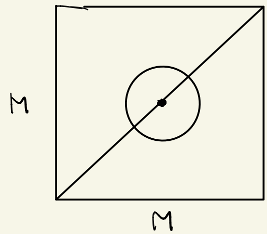
· Mazzeo-Melrose JFA 1987

Meromorphic Extension of Resolvent on Complete Spaces with Asymptotically Constant Negative Curvature.

Lecture 1. Construction of Parametrix.

① Closed comp. (M^n, g) Δ_g



$$\Delta_g \underset{\text{Green func.}}{G}(p,p') = \delta(p-p')$$

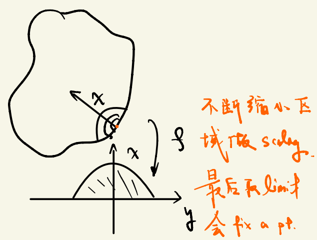
$$\Delta_g E(p,p') = \delta(p-p') + F(p,p'), \qquad F \in C^\infty(M)$$

key: $\mathbb{R}^n$, $\Delta_E G_E = \delta(p-p')$, $G = C_n |p-p'|^{2-n}$, $n \geq 3$

$h_{ij}(0,0) = \delta_{ij}$

· (X^{n+1}, g) Assume it's Asymptotically Hyperbolic, $\sec \to -1$.

x: bdy def, $g = \dfrac{d^2x + h_{ij}(x,y)\,dy^i dy^j}{x^2}$



Mode case: $\mathbb{H}^{n+1}$, $X = \{|z|<1\}$, $g_{\mathbb{H}} = \dfrac{4d^2z}{(1-|z|^2)^2}$

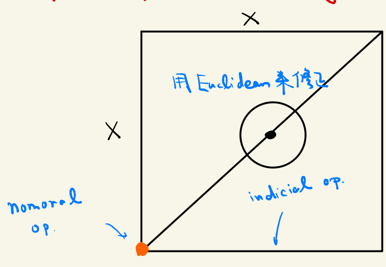
global. (无穷远紧化).

$$X = \mathbb{R}^+_x \times \mathbb{R}^n_y \qquad g_{\mathbb{H}} = \frac{d^2x + d^2y}{x^2}$$

Goal: $(\Delta_g - \lambda) E(p,p') = \delta(p-p') + F(p,p')$

内部很光滑，无穷远衰减

② Operators: if Laplacian: "$m=2$"



$$\Delta_{\mathbb{H}} = -(x\partial_x)^2 + n x \partial_x - \sum_{i=1}^n (x \partial_{y_i})^2$$

$$P = \sum_{i+|\alpha| \leq m} a_{i\alpha}(x,y)\,(x\partial_x)^i (x\partial_y)^\alpha$$

$$\downarrow$$

$$N(P) = \sum_{i+|\alpha|\le m} a_{i\alpha}(0,0)\,(x\partial_x)^i (x\partial_y)^\alpha$$

scaling 下不变

$P=(x,y)$, $P'=(x',y')$

分析 $x\to 0$, $x'\to 0$, $y-y'\to 0$

(Normal operator)

$$N(\Delta_g) = \Delta_{\mathbb{H}}$$

法向展开：$I(p) = \sum_{i\le m} a_{i,0}(0,y)(x\partial_x)^i$ (Indicial operator)

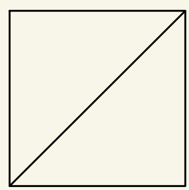
$$\lim_{x\to 0} x^{-\xi} P(x^{\xi} f(y)) = \sum_{i\le m} a_{i,0}(0,y)\,\xi^i f(y) = I(p)(\xi) f(y)$$

$$I(\Delta_g) = -(x\partial_x)^2 + nx\partial_x \;,\quad I(\Delta_g) = -\xi^2 + n\xi .$$

$$I(\Delta_g - \underset{\sim\sim}{s(n-s)}) = -(x\partial_x)^2 + nx\partial_x - s(n-s)$$

$(\Delta_g - \lambda)^{-1}$?

