

FUNCTIONAL ANALYSIS

LECTURE NOTES OF ZUOQIN WANG

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ABSTRACT. These lecture notes are compiled from the handwritten materials used by Professor Zuoqin Wang for his "Functional Analysis" course in 2015. The work consists of thirty lectures and includes a collection of exercises, supplementary notes with solutions, and answers to in-class quizzes. The original English content has been preserved in the main text. The TeX source files contain markers corresponding to the page numbers of the original manuscripts, and any corrections or clarifications made during the editing process are clearly indicated. The TeX version of these lecture notes was largely generated by ChatGPT-6.1 Sol. Please email me (yijiemiao0420@gmail.com or yijiemiao@seu.edu.cn) if you spot any typos or errors. Thank you!

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1. VECTOR SPACES, METRICS, AND NORMS

1.1. Course information (September 1, 2015). The lecturer is Zuoqin Wang. His office is Room 1611 in the Management and Research Building, and his email address is wangzuoq@ustc.edu.cn. Classes meet on Tuesdays, 9:45–11:20, and Fridays, 7:30–9:25, in Room 3A405. The teaching assistant is Yiran Cheng (story@mail.ustc.edu.cn). Question-answering sessions take place on Saturdays, 7:30–9:00 p.m., in Room 3A309.

The course website is <http://staff.ustc.edu.cn/~wangzuoq/Courses/15F-F>. The textbook is W. Rudin, *Functional Analysis*, second edition. Reference books are P. Lax, *Functional Analysis*, and M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*. Homework is assigned after class on the website, twice each week, and is collected on Tuesdays before class. Quizzes are given at random. There is one midterm and one final examination. The grade is computed as

$$15\% \text{ homework} + 15\% \text{ quizzes} + 30\% \text{ midterm} + 40\% \text{ final}.$$

The course covers the general theory of various types of spaces and linear maps between them, the theory of distributions, and spectral theory (mainly for operators on Hilbert spaces).

1.2. What is functional analysis? Functional analysis is also known as *modern analysis* or *soft analysis*. Classical analysis, including mathematical analysis, real analysis, and complex analysis, mainly concerns quantitative properties of individual functions: values, integrals, derivatives, and estimates. Functional analysis studies qualitative properties of spaces of functions. A function is treated as a point in an abstract space; for example, $L^2(\mathbb{R})$ is the space of square-integrable functions on $\mathbb{R}$.

Functional analysis may also be viewed as infinite-dimensional linear algebra. In finite-dimensional linear algebra, we study $\mathbb{R}^n$ and linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$ (or $\mathbb{C}^n \rightarrow \mathbb{C}^m$). One major result is the following.

Theorem (Finite-dimensional spectral theorem). *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map represented by a symmetric matrix. There is an orthogonal basis $v_1, \dots, v_n$ and scalars $\lambda_1, \dots, \lambda_n$ such that $Fv_i = \lambda_i v_i$. Consequently, if $v = \sum_{i=1}^n a_i v_i$, then*

$$Fv = \sum_{i=1}^n a_i \lambda_i v_i.$$

In linear functional analysis, we study infinite-dimensional vector spaces and linear maps between them. For example, $L^2(\mathbb{R})$ is a vector space: $f, g \in L^2(\mathbb{R})$ implies $\lambda f + \mu g \in L^2(\mathbb{R})$. Integration gives a linear map

$$I : L^1(\mathbb{R}) \longrightarrow \mathbb{R}, \quad f \longmapsto \int_{\mathbb{R}} f(x) \, dx$$

for real-valued functions. We will see how to extend the spectral theorem to infinite-dimensional settings.

Editorial note. The handwritten integration example appears to include absolute value bars around the integrand while calling the map linear. The linear map is integration of f ; the map $f \mapsto \int |f|$ is a norm and is not linear.

Functional analysis developed in the early twentieth century from problems in the calculus of variations, integral equations, and quantum physics. It was deeply influenced by the idea of axiomatization. It has extensive applications to PDE, probability, computational mathematics, applied mathematics, and mathematical physics. This course mainly concerns linear functional analysis.

1.3. Infinite-dimensional vector spaces.

Definition 1.1 (Vector space). A vector space V over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is a set with addition $V \times V \rightarrow V$ and scalar multiplication $\mathbb{F} \times V \rightarrow V$ satisfying

$$\begin{aligned} x + y &= y + x, & (x + y) + z &= x + (y + z), & x + 0 &= x, & x + (-x) &= 0, \\ \alpha(x + y) &= \alpha x + \alpha y, & (\alpha + \beta)x &= \alpha x + \beta x, \\ 1x &= x, & (\alpha\beta)x &= \alpha(\beta x). \end{aligned}$$

A set $\{u_1, \dots, u_n\}$ is a basis of V if every $x \in V$ has a unique expression $x = \alpha_1 u_1 + \dots + \alpha_n u_n$. In this case $\dim V = n$. A vector space with no finite basis is *infinite-dimensional*.

Example 1.1 (Polynomials). Let $\mathcal{P}$ be the space of all polynomials in one variable x . This is a vector space with no finite basis. Indeed, given finitely many polynomials $p_1, \dots, p_n$, choose N at least as large as all their degrees. A polynomial of degree $N + 1$ cannot be a linear combination of them. Thus no finite set spans $\mathcal{P}$.

Remark 1.1 (Hamel bases; cf. Rudin, p. 53). A Hamel basis of V is a linearly independent set such that every vector has a unique expression as a *finite* linear combination of its elements. For example, $\{1, x, x^2, x^3, \dots\}$ is a Hamel basis of $\mathcal{P}$. Every vector space admits a Hamel basis, by the axiom of choice. For most interesting infinite-dimensional spaces, a Hamel basis has uncountable cardinality. Other notions of basis are available; in this course we will study them only for Hilbert spaces.

Further infinite-dimensional examples are

$$C^\infty(\mathbb{R}), \quad L^2(\mathbb{R}), \quad C^\infty([0, 1]), \quad L^2([0, 1]).$$

Apart from $L^2(\mathbb{R})$, they contain copies of the polynomial space above. For $L^2(\mathbb{R})$ one instead uses $p(x)e^{-x^2}$ for polynomials p . An especially important example is

$$\ell^2 = \left\{ (a_1, a_2, \dots) : a_i \in \mathbb{R}, \quad \sum_{i=1}^{\infty} a_i^2 < \infty \right\}.$$

Vector subspaces are defined as in linear algebra, and results such as “the intersection of vector subspaces is a vector subspace” still hold.

