -2 Ric + \mathcal{L}_X \hat{g} \\ \hat{g}|_0 = g_0 \\ X^k = \hat{g}^{ij} (\hat{\Gamma}_{ij}^k - \tilde{\Gamma}_{ij}^k) \end{cases}$

$\tilde{g} = \text{some Ref metric}$

⊗ is strictly parabolic system

"S+d" PDE: In at case, it admits short-time solv on $M \times [0, T]$
 ?

In local coord.

$$\partial_t \hat{g}_{ij} = \hat{g}^{pq} \tilde{\nabla}_p \tilde{\nabla}_q \hat{g}_{ij} + \hat{g} \times \hat{g}^{-1} \times \tilde{g}^{-1} \times \tilde{R}_m + \hat{g}^{-1} \times \hat{g}^{-1} \times \tilde{\nabla} \hat{g} \times \tilde{\nabla} \hat{g}$$

unbounded, curvature.

Thm (Shi) If $g_0 =$ bdd curv. complete metric, then ^{RF} $\star$ admits short-time sol. w/ $\tilde{g} = g_0$. Moreover, $|R_m|/\hat{g}(t) \leq \tilde{C}(g_0)$

$\Downarrow$
Existence of RF. consider $\tilde{g}(t)$

$$\begin{cases} \partial_t \Psi_t = -X(\Psi_t(t), t) \\ \Psi_0 = Id \end{cases}$$

then $g(t) = \Psi_t^* \hat{g}(t) = \text{RF}$ from g_0 .

Important works about RDF $\Rightarrow$ many geometric applications related scalar curvature.

Thm (Simon, Koch-Lamm)

If $\tilde{g} =$ bdd geometry (i.e. $|\tilde{\nabla}^k \tilde{R}_m| \leq C_k, \forall k$).

$\exists \varepsilon_n > 0$ s.t. and $\|g_0 - \tilde{g}\|_{L^\infty} \leq \varepsilon_n$ (in L^∞ sense)

then $\exists!$ sol $g(t)$ to RDF remains L^∞ closed to $\tilde{g}$
 $\uparrow$
 unif. $C^\infty, \forall t > 0$

But PDE fail w/o bdd curvature background.

Progress of existence of RF.

$n=2$: Completely understood (Greim-Topping, Yin-Topping)

$n=3$: unclear even if $\text{Ric} \geq 0$.

(Lai: Yes for incomplete flow)

If RF exists w/ $\text{Ric}_t \geq 0$, Hamilton: $\text{Ker}(\text{Ric})$ is invariant under $\parallel$ transport. $\Rightarrow$ Liu

Special case. $n \geq 3$

Cabezas-Riva, Wilking: WPIC_2 (no geometric application?)

Well-understood in noncollapsing case

Bauer - Cabezas-Riva = Wilking, Lai, Simon-Topping, Li-Tam, Hochard

① almost PIC_1 , $V(x_0, 1) \geq V_0$, $\forall x$

② Kalhner, $BK \geq -1$

$\Rightarrow \exists g(t)$, $t \in [0, T]$, $-1 \leq \text{Ric} \leq \frac{C}{t}$, $\inf g \geq \sqrt{t}$.

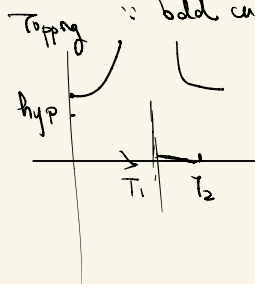
Back to pinching problem

Complete, noncpt w/ $\text{Ric} \geq \varepsilon R_0 \geq 0$, 3D

L-Topping: $\exists g(t)$, w/ $\varepsilon R \leq \text{Ric} \leq at^{-1}$

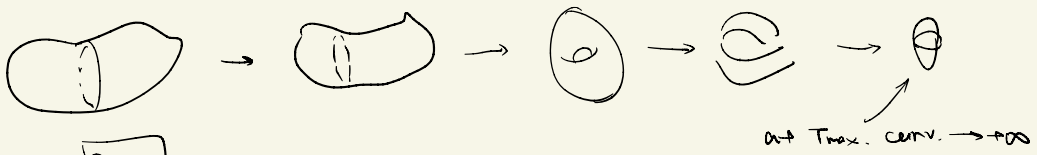
Only discuss bdd curvature. (not quantitative)

$\therefore$ bdd curv. $\therefore$ let $g(t)$ be RF constructed by Shi w/ bdd curv.



let $T_{\max}$ be maximal existence time and bdd curv.