4D



③ Setting.

$$\Delta_{g(p)} G(p,p') = \delta(p-p')$$

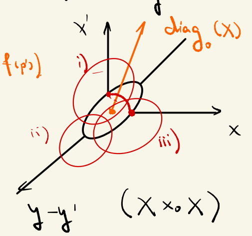
$$\Delta_{g(p)} \int_M G(p,p')\, f(p')\, dV_g(p') = f(p)$$

$$f(p) \sim f(p)\,|dV_g(p)|^{\frac12}$$

$$G(p,p') \sim G(p,p')\,|dV_g(p)\, dV_g(p')|^{\frac12}$$

$$f(p')G(p,p') \sim G(p,p')\, f(p')\, dV_g(p')\, |dV_g(p)|^{1/2}$$

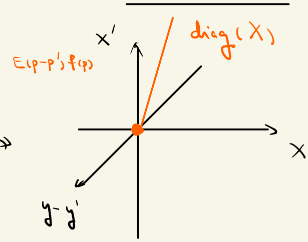
push forward $\downarrow \pi$



$(X\times_0 X)$

β blow up the pt.

p-fibration

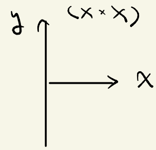


趋于0速度也可不同

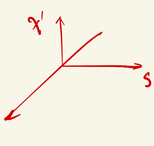
$$X\times X:\; dV_g(x,y)\, dV_g(x',y') = \frac{dx\,dy}{x\cdot x^n \quad x'(x')^n} = P$$

$$\left|\frac{dx}{x}\,\frac{dy}{xy}\right|^{\frac12}$$

$\downarrow \pi$ $(X\times X)$



i) $s = \dfrac{x}{x'}$, $z = \dfrac{y-y'}{x'}$,



$$\beta^* P = P_0 = \frac{ds}{s}\cdot\frac{dz}{s^n}\,\frac{dx'}{x'}\,\frac{dy'}{(x')^n}$$ ← coordinate

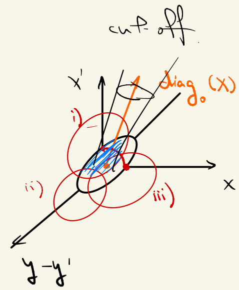
$s\partial_s \cdot s\partial_z$

iii) $r=|y-y'|$, $w=\frac{y-y'}{r}$, $\rho=\frac{x}{r}$, $\rho'=\frac{x'}{r}$

$$\beta^*\Gamma=\Gamma_0=\frac{d\rho}{\rho}\,\frac{d\rho'}{\rho'}\,\frac{dr}{r}\,\frac{dw\,dy}{\rho^n(\rho')^n r^n}$$

可用Schwarz kernel 表示算子

$$k(P)\in C^{-\infty}(X\times_0 X,\ \Gamma_0^{\frac{1}{2}})$$ (volume form)$^{1/2}$



$$\underbrace{k(x',y',s,z)\left|\frac{ds}{s}\,\frac{dz}{s^n}\,\frac{dx'}{x'}\,\frac{dy'}{(x')^n}\right|^{\frac12}}\ f(x',y')\underbrace{\left|\frac{dx'}{x'}\,\frac{dy'}{(x')^n}\right|^{\frac12}}$$

$k(\frac{x}{s}, y-\frac{x}{s}z, s, z)\,\frac{ds}{s}dz\left|\frac{dx}{x}\cdot\frac{dy}{x^n}\right|^{\frac12})$ 换 x', y' 为 x, y. $f(\frac{x}{s}, y-\frac{x}{s}z)$ 积 s, z 方向.

$$N(p)f:\ \int k(0,y,s,z)\, f(\tfrac{x}{s}, y-\tfrac{x}{s}z)\,\frac{ds}{s}dz\left|\frac{dx}{x}\cdot\frac{dy}{x^n}\right|^{\frac12}$$

$$N(Id):\ k=\delta(s-1)\,\delta(z)\,|\quad|^{\frac12}$$

z-s 平面即为新的 hyperbolic space.