1.4. Metric structures on vector spaces. We would like to introduce distance between elements of an abstract vector space.

Definition 1.2 (Metric). A metric on a set X is a real-valued function $d : X \times X \rightarrow \mathbb{R}$ such that

- (1) $d(x, y) \geq 0$, with equality if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$;
- (3) $d(x, y) \leq d(x, z) + d(z, y)$.

If the requirement that $d(x, y) = 0$ imply $x = y$ is removed, d is called a *pseudo-metric*.

Example 1.2. On $\mathbb{R}^n$, the Euclidean distance

$$d(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}$$

is a metric. On $C([0, 1])$, the space of continuous functions on $[0, 1]$, each of the following is a metric:

$$\begin{aligned} d_0(f, g) &= \max_{0 \leq x \leq 1} |f(x) - g(x)|, \\ d_1(f, g) &= \int_0^1 |f(x) - g(x)| \, dx, \\ d_2(f, g) &= \left(\int_0^1 |f(x) - g(x)|^2 \, dx \right)^{1/2}. \end{aligned}$$

Compatible structures are particularly useful. Which metrics are compatible with vector addition and scalar multiplication?

Definition 1.3. A metric d on a vector space X is *translation-invariant* if

$$d(x + z, y + z) = d(x, y) \quad (x, y, z \in X).$$

It is *absolutely homogeneous* if

$$d(\alpha x, \alpha y) = |\alpha| d(x, y) \quad (\alpha \in \mathbb{F}).$$

(The handwritten notes use the term “positively homogeneous” for this last identity.) The preceding metrics have both properties.

Proposition 1.1. Suppose d is a translation-invariant, absolutely homogeneous metric on a vector space X . Define $\|x\| = d(x, 0)$. Then

- (i) $\|x + y\| \leq \|x\| + \|y\|$;
- (ii) $\|\alpha x\| = |\alpha| \|x\|$;
- (iii) $\|x\| > 0$ if $x \neq 0$.

Proof. The triangle inequality and translation invariance give

$$\|x + y\| = d(x + y, 0) \leq d(x + y, y) + d(y, 0) = d(x, 0) + d(y, 0) = \|x\| + \|y\|.$$

Also,

$$\|\alpha x\| = d(\alpha x, \alpha 0) = |\alpha| d(x, 0) = |\alpha| \|x\|.$$

Finally, $x \neq 0$ implies $d(x, 0) > 0$. □

Definition 1.4 (Normed vector space). A function $\|\cdot\| : X \rightarrow \mathbb{R}$ satisfying the preceding three properties is a *norm*. The pair $(X, \|\cdot\|)$ is a *normed vector space*.

Proposition 1.2. If $(X, \|\cdot\|)$ is a normed vector space, then $d(x, y) = \|x - y\|$ is a translation-invariant, absolutely homogeneous metric on X .

Proof. Exercise, as in the handwritten notes. □

Thus a norm corresponds precisely to a metric satisfying the two compatibility identities above.

2. COMPLETE METRIC SPACES

2.1. Review (September 6, 2015). The previous lecture introduced infinite-dimensional vector spaces, metric spaces, and the correspondence between norms and metrics satisfying translation invariance and absolute homogeneity. Today we study complete metric spaces.

2.2. Convergence. Let (X, d) be a metric space; X need not be a vector space.

Definition 2.1. A sequence $\{x_i\}_{i=1}^\infty$ in X *converges* to $x_0 \in X$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall i > N : d(x_i, x_0) < \varepsilon.$$

We write $x_i \rightarrow x_0$. Convergence in more general spaces will be introduced later.

Example 2.1. Let $X = C([0, 1])$ and $f_n(x) = x^n$, $f_0(x) = 0$. For the integral metric,

$$d_1(f_n, f_0) = \int_0^1 x^n dx = \frac{1}{n+1} \rightarrow 0,$$

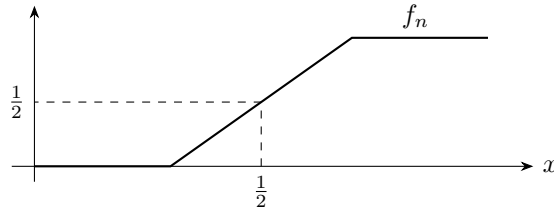
so $f_n \rightarrow f_0$ in (X, d_1) . For the uniform metric,

$$d_0(f_n, f_0) = \max_{0 \leq x \leq 1} x^n = 1,$$

so $f_n \not\rightarrow f_0$ in (X, d_0) . In general $d_1(f, g) \leq d_0(f, g)$. Thus d_0 is stronger than d_1 : convergence in d_0 implies convergence to the same element in d_1 .

Example 2.2. Still use d_1 on $C([0, 1])$. For $n \geq 2$, set

$$f_n(x) = \begin{cases} 0, & x < \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}(x - \frac{1}{2}), & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + \frac{1}{n}, \\ 1, & x > \frac{1}{2} + \frac{1}{n}, \end{cases} \quad f_0(x) = \begin{cases} 0, & x \leq \frac{1}{2}, \\ 1, & x > \frac{1}{2}. \end{cases}$$



The area between the graphs is

$$\int_0^1 |f_n(x) - f_0(x)| dx = \frac{1}{2n} \rightarrow 0.$$

Nevertheless, we cannot say $f_n \rightarrow f_0$ in (X, d_1) , since $f_0 \notin X$.

Definition 2.2 (Bounded set). A subset A of a metric space is bounded if there exists $C > 0$ such that $d(x, y) < C$ for every $x, y \in A$. Equivalently, for a fixed base point $x_0 \in X$ there is $C' > 0$ such that $d(x, x_0) < C'$ for every $x \in A$.

Editorial note. The base-point formulation is written for a metric space, where there need not be a distinguished zero. The handwritten margin also uses $d(x, 0)$ in the vector-space setting.

Proposition 2.1. Every convergent sequence in a metric space is bounded, and its limit is unique.