Idea back to Hamilton



$$\boxed{R > 0}$$

lemma $Ric > 0$, $Ric \geq \varepsilon R$ is preserved under RF. But $\boxed{Ric \rightarrow 0}$

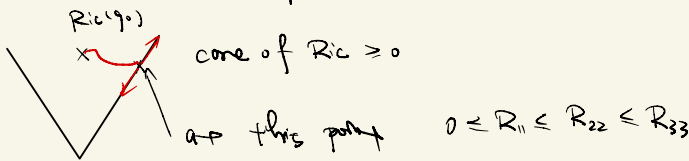
Pf: Recall evo. eqn. of Ric (Modules vector bundle trick).

$$\square Ric_j = 2 Ric_{ikj} Ric_k \quad (\text{might pretend at } g_{ij} = 0)$$

$$\square R = 2 \langle Ric, Ric \rangle \geq 0.$$

MP (need justification) $\Rightarrow R > 0$ is preserved.

For tensor (developed by Hamilton)



And $\square R_1 \geq 0$ at this pt. $\Rightarrow Ric \geq 0$ is preserved.

$$A := Ric - \varepsilon R g \geq 0$$

evolution eqn of A_{ii} , $\square A_{ii} \geq 0$ ($\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ eigenvalue of Ric)

$$\square A_{ii} = 2 \sum_{k=1}^3 (Ric_{ik} Ric_k) - 2 \varepsilon |Ric|^2$$

$$3D: \boxed{Ric_{1221} + Ric_{3311} = Ric_{11}}$$

$$= (\lambda_1 + \lambda_2 - \lambda_3) \lambda_2 + (\lambda_1 + \lambda_3 - \lambda_2) \lambda_3 - 2 \varepsilon (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$$

$Ric \geq 0$ (?) $\varepsilon = 0$, $\lambda_1 = 0$ (because in this point, ~~at this point~~)

$$= (\lambda_2 - \lambda_3)^2 \geq 0.$$

$$\varepsilon > 0, Ric \geq \varepsilon R$$

$$\lambda_1 = \varepsilon (\lambda_1 + \lambda_2 + \lambda_3)$$

$$\square A_n = \frac{\varepsilon}{1-\varepsilon} (\lambda_2 + \lambda_3)^2 + (\lambda_2 - \lambda_3)^2 - 2\varepsilon (\lambda_2^2 + \lambda_3^2 + \underbrace{\left(\frac{\varepsilon}{1-\varepsilon}\right)^2 (\lambda_2 + \lambda_3)^2}_{\downarrow O(\varepsilon^2)}) \geq 0.$$

□

Lemma (Curvature becomes homogeneous at high curvature pt.)

If $\text{Ric} \geq \varepsilon R_k > 0$, then $\exists C, \sigma > 0$ s.t.

↓
boom up at this pt.?

$$|\text{Ric} - \frac{1}{3} R_k|^2 \leq \frac{C}{t^{2-\sigma}} R^\sigma.$$

↑ (MP)

Pf (Skip) consider $f = R^{-\sigma} (|\text{Ric}|^2 - \frac{1}{3} R^2)$, then

$$\square f \leq \frac{\sigma-1}{2-\sigma} \frac{|\nabla f|^2}{f} - C f^{\frac{3-\sigma}{2-\sigma}}$$

Prop: $\exists \alpha > 0$, s.t. $g(t)$ satisfies $|R_m| \leq \alpha t^{-1}$ on $M \times [0, T_{\max})$

(In particular, $T_{\max} = +\infty$).

Pf: (Baby pt. picking).

Suppose not, $\exists (M_k^3, g_k(t), x_k)$

$$\alpha_k \rightarrow +\infty, t \in [0, t_k)$$

$$\text{s.t. } ① |R_m(x_k, t_k)| \approx \sup_{M_k} |R_m(t_k)|$$

$$\parallel \alpha_k t_k^{-1}$$

$$② |R_m| \leq \alpha_k t^{-1} \text{ on } M_k \times (0, t_k)$$

$$③ \text{Ric}_k \geq \varepsilon R_k$$

$$④ |\text{Ric}_k - \frac{1}{3} R_k|^2 \leq L \cdot t^{\sigma-2} R_k^\sigma$$

$$\text{Let } \tilde{g}_k(t) := \lambda_k(g_k(t_k + t\lambda_k^{-1})).$$

$$t \in (\lambda_k^{-1} t_k, 0] \text{ at } \alpha_k$$

$$② |\tilde{R}_m| \leq 2, \leadsto |\tilde{\nabla}^k \tilde{R}_m| \leq C_k$$

$$\textcircled{1} |\widetilde{R}_m(x_k, 0)| = 1$$

$$\textcircled{4} |\widetilde{\text{Ric}} - \frac{1}{3} \widetilde{R}|^2 \leq L \alpha_k^{\sigma-2} \rightarrow 0.$$

If $\widetilde{g}_k$ is non-collapsing.