Lem: $N(\Delta_g)=$ (s, z) $\vdash \Delta_{\mathbb{H}}$ $R=\sqrt{|x|^2+|x'|^2+|y-y'|^2}$

$R=0$ Front face.

Step 1: $\exists E_1$ s.t. $(\Delta_g - s(n-s))E_1 = Id - Q_1$

$Q_1\in \Psi_0^{-\infty}$ → order

在 不衰减.

解 Green func. 可解即要求 $\Delta_{\mathbb{H}}$ 可逆.

Step 2: $\underline{(\Delta_{\mathbb{H}} - s(n-s))}E_{2,k} = Q_{1,k}$, $E_2 = E_1 + \sum R^k E_{2,k}$.

$$\Rightarrow (\Delta_g - s(n-s))E_2 = Id - Q_2$$

$Q_2\in R^\infty \Psi^{-\infty,\boxed{s.s}}$

Step 3: $x\to 0$'s surface. 找渐近展开, 有无穷阶衰减

$$(\Delta_g - s(n-s))(x^\xi f(y)) = x^\xi(+\xi(n-s) - s(n-s))f(y) + O(x^{\xi+1})$$

Prop. $(\Delta_g - s(n-s))E = Id - Q$, $Q\in R^\infty \Psi_0^{-\infty,\infty,s}$

$$\left(\Delta_{\mathbb{H}} - s(n-s)\right) G_s(\omega,\omega') = \mathrm{Id}, \qquad G_s = g_s(d) \qquad \omega = (x, y)$$

$\mathrm{Re}(s) > n/2$ $$g_s'' + n\frac{\cosh d}{\sinh d} g_s' + s(n-s) g_s = 0, \qquad \cosh(d) = \frac{|\omega-\omega'|^2}{2xx'} + 1$$

$$G_s : \dot{C}^\infty(B^{n+1}) \longrightarrow \rho^s C^\infty(\overline{B}^{n+1})$$

迅速衰减

Prop: $n = $ even : G_s entire (自然延拓至整张复平面).

$n = $ odd ; G_s meromophic pole $0, -1, -2, \cdots$

$g = \dfrac{dx^2 + h(x)}{x^2}$ 如果展开只有 even, 会有类似表现

if not, 有本性奇点. (Guillarmou 博士毕业论文)

Reference: • Mazzeo 1988 JDG.

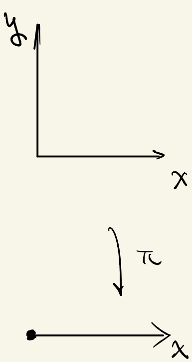
• Graham-Zworski 2003 Invent

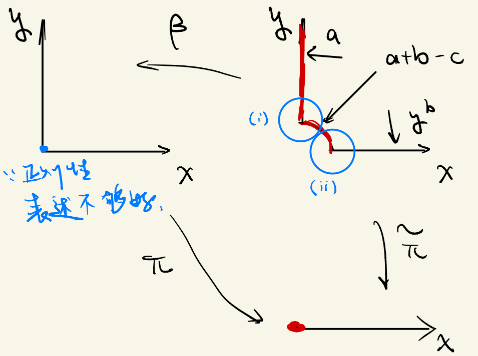
• Guillarmou 2005 Duke.

① $$\int_0^1 \frac{x^a y^b}{(x^2+y^2)^{c/2}} f(y) \frac{dy}{y} = g(x) \qquad f(0) = 1$$

$$\pi_*\left(\frac{x^a y^b f(y)}{(x^2+y^2)^{c/2}} \left|\frac{dy}{y}\frac{dx}{x}\right|^{\frac{1}{2}} \left|\frac{dy}{y}\right|^{\frac{1}{2}} \right)$$

$$= g(x)\left|\frac{dx}{x}\right|^{\frac{1}{2}}$$





∵正则性表述不够好.