Proof. Suppose $x_i \rightarrow x_0$. Choose N such that $d(x_i, x_0) < 1$ for $i > N$. Put

$$C = 1 + \max\{d(x_0, x_1), \dots, d(x_0, x_N)\}.$$

Then $d(x_i, x_0) < C$ for every i , proving boundedness. If also $x_i \rightarrow x'_0$, then for every $\varepsilon > 0$ and sufficiently large i ,

$$d(x_0, x'_0) \leq d(x_0, x_i) + d(x_i, x'_0) < 2\varepsilon.$$

Hence $d(x_0, x'_0) = 0$, and $x_0 = x'_0$. $\square$

2.3. Completeness. Example 2.2 suggests that a sequence can approach a function outside the original space. As with $\mathbb{Q}$, this leads to incompleteness. For analysis, complete spaces are especially useful.

Definition 2.3 (Cauchy sequence). A sequence $\{x_i\}$ in (X, d) is *Cauchy* if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall i, j > N : d(x_i, x_j) < \varepsilon.$$

Definition 2.4 (Complete metric space). A metric space (X, d) is *complete* if every Cauchy sequence in X converges to an element of X .

Definition 2.5 (Banach space). A *Banach space* is a normed vector space complete for the metric $d(x, y) = \|x - y\|$ induced by its norm. Banach spaces are a main object of study in this course.

Example 2.3. Euclidean space $\mathbb{R}^n$ is complete for

$$d(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

Thus $\mathbb{R}^n$ with its Euclidean norm is a Banach space.

Example 2.4. For $f_n(x) = x^n$ in $(C([0, 1]), d_1)$,

$$d_1(f_n, f_m) \leq \int_0^1 x^n dx + \int_0^1 x^m dx = \frac{1}{n+1} + \frac{1}{m+1}.$$

Thus $\{f_n\}$ is Cauchy, as is expected from its convergence to zero.

Proposition 2.2. *Every convergent sequence in a metric space is Cauchy.*

Proof. If $x_i \rightarrow x_0$, then for any $\varepsilon > 0$ there is N such that $d(x_i, x_0) < \varepsilon$ for $i > N$. Consequently, for $i, j > N$,

$$d(x_i, x_j) \leq d(x_i, x_0) + d(x_0, x_j) < 2\varepsilon.$$

$\square$

Example 2.5. In the uniform metric, the sequence $f_n(x) = x^n$ is not Cauchy. For fixed n , as $m \rightarrow \infty$,

$$\max_{0 \leq x \leq 1} |x^n - x^m| \longrightarrow 1.$$

Proposition 2.3. *$(C([0, 1]), d_0)$ is complete. In particular, $C([0, 1])$ is a Banach space with the norm $\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|$.*

Proof. Let $\{f_n\}$ be Cauchy in d_0 . For every $\varepsilon > 0$ there is N such that

$$|f_n(x) - f_m(x)| < \varepsilon \quad (n, m > N, x \in [0, 1]).$$

For each fixed x , the scalar sequence $\{f_n(x)\}$ is Cauchy. By completeness of $\mathbb{R}$, let $f_0(x) = \lim_{m \rightarrow \infty} f_m(x)$. Taking the limit above gives

$$|f_n(x) - f_0(x)| \leq \varepsilon \quad (n > N, x \in [0, 1]).$$

Thus $f_n \rightarrow f_0$ uniformly. The uniform limit of continuous functions is continuous, so $f_0 \in C([0, 1])$, and $d_0(f_n, f_0) \rightarrow 0$. $\square$

By contrast, the ramp sequence in Example 2.2 is Cauchy in d_1 , since

$$d_1(f_n, f_m) \leq \int_0^1 |f_n - f_0| dx + \int_0^1 |f_0 - f_m| dx = \frac{1}{2n} + \frac{1}{2m}.$$

It has no continuous limit. Hence $(C([0, 1]), d_1)$ is not complete.

Editorial note. To justify the last assertion explicitly, a hypothetical continuous limit g would satisfy $\int_0^1 |g - f_0| = 0$, so $g = 0$ on $[0, \frac{1}{2})$ and $g = 1$ on $(\frac{1}{2}, 1]$, contradicting continuity. This step is implicit in the handwritten notes.

We will see that $L^1([0, 1])$ is complete. More generally, every incomplete metric space (X, d) has a completion $(\tilde{X}, \tilde{d})$ in which X is dense and $\tilde{d}|_{X \times X} = d$. This is analogous to the construction of $\mathbb{R}$ from $\mathbb{Q}$; density will be defined next time.

2.4. The Banach space ℓ^1 .

Proposition 2.4. *Let*

$$\ell^1 = \left\{ x = (a_1, a_2, \dots) : \sum_{i=1}^{\infty} |a_i| < \infty \right\}, \quad \|x\|_1 = \sum_{i=1}^{\infty} |a_i|.$$

Then $(\ell^1, \|\cdot\|_1)$ is a Banach space.

Proof. First, ℓ^1 is a vector space. For $x = (a_i)$ and $y = (b_i)$ in ℓ^1 ,

$$\sum_i |a_i + b_i| \leq \sum_i |a_i| + \sum_i |b_i| < \infty, \quad \sum_i |\alpha a_i| = |\alpha| \sum_i |a_i| < \infty.$$

The vector-space axioms hold coordinatewise. The same inequalities give the triangle inequality and absolute homogeneity of the norm. If $x \neq 0$, then some $a_i \neq 0$, so $\|x\|_1 > 0$.