$\rightarrow \widetilde{g}_\infty(t)$ s.t. complete RF s.t.

① non-flat ② bdd curv.
 $|R_m| = 1$

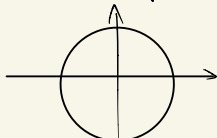
$$\textcircled{3} \text{Ric} = \frac{1}{3} R g \text{ at } t=0.$$

$\Downarrow$ Bianchi

$$R \equiv \text{const}(n) > 0$$

$$\text{Ric} \equiv \text{const} > 0 \Rightarrow M_\infty = \text{cpt} \Rightarrow M_i = \text{cpt}.$$

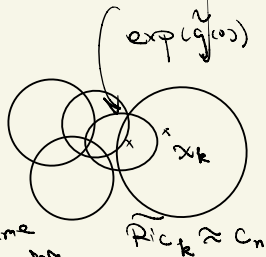
Trick in collapsed case:



$$\exp_{\widetilde{g}_k(0)}^* \widetilde{g}_k(t)$$



$$\text{Lemma: } \widehat{g}_\infty(t).$$



$$\widehat{\text{Ric}}_\infty \equiv C_n > 0.$$

Same
Evolution
constants.

$$\Rightarrow \forall r > 0, \exists N \text{ s.t. if } k > 0, \text{Ric}(\widetilde{g}_k(0)) > \frac{1}{2} C_n \text{ on } \widetilde{B}_{\widetilde{g}_k(0)}(x_k, r).$$

From pt-picking: $\exists \alpha, T > 0$, s.t. if $\text{Ric} \geq \varepsilon R > 0$, then

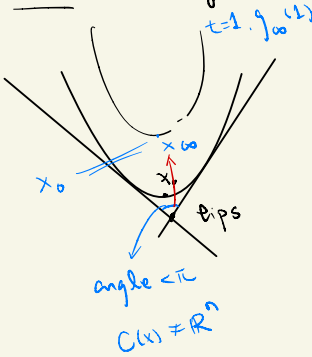
$$|R_m| \leq \alpha t^{-1} \text{ on } [0, T_{\max} \wedge T]$$

□

Thm (DLSST) $g_\infty = g_0$

Suppose $g(t), t \in [0, +\infty)$ in RF on M^n s.t. $0 \leq Ric_t, |Rm| \leq at^{-1}$ and $\exists x_0 \in M$ s.t. $Ric_0(x_0) > 0$, then $\lim_{t \rightarrow \infty} tR(x_0, t) > 0 = R_{g_\infty^{(1)}}(x_0)$

Model: this $g(t)$ is modeling smoothing of target cone at ∞ , $G(x)$



$$G(x) = R_i^{-2} g, R_i \rightarrow +\infty$$

info of $G(x)$
info of M

$$\text{Consider } R_i^{-2} g(R_i t) := g_i(t)$$

$$R_i^2 \quad w \mid \begin{array}{l} 0 \leq Ric_i \\ |Rm| \leq at^{-1} \end{array}$$

Remark: Thm is known if $\underline{AVR}_0 > 0$ (Wang(?))
about in our setting
if $Rm_t \geq 0$ (Li-Yau-Hamilton)

Hamilton: Harnack ineq. for RF

$\Downarrow$
 $tR(x, t)$ is monotone under $Rm \geq 0$.

AND: equality holds iff $g(t) = ERS$.

PROP (Wang)

Suppose (M^n, g_0) is complete, s.t. $Ric_0 > 0$.

$$\bullet AVR > 1 - \varepsilon_n$$

$$\bullet AVR < 1 - \sigma$$

$$\text{then } \lim_{t \rightarrow \infty} tR > \eta(0)$$

$$Ric \geq \varepsilon R \xrightarrow{n \geq 4} Ric = 0$$

$$\downarrow$$

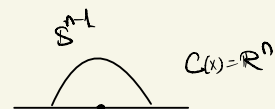
$$Rm - 2R1 \in C_{pic_1} \rightarrow Rm = 0$$

Pf (Sketch) Issue: collapsing!!