Two surface ⟶ one point.

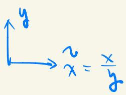
$$\Longrightarrow\ g(x) = x^a g_1(x) + x^{a+b-c} g_2(x)$$

$$g_1, g_2 \in C^\infty([0,1])$$

一个func.在normal方向的渐近展开.

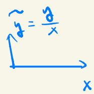
$$\Longleftrightarrow\ \prod_{i=0}^{\infty}\left(x\partial_x - a - i\right)\prod_{j=0}^{\infty}\left(x\partial_x - (a+b-c) - j\right) g(x) = O(x^\infty)$$

$b-c \notin \mathbb{Z}$, 否则会有 $\log(\cdot)$.

(i) 

$$\tilde{\pi}_*(y\partial_y) = x\partial_x$$

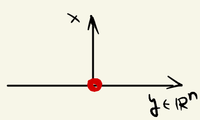
$$\tilde{\pi}_*(\tilde{x}\partial_{\tilde{x}}) = x\partial_x$$

(ii) 

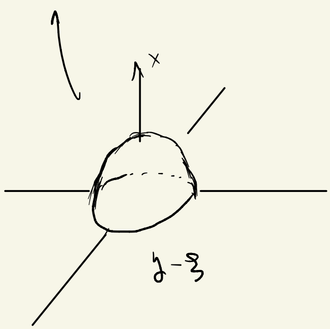
$$\tilde{\pi}_*(x\partial_x) = x\partial_x$$

例.
$$\int_{\mathbb{R}^n} \frac{x^{2\lambda} f(\xi)\, d\xi}{(|y-\xi|^2 + x^2)^{n/2+\lambda}} = U(x,y)$$

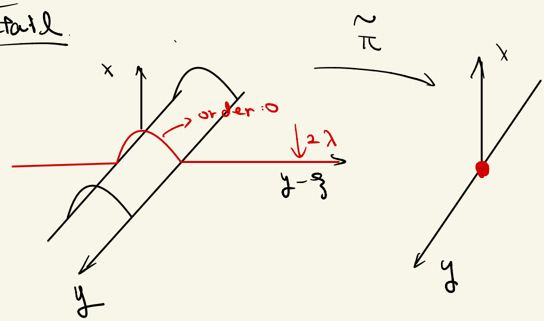
$$\left(\Delta_{\mathbb{H}} - \left(\tfrac{n^2}{4} - \lambda^2\right)\right) u = 0$$



$$u = x^{\frac{n}{2}-\lambda}\, U(x,y)$$



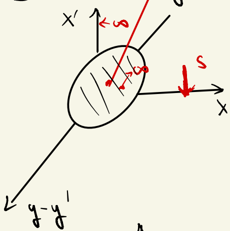
detail



$$R = |y-\xi| \qquad \pi^*\left(\frac{x^{2\lambda}\cdot R^n f}{(R^2+x^2)^{n/2+\lambda}}\,\frac{dR}{R}\,d\theta\right)$$

$$U(x,y) = x^0 \overset{f}{\overbrace{u_1(x,y)}} + x^{2\lambda} u_2(x,y) \longrightarrow \text{分数阶 Laplacian.}$$

② (X^{n+1}, g) A.H. $\cdot$ $P = \Delta_g - s(n-s)$ $\quad Re(s) > \frac{n}{2}$.



$$M_s = M_s(x,y;x',y') \in \Psi_0^{-2,s,s}$$

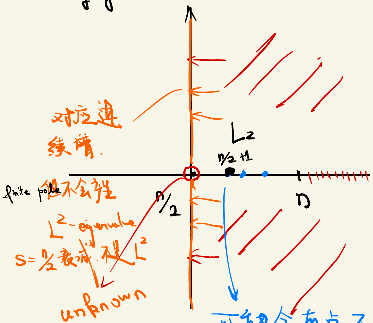
$$PM_s - Id = E_s \in R^{\infty}\Psi_0^{-\infty,+\infty,s}$$

- $E_s : x^{\alpha} H_0^{m}(X) \longrightarrow x^{\alpha+N} H_0^{m+N}$

$\forall \alpha \geq 0, \ \forall m \geq 0, \ \forall N$, Compact.