For completeness, let $x^j = (a_i^j)$ be Cauchy in ℓ^1 . Given $\varepsilon > 0$, choose N such that

$$\sum_{i=1}^{\infty} |a_i^j - a_i^k| < \varepsilon \quad (j, k \geq N).$$

For each fixed i , $\{a_i^j\}_j$ is a scalar Cauchy sequence; write $a_i^0 = \lim_{j \rightarrow \infty} a_i^j$ and $x^0 = (a_i^0)$. Choose M such that $\sum_{i=M}^{\infty} |a_i^N| < \varepsilon$. Then for $j \geq N$,

$$\sum_{i=M}^{\infty} |a_i^j| \leq \sum_{i=M}^{\infty} |a_i^j - a_i^N| + \sum_{i=M}^{\infty} |a_i^N| < 2\varepsilon.$$

For any finite $K \geq M$, take the limit $j \rightarrow \infty$ in the finite sum:

$$\sum_{i=M}^K |a_i^0| \leq 2\varepsilon.$$

Now let $K \rightarrow \infty$, obtaining $\sum_{i=M}^{\infty} |a_i^0| \leq 2\varepsilon$. It follows that $x^0 \in \ell^1$. Finally, for sufficiently large j ,

$$\begin{aligned} \|x^j - x^0\|_1 &\leq \sum_{i=1}^{M-1} |a_i^j - a_i^0| + \sum_{i=M}^{\infty} |a_i^j| + \sum_{i=M}^{\infty} |a_i^0| \\ &< \varepsilon + 2\varepsilon + 2\varepsilon = 5\varepsilon. \end{aligned}$$

Thus $x^j \rightarrow x^0$ in ℓ^1 . □

Editorial note. The handwritten proof passes directly to the limit in an infinite tail sum. The finite-truncation argument above supplies the justification. Absolute homogeneity is written with $|\alpha|$, correcting a missing absolute value in the source.

3. METRIC TOPOLOGY AND THE BAIRE CATEGORY THEOREM

3.1. Review (September 8, 2015). Every convergent sequence is Cauchy. In a complete metric space, every Cauchy sequence converges. Examples of Banach spaces include $C([0, 1])$ with the uniform norm and ℓ^1 with its usual norm. Today's topic is the Baire category theorem for complete metric spaces.

3.2. Metric topology. Let (X, d) be a metric space and $A \subset X$.

Definition 3.1 (Interior points and open sets). A point $x \in A$ is an *interior point* of A if there is $\varepsilon > 0$ such that

$$B(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\} \subset A.$$

The ball $B(x, \varepsilon)$ is also called an ε -neighborhood of x . A subset A is *open* if each of its points is an interior point.

Example 3.1. For every $x \in X$ and $r > 0$, the ball $B(x, r)$ is open. Indeed, if $y \in B(x, r)$, take $0 < \varepsilon < r - d(x, y)$. For $z \in B(y, \varepsilon)$,

$$d(z, x) \leq d(z, y) + d(y, x) < \varepsilon + d(x, y) < r.$$

Thus $B(y, \varepsilon) \subset B(x, r)$.

Proposition 3.1 (Properties of open sets). (1) $\emptyset$ and X are open.

(2) Any union of open sets is open, even for an uncountable family.

(3) Every finite intersection of open sets is open.

Proof. The first assertion is immediate. If $x \in \bigcup_{\alpha} A_{\alpha}$, then $x \in A_{\alpha}$ for some α . Since A_{α} is open, a ball about x lies in A_{α} and hence in the union. If $x \in A \cap B$, choose $\varepsilon_1, \varepsilon_2 > 0$ such that $B(x, \varepsilon_1) \subset A$ and $B(x, \varepsilon_2) \subset B$. Then $B(x, \min\{\varepsilon_1, \varepsilon_2\}) \subset A \cap B$. Induction proves the finite-intersection assertion. $\square$

Remark 3.1. A countable intersection of open sets need not be open. For example,

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is not open in $\mathbb{R}$.

Proposition 3.2 (Convergence in terms of open sets). A sequence x_i converges to x_0 if and only if, for every open set A containing x_0 , there is N such that $x_i \in A$ for $i > N$. Such an open set A is called an *open neighborhood* of x_0 .

Proof. If $x_i \rightarrow x_0$ and $x_0 \in A$ with A open, choose $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset A$. For large i , $x_i \in B(x_0, \varepsilon) \subset A$. Conversely, apply the condition to $A = B(x_0, \varepsilon)$ for every $\varepsilon > 0$ to get $d(x_i, x_0) < \varepsilon$ eventually. $\square$

Definition 3.2 (Closed set). A subset $A \subset X$ is *closed* if $X \setminus A$ is open.

Properties of closed sets follow by taking complements of the corresponding statements about open sets.

Example 3.2. For $x \in X$ and $r > 0$, the closed ball

$$\overline{B}(x, r) = \{y \in X : d(y, x) \leq r\}$$

is closed. The triangle inequality shows that its complement $\{y \in X : d(y, x) > r\}$ is open, as above.

Proposition 3.3 (Sequential characterization of closed sets). *A subset A is closed if and only if every sequence $x_n \in A$ converging to some $x \in X$ has its limit x in A .*

Proof. Suppose A is closed and $x_n \in A$, $x_n \rightarrow x$. If $x \notin A$, then $X \setminus A$ is an open neighborhood of x , so eventually $x_n \in X \setminus A$, a contradiction. Conversely, suppose every such limit lies in A . Take $y \notin A$. If no ball about y is disjoint from A , choose $x_n \in B(y, 1/n) \cap A$. Then $x_n \rightarrow y$, implying $y \in A$, a contradiction. Thus some ball about y lies in $X \setminus A$, proving that this complement is open. $\square$

Definition 3.3 (Closure). The *closure* of $A \subset X$ is

$$\overline{A} = \{x \in X : \text{there is a sequence } x_n \in A \text{ with } x_n \rightarrow x\}.$$

Example 3.3. $\overline{\mathbb{Q}} = \mathbb{R}$, since every real number is a limit of rational numbers. Let $\mathcal{P}([0, 1])$ denote polynomial functions restricted to $[0, 1]$. Every continuous function on $[0, 1]$ can be approximated uniformly by polynomials (for example, by Bernstein polynomials). Thus, in the uniform metric on $C([0, 1])$,

$$\overline{\mathcal{P}([0, 1])} = C([0, 1]).$$

Proposition 3.4. $\overline{A}$ is closed.

Proof. Suppose $x_i \in \overline{A}$ and $x_i \rightarrow x \in X$. For each n , choose an index i_n with $d(x_{i_n}, x) < 1/n$. Since $x_{i_n} \in \overline{A}$, choose $y_n \in A$ with $d(y_n, x_{i_n}) < 1/n$. Then

$$d(y_n, x) \leq d(y_n, x_{i_n}) + d(x_{i_n}, x) < \frac{2}{n}.$$

Hence $y_n \rightarrow x$ and $x \in \overline{A}$. $\square$

If B is closed and contains A , then B contains $\overline{A}$. Why? Consequently, $\overline{A}$ is the smallest closed subset of X containing A .