No metric cone picture.

$\downarrow$ thm

Let $R_i \rightarrow +\infty$ s.t. $R_i R(x_0, R_i^2) \rightarrow \lambda = 0$
 $\parallel$
 t_i



$$g_i(t) = R_i^{-2} g(R_i^2 t)$$

$$Ric_i \geq 0, |Rm| \leq \alpha t^{-1}$$

From assumption (WLOG) $\exists \eta > 0, Ric(g_{i(0)}) \geq \eta R_i^2$ on $B_{g_{i(0)}}(x_0, R_i^{-1})$
 "Scaling invariant"

Can't take $i \rightarrow +\infty$ globally!!

$\therefore$ curvature bounded

$$\therefore \text{lift to } B_{\mathbb{R}^n}(\frac{\pi}{\sqrt{\alpha}}) \xrightarrow{\phi_i} (M, g_{i(t)}, x_0)$$

$$h_i(t) := \phi_i^* g_{i(t)} \quad \cdot \text{RF (locally)}$$

$$\text{on } B(\frac{\pi}{\sqrt{\alpha}}) \times [0, +\infty)$$

$$\cdot \text{unif. } C^\infty \text{ on } \Omega \times [a, b] \subset B(\frac{\pi}{\sqrt{\alpha}}) \times \mathbb{R}_{>0}$$

$$\textcircled{1} h_i(t) \sim g_{\text{Euc.}}$$

$$\textcircled{2} Ric(h_{i(0)}) \geq \eta R_i^2 \text{ on } B_{h_{i(0)}}(x_0, R_i^{-1}).$$

In general, $0 \leq Ric_t, |Rm| \leq \frac{\alpha(t)}{t}$

Under $0 \leq Ric_t \leq \frac{\alpha}{t}$, the distance geom. of RF is very nice!!

$$\Rightarrow d_0 - C\sqrt{t} \leq d_t \leq d_0$$

$$\Rightarrow d_t \sim d_0 \text{ locally}$$

Rug
TAMS.

$$\Rightarrow C d_0^\beta \leq d_t \leq d_0, \beta > 1$$

$$h_i(t) \xrightarrow{C^\infty} h_\infty(t) \text{ on } B(\frac{\pi}{\sqrt{\alpha}}) \times \mathbb{R}_+$$

$$\textcircled{2} \Rightarrow \exists \delta > 0 \text{ s.t. } V_{h_{i(0)}}(0^n, R_i^{-1}) \leq (1-\delta) \omega_n (R_i^{-1})^n$$

(from Volume comp.)

$$\Rightarrow V_{h_{i,j}}(0^n, r) \leq (1-\delta) w_n r^n, \quad \forall r > R_i^{-1} \rightarrow 0^+$$

↑
no limit

$$\therefore \partial_t d\mu_t = -R_t d\mu_t \leq 0$$

$$\therefore V_{h_{i,j}}(0^n, r) \leq (1-\delta) w_n (r + C\sqrt{t})^n \quad \forall r > 0 \quad (*)$$

$$i \rightarrow \infty$$

no contradiction in general

But we assume $R(h_\infty(0^n, 1)) = 1$

$$\text{SMP}(\square R = 2|\text{Ric}|^2)$$

$$\parallel \lambda = \lim_{t \rightarrow \infty} t R(x_0, t)$$

$$\Rightarrow \text{Ric}_t \equiv 0 \Rightarrow h_{\infty(t)} = h_{\infty(1)}$$

$$\therefore (*) \text{ became } V_{h_{\infty(1)}}(0^n, r) \leq (1-\delta) w_n (r + C\sqrt{t})^n$$

$$\text{Let } t \rightarrow 0^+, r \rightarrow 0^+.$$

$$\therefore V_{h_{\infty(1)}}(r) \leq (1-\delta) w_n r^n$$

↑
smooth metric

→ ← ✱

$$\therefore \boxed{\lim_{t \rightarrow \infty} t R(x_0, t) > 0}$$

Recall

$$\|R_m\|_\infty < +\infty \quad (\text{removable})$$

$$\textcircled{1} (M^3, g_0) \quad \omega \mid \begin{array}{l} \Delta \varphi > 0 \\ \text{Ric} \geq \varepsilon R \end{array} \Rightarrow (M^3, g_0) \cong \mathbb{R}^3$$

$$\textcircled{2} (M^3, g_0) \quad \omega \mid \text{Ric}_0 \geq \varepsilon R_0$$

$$\|R_m\|_\infty \Rightarrow \exists g(t) \quad \text{Ric} \geq \varepsilon R$$

(removable)

$$\omega \mid |R_m| \leq \alpha t^{-1}$$

Pf of pinching conj.

might assume $Ric_0(x_0) > 0$.