- $P: x^{\alpha} H_0^{m+2} \longrightarrow x^{\alpha} H_0^{m}$ Fredholm.

$\forall \alpha \in (\frac{n}{2}-s, s-\frac{n}{2})$



可能会有点不太好, 零点, 可提供一些 L^2-eigenvalue 信息

- $s \in (\frac{n}{2}, n)$ $\qquad M_s^* P^* - Id = M_s^* P - Id = E_s^* \ \cancel{\in R^{\infty}\Psi^{-\infty,s,+\infty}}$

$\Downarrow$

$\cancel{Pu = 0, \ u \in x^{-\alpha} H_0^{-m} \Rightarrow -u - E_s^* u \in x^{s} C^{\infty}(X)}$

$E_s^* . \ x^{\alpha} H_0^{-m} \longrightarrow x^{s} C^{\infty}(\bar{X})$

$u \in L^2$, $Pu = 0$, $-u = E_s^* u \in x^{s} C^{\infty}(\bar{X})$. (即其 L^2 kernel $\in x^{s} C^{\infty}(\bar{X})$).

$\Longrightarrow$ finite dimension. $\qquad$ ($\downarrow$ 落在 $x^{\alpha} H_0^{m+2}$ 中 $\to$ P 有 Fredholm)

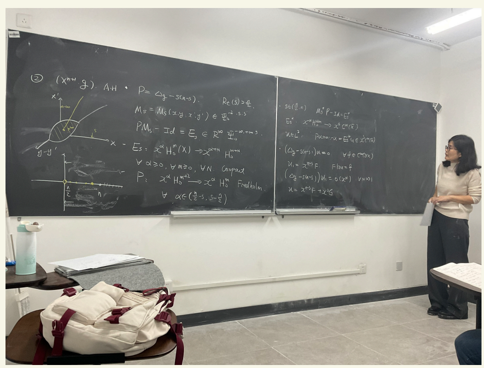
- $(\Delta_g - s(n-s))u = 0$, $\quad \forall f \in C^{\infty}(\partial X)$

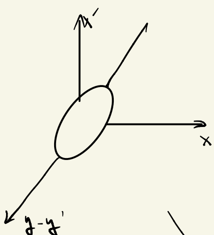
$$u_1 = x^{n-s} F, \quad F|_{\partial X} = f$$

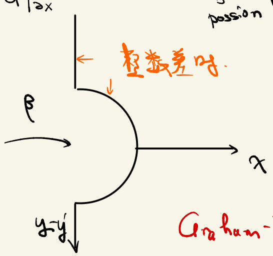
$$(\Delta_g - s(n-s))\, u_1 = O(x^N) \quad \forall N \gg 1.$$

$$u = x^{n-s} F + x^{s} G = \Phi_s(f)$$

$S(s)$: $F|_{\partial X} \longrightarrow G|_{\partial X}$ $\qquad$ possion kernel.

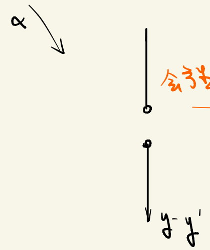






$$\Phi_s = \beta_*(x^{1-s} R_s)$$

Graham-Zworski $\quad R_s = (\Delta_g - s(n-s))^{-1}$



$S(s) = \alpha_*(x^{1-s}\, x^{-s}\, R_s)$

会产生 log. Joshi - SaBaneto.

为拟微分椭圆算子.

$\sigma_{2s-n}(S(s)) = (\Delta_h)^{s-\frac{n}{2}} + \text{L.O.T.}$

- $\mathrm{Re}(s) = \frac{n}{2}$, $s \neq \frac{n}{2}$, R_s analytic $\Longrightarrow$ $\boxed{S(s)\, S(n-s) = \mathrm{Id}.}$
- Guillarmous. R_s meromorphic Extension

  $\cong$ $S(s)$ meromorphic. Extension.