3.3. The Baire category theorem.

Definition 3.4 (Dense subset). A subset $A \subset X$ is *dense* if $\overline{A} = X$. Equivalently, each point of X is a limit of a sequence in A .

Example 3.4. Both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$. The polynomials are dense in $C([0, 1])$ with the uniform metric.

Proposition 3.5. A subset $A \subset X$ is dense if and only if $A \cap B \neq \emptyset$ for every nonempty open subset $B \subset X$.

Proof. Suppose A is dense and B is nonempty and open. Choose $x \in B$ and $\varepsilon > 0$ with $B(x, \varepsilon) \subset B$. Since $x \in \overline{A}$, some $y \in A$ satisfies $d(y, x) < \varepsilon$. Hence $y \in A \cap B$. Conversely, for each $x \in X$ and $n \geq 1$, choose $x_n \in B(x, 1/n) \cap A$. Then $x_n \rightarrow x$, so $\overline{A} = X$. $\square$

Definition 3.5 (Nowhere dense subset). A subset $A \subset X$ is *nowhere dense* if its closure has no interior point.

Example 3.5. The integers $\mathbb{Z}$ and the Cantor set are nowhere dense in $\mathbb{R}$. The subset

$$A = \{f \in C([0, 1]) : f(0) = 0\}$$

is nowhere dense in $C([0, 1])$ with the uniform metric.

Definition 3.6 (First and second category). A subset $A \subset X$ is of *first category* if it is a union of countably many nowhere dense subsets of X . It is of *second category* if it is not of first category.

Example 3.6. The rationals $\mathbb{Q}$ are of first category in $\mathbb{R}$. The following theorem and its corollary imply that $\mathbb{R} \setminus \mathbb{Q}$ is of second category in $\mathbb{R}$.

Theorem 3.6 (Baire). *Let (X, d) be a complete metric space. The intersection of every countable collection of dense open subsets of X is dense in X (although it need not be open).*

Corollary 3.1 (Baire). *Every nonempty complete metric space is of second category in itself.*

Proof. If $X = \bigcup_{i=1}^{\infty} A_i$ with each A_i nowhere dense, then also $X = \bigcup_{i=1}^{\infty} \overline{A_i}$. Each $X \setminus \overline{A_i}$ is open and dense. The theorem says that

$$\bigcap_{i=1}^{\infty} (X \setminus \overline{A_i})$$

is dense in X , hence nonempty. But this intersection is empty by the assumed covering, a contradiction. $\square$

Editorial note. The nonemptiness assumption in the corollary is added explicitly; it is implicit in the handwritten proof.

Proof of Theorem 3.6. Let $A_1, A_2, \dots$ be dense open subsets of X , and let $B \subset X$ be nonempty and open. We must show $B \cap \bigcap_{i=1}^{\infty} A_i \neq \emptyset$. Since $A_1 \cap B$ is nonempty and open, choose $x_1 \in X$ and $0 < r_1 < 1$ with

$$\overline{B}(x_1, r_1) \subset A_1 \cap B.$$

Continue inductively. Since A_n is dense and open, $A_n \cap B(x_{n-1}, r_{n-1})$ is nonempty and open. Choose $x_n \in X$ and $0 < r_n < 1/n$ such that

$$\overline{B}(x_n, r_n) \subset A_n \cap B(x_{n-1}, r_{n-1}).$$

The sequence $\{x_n\}$ is Cauchy: if $n, m > N$, both points lie in $B(x_N, r_N)$, so

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < 2r_N < \frac{2}{N}.$$

By completeness there exists $x_0 \in X$ with $x_n \rightarrow x_0$. For every fixed N , the tail of the sequence is contained in $\overline{B}(x_N, r_N)$, which is closed. Hence $x_0 \in \overline{B}(x_N, r_N) \subset A_N$; also $x_0 \in B$ by the first closed ball. Therefore

$$x_0 \in B \cap \bigcap_{n=1}^{\infty} A_n,$$

as required. $\square$

Remark 3.2. The theorem has many applications. Some appear in the problem sets, and others will be discussed in future lectures.

4. TOPOLOGICAL VECTOR SPACES

4.1. Review and motivation (September 11, 2015). Last time we studied open and closed sets, closure and density in metric spaces. The Baire category theorem says that a countable intersection of dense open subsets of a complete metric space is dense. In particular, a nonempty complete metric space is not a countable union of nowhere dense sets.

There are important vector spaces whose natural structure does not come from a complete metric. For example, let

$$X = C_c(\mathbb{R}) = \{f \in C(\mathbb{R}) : \text{supp } f \text{ is compact}\}, \quad \text{supp } f = \overline{\{x : f(x) \neq 0\}}.$$

The source denotes this space by $C_0^0(\mathbb{R})$. Let

$$X_n = \{f \in C_c(\mathbb{R}) : \text{supp } f \subset [-n, n]\}.$$

Then $X = \bigcup_{n=1}^{\infty} X_n$. Identified by restriction, X_n is the closed subspace

$$\{f \in C([-n, n]) : f(-n) = f(n) = 0\}$$

of the Banach space $C([-n, n])$. It is nowhere dense in $C([-n, n])$, and X_n is nowhere dense in X_m when $m > n$. A natural structure on $C_c(\mathbb{R})$ should give each X_n its usual Banach space topology. The Baire theorem motivates seeking structures more general than complete metric spaces.

Editorial note. For the Baire obstruction on X itself one needs each X_n to be closed and nowhere dense in X for the chosen topology. Merely requiring the correct subspace topology on X_n is not by itself the full stated hypothesis. For example, in the compact-open topology introduced in Lecture 6, these sets are indeed closed and nowhere dense, so that topology cannot be induced by a complete metric.

4.2. Topological spaces.

Definition 4.1. A *topological space* is a set X with a collection $\mathcal{T}$ of subsets, called a *topology*, such that

- (1) $\emptyset, X \in \mathcal{T}$;
- (2) every union of members of $\mathcal{T}$ lies in $\mathcal{T}$;
- (3) the intersection of two members of $\mathcal{T}$ lies in $\mathcal{T}$.

The members of $\mathcal{T}$ are the *open sets*.

Example 4.1. The weakest topology on X is $\{\emptyset, X\}$, and the strongest is the collection of all subsets of X . A metric induces a topology. On $X = \{-1, 0, 1\}$, the collection

$$\{\emptyset, \{0\}, \{0, 1\}, \{-1, 0\}, \{-1, 0, 1\}\}$$

is a topology.