Prop. $\Rightarrow \lim_{t \rightarrow \infty} tR(x_0, t) = \lambda > 0$.

tracing the proof of Prop.

$h_\infty(t)$ RF defined on $B_{\mathbb{R}^n}(\frac{\pi}{\sqrt{\lambda}}) \times \mathbb{R}_+$

$$\begin{cases} 0 \leq Ric_t \leq \alpha t^{-1} \\ Ric_t \geq \varepsilon R_t \\ h_\infty(t) \sim g_{Euc.} \end{cases}$$

From distance geom. comparison ($d_0 \leq d_t \leq d_0$)

$\Rightarrow \inf_{d_0(t)} (0^*) \geq C\sqrt{t}$ from $V_{h_\infty}(r_0) \geq V_0 \quad \exists r_0 > 0$.

And $tR(h_\infty(t)) = \lambda > 0$.

($\because \lambda = \liminf_{t \rightarrow \infty}$)

look at tge flow $H(t) := \lim_{s \rightarrow 0^+} s^{-1} h_\infty(st)$

(tangent)

(= limit in Cheeger-Gromov thm).

At "t=1" $\begin{cases} AVR(H) > 0 \\ Ric(H) \geq \varepsilon R \\ |Rm| \leq C \\ R_H(x_\infty) \geq \lambda > 0. \end{cases}$

$\downarrow$
scalar curv. of H $\rightarrow \textcircled{1}$

Last ingredient

Most general existence.

Thm (L.-Topping) (M^3, g_0) complete non-cpt w/ $\text{Ric}_0 \geq \varepsilon R_0$.

admits RF $g(t) \quad \omega | g(0) = g_0$

$$0 \leq \mathcal{R}R_t \leq Ric_t \leq \alpha t^{-1} \quad \text{on } M \times [0, t_\infty)$$

Issue: Recall the $\|R_m\|_\infty$ setting

Existence of $g(t)$ ← short-time existence.

ω | curv. bdd.

$$|R_m| \leq \alpha t^{-1}$$

G_{MP}

9 P.P. picking

preserved $R/c \geq \varepsilon R$

$$\overset{MP}{\curvearrowright} |\overset{\circ}{Ric}|^2 \leq \frac{C}{t^{2-\sigma}} R^{\sigma}$$

MP by L.-Tam

Suppose $g(t) = RF$ on $B_0(x_0, 1) \times [0, T]$

$$\omega / |R_m| \leq \alpha t^{-1}$$

And if f is fun s.t. $(\partial_t - \Delta)f \leq Lf$, where $L, f \leq \alpha t^{-1}$

Smooth up to $t=0$, $f \geq 0$

only on m, α, n

$$f(0) = 0 \Rightarrow f(x_0, t) \leq t^m, \quad \forall t \in [0, T \wedge T_m)$$

Lemma 1: $\text{Ric}_g \geq \varepsilon R_0$, $|R_m| \leq \alpha t^{-1}$ on $B_0(x_0, 1)$

$$\Rightarrow Ric_t \geq \underbrace{\varepsilon R_t - 1}_{\text{error}} \text{ at } x_0, t \in [0, T \wedge T_1]$$

Lemma 2: $|R_m| \leq \alpha t^{-1}$, $Ric \geq \varepsilon R^{-1}$ on $B_\varepsilon(x_0, 1) \times [0, T]$

$$\Rightarrow |Ric|^2 \leq \frac{C}{t^{2-\sigma}} (R+4)^{\sigma} \quad \text{at } x_0, t \in [0, T \wedge T_2]$$

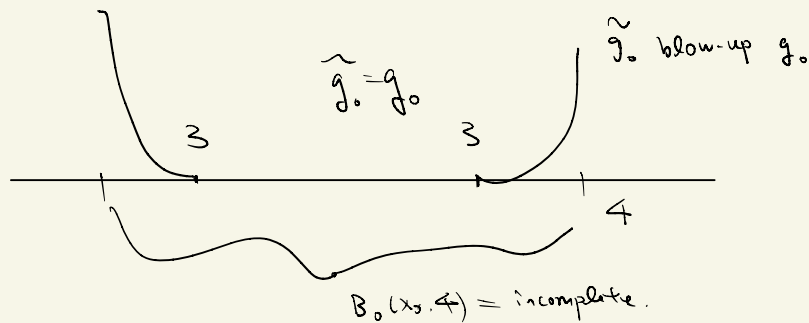
Lemma 3: $|\mathring{\text{Ric}}|^2 \leq \frac{C}{t^{2-\sigma}} (R_+ t)^{\sigma}$, $R_t \geq -1$ on $B_0(x_0, 1) \times [0, T]$

$$\Rightarrow |R_m| \leq \beta t^{-1} \text{ at } x_0, t \in [0, T \wedge T_3], \beta \text{ is independent of } \alpha.$$