Ⓑ (X^{n+1}, g), CCE $\mathrm{Ric}_g = -ng$ $\quad x^2 g|_{\partial X} = h$

John Lee: $\underline{\overset{\text{Yamabe}}{Y}(\partial X, [h]) \geq 0}$ $\Longrightarrow$ $\mathrm{Spec}(\Delta_g) \geq \frac{n^2}{4}$

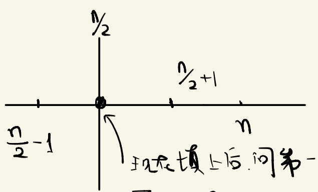
$S_1 = \frac{n}{2} + 1$, $\mathrm{Res}\big|_{s=s_1} = c(\Delta_h + C_h R_h)$

$\Downarrow Y(\ ) > 0$

Qing - Guillarmous: • $\forall s \in (\frac{n}{2}, \frac{n}{2}+1)$ $S(s)$ positive.

• $s = \frac{n}{2}$ "analytic" $\Rightarrow$ $S(\frac{n}{2}) = -\mathrm{Id}$

$S(s)\, S(n-s) = \mathrm{Id}$ $\quad\Longleftrightarrow\quad$ $(\Delta_g - \frac{n^2}{4})$ 没有 $x^{\frac{n}{2}} C^\infty(\overline{X})$ 解.



现在填上后，问第一次遇到pole是何时?

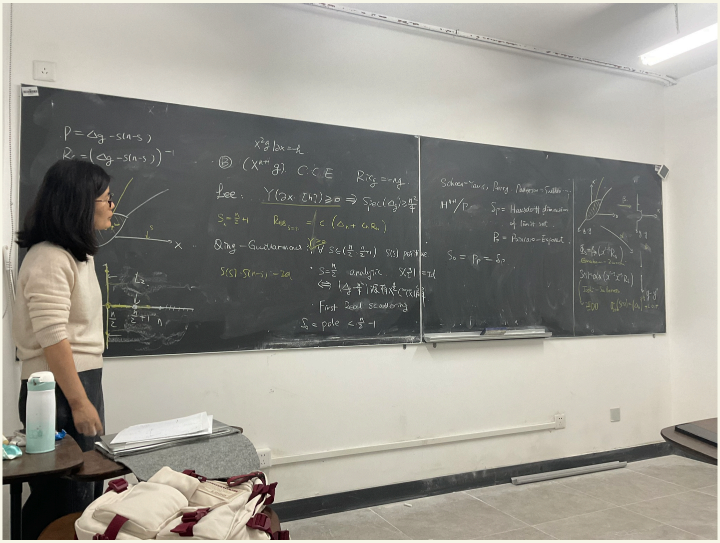
至少 $\frac{n}{2} - 1$ 之后. ($\because$ $\mathrm{Id} = S(s) S(n-s)$ + $s \in (\frac{n}{2}, \frac{n}{2}+1)$ $S(s)$ positive).

s_0 = First Real scattering pole $< \frac{n}{2} - 1$.

Schoen - Yau; Perry. Patterson - Smith....

$\underset{\sim\sim}{\mathbb{H}^{n+1}/\Gamma}$ $\qquad \delta_p$ = Hausdorff dimension of limit set.

$S_0 = P_\Gamma = \delta_p$ $\qquad P_p$ = Poincare - Exponent.



P = Δg − s(n−s)
R = (Δg − s(n−s))^−1
(X^n+1, g) C.C.E
Ric_g = −ng
Lee:
Spec(Δg) ≥ n²/4
Qing − Guillarmou
S(s) positive
analytic
First Real scattering
Schoen-Yau, Perry, Anderson-Sullivan
Hausdorff dimension of limit set
Poincaré-Exponent