Editorial note. A marginal example in the source mentions the Zariski topology on $\mathbb{R}^2$ using polynomial zero sets. These are closed sets: the open sets are their complements, with closed sets given by common zeros of families of polynomials.

Remark 4.1. If $\mathcal{T}$ is induced by a metric d , we call $\mathcal{T}$ and d *compatible*, and say $\mathcal{T}$ is *metrizable*. If $\mathcal{T}_1 \subset \mathcal{T}_2$, then $\mathcal{T}_1$ is *weaker* than $\mathcal{T}_2$. The intersection of any collection of topologies on X is again a topology. Consequently, for any collection $\mathcal{S}$ of subsets,

$$\bigcap \{\mathcal{T} : \mathcal{T} \text{ is a topology on } X, \mathcal{S} \subset \mathcal{T}\}$$

is the weakest topology containing $\mathcal{S}$, called the topology *generated* by $\mathcal{S}$.

The definitions of neighborhoods, closed sets, closure, and convergence extend to topological spaces:

- (1) In these notes, an open neighborhood of x is an open set containing x . Write $\mathcal{T}_x = \{U \in \mathcal{T} : x \in U\}$.
- (2) A is closed if $X \setminus A$ is open.
- (3) $\overline{A}$ is the smallest closed set containing A . Thus A is closed if and only if $A = \overline{A}$.
- (4) $x_i \rightarrow x$ if every open neighborhood of x contains all sufficiently late terms of the sequence.

Example 4.2. In the weakest topology, every sequence converges to every point.

Definition 4.2 (Hausdorff space). A topological space is *Hausdorff* if any two distinct points x, y have disjoint open neighborhoods U, V .

Proposition 4.1. In a Hausdorff space, the limit of a convergent sequence is unique, and every singleton is closed.

Proof. If $x_n \rightarrow x$ and $y \neq x$, choose disjoint neighborhoods U of x and V of y . Eventually $x_n \in U$, hence $x_n \notin V$, so x_n cannot converge to y . For each $y \neq x$, choose a neighborhood V_y of y avoiding x . Then $X \setminus \{x\} = \bigcup_{y \neq x} V_y$ is open. $\square$

Definition 4.3 (Base and local base). A subcollection $\mathcal{B} \subset \mathcal{T}$ is a *base* if every open set is a union of members of $\mathcal{B}$. A subcollection $\mathcal{B}_x \subset \mathcal{T}_x$ is a *local base at x* if every neighborhood of x contains a member of $\mathcal{B}_x$. A neighborhood need not itself be a union of members of a local base at one fixed point.

Example 4.3. For a metric topology, all open balls form a base, and

$$\{B(x, r) : r > 0\}, \quad \{B(x, 1/n) : n \in \mathbb{N}\}$$

are local bases at x . The latter is countable. Different bases can generate the same topology. If $\mathcal{B}$ is a base for $\mathcal{T}$, it generates $\mathcal{T}$.

Definition 4.4 (Product topology). For topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{G})$, the *product topology* on $X \times Y$ is generated by

$$\{U \times V : U \in \mathcal{T}, V \in \mathcal{G}\}.$$

Example 4.4. The usual topology on $\mathbb{R}^2$ is the product of the usual topologies on $\mathbb{R}$, since every open subset of $\mathbb{R}^2$ is a union of open rectangles.

Definition 4.5 (Continuity). A map $f : X \rightarrow Y$ is *continuous at x* if the inverse image of each open neighborhood of $f(x)$ contains an open neighborhood of x . It is *continuous* if it is continuous at every point; equivalently, $f^{-1}(V)$ is open for every open subset $V \subset Y$.

Proposition 4.2. If f is continuous at x and $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$.

Proof. For an open neighborhood V of $f(x)$, the set $f^{-1}(V)$ contains an open neighborhood of x . Thus eventually $x_n \in f^{-1}(V)$, so eventually $f(x_n) \in V$. $\square$

Definition 4.6 (Homeomorphism). A map $f : X \rightarrow Y$ is a *homeomorphism* if it is bijective and both f and f^{-1} are continuous.

4.3. Topological vector spaces. A topological vector space is a vector space endowed with a topology compatible with its vector operations.

Definition 4.7. Let X be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with a Hausdorff topology. It is a *topological vector space* (TVS) if the maps

$$X \times X \rightarrow X, \quad (x, y) \mapsto x + y, \quad \mathbb{F} \times X \rightarrow X, \quad (\alpha, x) \mapsto \alpha x$$

are continuous, with the product topologies on their domains. Some books do not impose the Hausdorff requirement.

For $A, B \subset X$ and $I \subset \mathbb{F}$, write

$$A + B = \{x + y : x \in A, y \in B\}, \quad I \cdot A = \{\alpha x : \alpha \in I, x \in A\}.$$

For example, in $\mathbb{R}^2$, if $A = \{(x, 0) : -1 \leq x \leq 1\}$ and $B = \{(1, y) : -1 \leq y \leq 1\}$, then

$$A + B = \{(x, y) : 0 \leq x \leq 2, -1 \leq y \leq 1\}.$$

If instead $A = \{(1, 0), (2, 0)\}$, then

$$2A = \{(2, 0), (4, 0)\} \neq A + A = \{(2, 0), (3, 0), (4, 0)\}.$$

Continuity of the two vector operations means:

- For $x, y \in X$ and any neighborhood V of $x + y$, there are neighborhoods U_x, U_y of x, y with $U_x + U_y \subset V$.
- For $\alpha \in \mathbb{F}$, $x \in X$, and a neighborhood V of αx , there are $\varepsilon > 0$ and a neighborhood U_x of x such that $\{\beta : |\beta - \alpha| < \varepsilon\} \cdot U_x \subset V$.

For $a \in X$ and $0 \neq \alpha \in \mathbb{F}$, define

$$T_a(x) = a + x, \quad M_\alpha(x) = \alpha x.$$

Proposition 4.3. T_a and M_α are homeomorphisms.

Proof. Their inverses are T_{-a} and $M_{1/\alpha}$. All four maps are continuous by continuity of the vector operations. $\square$

Corollary 4.1. A subset A is open if and only if $a + A$ is open. Consequently, the topology of a TVS is determined by any local base at 0.