Thm Under assumptions, $x_0 \in M$, $\exists g(t)$ on $B_0(x_0, 1) \times [0, \tilde{T}_2]$

$$\text{s.t. } \begin{cases} |R_m| \leq \alpha_2 t^{-1} \\ g_{(0)} = g_0 \\ Ric \geq \varepsilon R - 1. \end{cases}$$

Rescaling yield global soln. apply to $R_i^{-2} g_0$. $R_i \rightarrow +\infty$



$$\tilde{g}_0 := e^{2f} g_0 = \text{complete w/ bdd geom.}$$

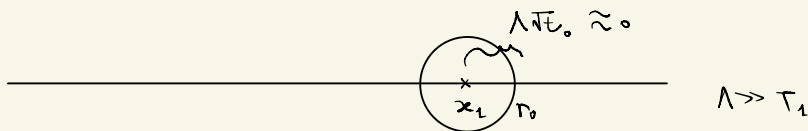
$$\Rightarrow \exists \text{ Shi's RF } g(t) \text{ on } B_{\tilde{g}_0}(x_0, 3) \times [0, t_0]$$

un-controlled.

Fix $\alpha \gg 1$ (to be determined)

might assume $|R_m| \leq \alpha t^{-1}$

$$r_0 = 3, t_0 = t_0, \text{ let } \Lambda \gg 1 \text{ (to be determined)}$$

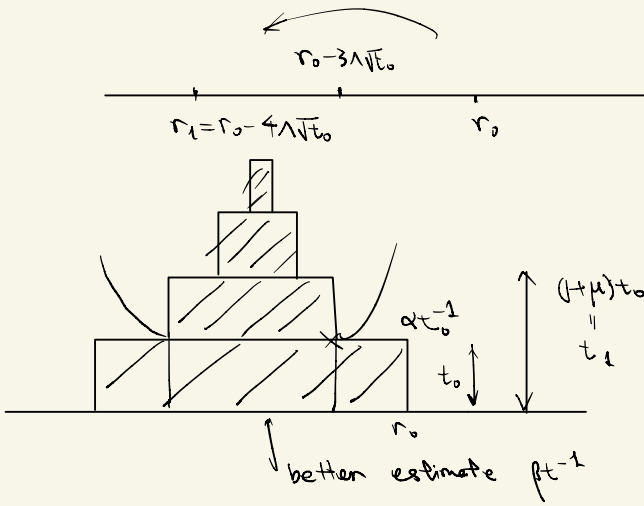


$$\Rightarrow Ric_t \geq \varepsilon R_t - (\Lambda \sqrt{t_0})^{-2} \text{ on } [0, t_0 \wedge T_1 (\Lambda \sqrt{t_0})^2]$$

$$\text{Hence } Ric \geq \varepsilon R - (\Lambda \sqrt{t_0})^{-2} \text{ on } B_0(r_0 - \Lambda \sqrt{t_0}) \times [0, t_0]$$

Lemma 2 $\Rightarrow |Ric|^2 \leq \frac{C}{t^{2-\alpha}} (R+4)^\alpha \text{ on } B_0(r_0 - 2\Lambda \sqrt{t_0}) \times [0, t_0] \quad \Lambda \gg T_2$

$$\Rightarrow |R_m| \leq \beta t^{-1} \leq \alpha t^{-1} \text{ on } B(r_0 - 3\Lambda \sqrt{t_0}) \times [0, t_0]$$



Inductively, $\exists RF=g(\alpha)$ on

$$B_0(x_0, r_k) \times [0, t_k]$$

$$|R_m| \leq \alpha t^{-1}$$

where $t_{k+1} = \underbrace{(1+\mu)t_k}_{>1}$

$$r_{k+1} = r_k - 4\sqrt{t_k}$$

look at $r_n \approx 1$.

$$\Rightarrow t_n \geq \tilde{T}(\varepsilon)$$

□