5. LOCAL GEOMETRY AND METRIZATION

5.1. Review (September 15, 2015). A topology specifies open sets and therefore determines continuity of maps: inverse images of open sets are open. A TVS is a Hausdorff space with continuous vector operations.

5.2. Local geometry of topological vector spaces. Recall that a base $\mathcal{B}$ for a topology $\mathcal{T}$ has the property that for every open U and $x \in U$, some $V \in \mathcal{B}$ satisfies $x \in V \subset U$. Equivalently, open sets are unions of base elements. A local base $\mathcal{B}_x$ at x consists of open neighborhoods of x , with each neighborhood of x containing one of them. In a metric space, the balls form a base, and both $\{B(x, r) : r > 0\}$ and $\{B(x, 1/n) : n \in \mathbb{N}\}$ are local bases at x .

For a TVS, $T_a(x) = x + a$ and $M_\alpha(x) = \alpha x$ are homeomorphisms when $\alpha \neq 0$. Thus A is open if and only if $a + A$ is open, or if and only if αA is open. Any local base at zero determines the whole topology. We seek local bases with useful geometric properties.

Definition 5.1 (Local convexity). A TVS is *locally convex* if zero has a local base consisting of convex sets.

Example 5.1. Normed vector spaces are locally convex. If $x, y \in B(0, r)$ and $0 \leq t \leq 1$, then

$$\|tx + (1-t)y\| \leq t\|x\| + (1-t)\|y\| < r.$$

Definition 5.2 (Absorbing set). A subset $E \subset X$ is *absorbing* if for each $x \in X$ there is $\delta > 0$ such that $\alpha x \in E$ whenever $|\alpha| < \delta$. In particular, $0 \in E$.

Proposition 5.1. Every neighborhood of zero in a TVS is absorbing.

Proof. For fixed $x \in X$, the map $\mathbb{F} \rightarrow X$, $\alpha \mapsto \alpha x$, is continuous, as the composition $\alpha \mapsto (\alpha, x) \mapsto \alpha x$. The inverse image of an open neighborhood U of zero is an open scalar set containing zero. It therefore contains $\{\alpha : |\alpha| < \delta\}$ for some $\delta > 0$. $\square$

Corollary 5.1. For every neighborhood U of zero, $X = \bigcup_{k=1}^{\infty} kU$.

Proof. For $x \in X$, choose k with $1/k < \delta$ in the absorbing property. Then $x/k \in U$, so $x \in kU$. $\square$

Definition 5.3 (Symmetric set). A set E is *symmetric* if $E = -E$.

Proposition 5.2. Every neighborhood U of zero contains a symmetric neighborhood V of zero such that $V + V \subset U$.

Proof. By continuity of addition at $(0, 0)$, choose neighborhoods U_1, U_2 of zero with $U_1 + U_2 \subset U$. Set

$$V = U_1 \cap U_2 \cap (-U_1) \cap (-U_2).$$

Then V is a symmetric open neighborhood of zero and $V + V \subset U$. $\square$

Iterating this construction gives a neighborhood V with $V + V + V + V \subset U$.

Definition 5.4 (Balanced set). A subset E is *balanced* if $\alpha E \subset E$ whenever $|\alpha| \leq 1$. Every balanced set is symmetric: $-E \subset E$ and $E \subset -E$. If A and B are balanced, so is $A + B$.

Proposition 5.3. Every neighborhood of zero in a TVS contains a balanced neighborhood of zero.

Proof. Let U be an open neighborhood of zero. By continuity of scalar multiplication at $(0, 0)$, choose $\delta > 0$ and a neighborhood V_1 of zero such that $\beta V_1 \subset U$ for $|\beta| < \delta$. Set

$$V = \bigcup_{0 < |\beta| < \delta} \beta V_1.$$

This is an open neighborhood of zero, since each nonzero dilation is a homeomorphism. It is contained in U and is balanced: if $|\alpha| \leq 1$, then $|\alpha\beta| < \delta$, and the case $\alpha = 0$ is immediate. $\square$

Corollary 5.2. *Every TVS has a balanced local base at zero.*

Remark 5.1. Similarly, every convex neighborhood of zero contains a balanced convex neighborhood of zero. Thus every locally convex TVS has a balanced convex local base.

Definition 5.5 (Bounded set in a TVS). A subset $E \subset X$ is *bounded* if for every neighborhood U of zero there exists $s > 0$ such that $E \subset tU$ for all real $t > s$.

Proposition 5.4. *A set E is bounded if and only if, for every sequence $x_n \in E$ and every scalar sequence $\alpha_n \rightarrow 0$, one has $\alpha_n x_n \rightarrow 0$.*

Proof. Exercise, as in the source. $\square$

In a normed vector space, this is equivalent to $E \subset B(0, C)$ for some $C > 0$.

Editorial note. The source says “metric space” at this point. The assertion requires a norm metric (or suitable homogeneity); for a general compatible metric, metric boundedness need not coincide with TVS boundedness.

Not every TVS has a bounded open neighborhood.

Proposition 5.5. *If V is a bounded open neighborhood of zero and $\alpha_k \neq 0$, $\alpha_k \rightarrow 0$, then $\{\alpha_k V : k \geq 1\}$ is a local base at zero.*

Proof. Exercise, as in the source. $\square$

Editorial note. The requirement $\alpha_k \neq 0$ is made explicit so each $\alpha_k V$ is an open neighborhood.

Definition 5.6 (Local boundedness). A TVS is *locally bounded* if zero has a bounded neighborhood.

A locally bounded TVS has a countable local base. The following theorem shows that it is metrizable.

5.3. Metrization. Every metric space has a countable local base at each point, so this is necessary for a TVS to be metrizable. For TVSs it is also sufficient.

Theorem 5.6 (Metrization of topological vector spaces). *Let X be a TVS with a countable local base at zero. Then its topology is induced by a metric d which is translation-invariant and whose open balls centered at zero are balanced. If X is also locally convex, d can be chosen so that all its open balls are convex.*

Proof. Start with a countable local base $\{U_n\}$. Choose a balanced neighborhood $V_1 \subset U_1$. Choose a neighborhood $\tilde{U}_2$ with $\tilde{U}_2 + \tilde{U}_2 + \tilde{U}_2 + \tilde{U}_2 \subset V_1 \cap U_2$, and then a balanced $V_2 \subset \tilde{U}_2$. Continuing inductively gives a balanced local base $\{V_n\}$ with

$$V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n.$$

If X is locally convex, choose each V_n convex as well. Let D consist of the dyadic rationals in $[0, 1)$ with finite binary expansion:

$$D = \left\{ r = \sum_{n=1}^{\infty} c_n(r) 2^{-n} : c_n(r) \in \{0, 1\}, \quad c_n(r) = 0 \text{ for all but finitely many } n \right\}.$$

This set is dense in $[0, 1)$.

For $r \in D$, define the finite Minkowski sum

$$A(r) = c_1(r)V_1 + c_2(r)V_2 + \cdots,$$

where a term with coefficient zero contributes $\{0\}$. Put $A(r) = X$ for $r \geq 1$. The key property is

$$A(r) + A(s) \subset A(r + s) \quad (r, s \in D).$$

The source refers to Rudin, p. 20, for its proof. Each $A(r)$ is balanced. Also $r < s$ implies $A(r) \subset A(s)$, since $A(r) + A(s - r) \subset A(s)$ and $0 \in A(s - r)$. Define

$$f(x) = \inf\{r \in D \cup [1, \infty) : x \in A(r)\}.$$

Then $0 \leq f(x) \leq 1$ and $f(0) = 0$. If $x \neq 0$, the Hausdorff property and the local base give N with $x \notin V_n$ for $n \geq N$. Since $A(2^{-N}) = V_N$ and $A(r) \subset A(2^{-N})$ for $r < 2^{-N}$, we have $f(x) \geq 2^{-N} > 0$. Balancedness gives $f(-x) = f(x)$. To prove subadditivity, it suffices to consider $f(x) + f(y) < 1$. For arbitrarily small $\varepsilon > 0$, choose dyadics r, s with

$$f(x) < r < f(x) + \varepsilon, \quad f(y) < s < f(y) + \varepsilon, \quad r + s < 1.$$

By the definition of the infimum and monotonicity, $x \in A(r)$ and $y \in A(s)$. Hence

$$f(x + y) \leq r + s < f(x) + f(y) + 2\varepsilon.$$

Let $\varepsilon \downarrow 0$. If $f(x) + f(y) \geq 1$, subadditivity follows from $f(x + y) \leq 1$. Thus

$$d(x, y) = f(x - y)$$

is a translation-invariant metric (compare Problem Set 1, Part 2, Problem 1). For $0 < \delta \leq 1$, its open balls centered at zero are

$$B_d(0, \delta) = \{x : f(x) < \delta\} = \bigcup_{\substack{r \in D \\ 0 < r < \delta}} A(r).$$

These are open in the original topology, balanced, and form a local base. Indeed, for $r < 2^{-n}$ we have $A(r) \subset V_n$, so $\delta < 2^{-n}$ implies $B_d(0, \delta) \subset V_n$; conversely, $V_n = A(2^{-n}) \subset B_d(0, \delta)$ whenever $2^{-n} < \delta$. Balls of radius greater than one are all of X . Thus d induces the original topology. If the V_n are convex, so are the $A(r)$, and their increasing unions are convex. Translation then gives convexity of every open ball. $\square$

Editorial note. The bound is $f \leq 1$, rather than the strict inequality written in the source. The ball formula above treats radii greater than one separately, and omits $A(0) = \{0\}$ from the open union.

6. SEMINORMS AND FRÉCHET SPACES

6.1. Review (September 18, 2015). A TVS has a balanced, absorbing local base $\mathcal{B}_0$ with the following properties: for $U, V \in \mathcal{B}_0$, there is $W \in \mathcal{B}_0$ with $W \subset U \cap V$; for $U \in \mathcal{B}_0$, there is $V \in \mathcal{B}_0$ with $V + V \subset U$. A TVS is metrizable if and only if it has a countable local base at zero. The compatible metric can be chosen translation-invariant with balanced balls at zero, and with convex balls if the TVS is locally convex. In particular, locally bounded TVSs are metrizable.

Editorial note. The source also says conversely that such collections define a TVS structure. A full converse requires appropriate separation and scalar-compatibility conditions; the displayed list is used here only as a review of properties of an existing TVS.

6.2. Topologies defined by seminorms. Return to $X = C_c(\mathbb{R})$, written $C_c^0(\mathbb{R})$ in the source. Let

$$X_n = \{f \in C_c(\mathbb{R}) : \text{supp } f \subset [-n, n]\}.$$

Restriction identifies X_n with the functions in $C([-n, n])$ vanishing at both endpoints, a closed Banach subspace. Its topology is determined by balls

$$\left\{ g \in X_n : \sup_{x \in [-n, n]} |f(x) - g(x)| < r \right\}.$$

Put $p_n(f) = \sup_{x \in [-n, n]} |f(x)|$. At zero the sets $\{f \in X_n : p_n(f) < 1/k\}$ form a local base. On X consider

$$U_{n,k} = \{f \in C_c(\mathbb{R}) : p_n(f) < 1/k\}.$$

Using finite intersections of these sets as a local base at zero, and translating, gives a locally convex TVS topology inducing the prescribed topology on every X_n . The function p_n is not a norm on $C_c(\mathbb{R})$: a nonzero function supported outside $[-n, n]$ has $p_n(f) = 0$. Nevertheless,

$$p_n(f) \geq 0, \quad p_n(f + g) \leq p_n(f) + p_n(g), \quad p_n(\alpha f) = |\alpha|p_n(f).$$

There are many spaces of this type.

Editorial note. The source describes the topology as “generated by $U_{n,k}$ ”. Here this is interpreted as a translation-invariant topology with the specified local base, as in the theorem below. Merely taking the smallest topology containing the sets $U_{n,k}$ without their translates would not give the intended vector-space topology.

Definition 6.1 (Seminorm; separating family). A *seminorm* on a vector space X is a function $p : X \rightarrow [0, \infty)$ satisfying

$$p(x + y) \leq p(x) + p(y), \quad p(\alpha x) = |\alpha|p(x).$$

A family $\mathcal{P} = \{p_\lambda\}$ is *separating* if every $x \neq 0$ has $p_\lambda(x) \neq 0$ for some λ .

Example 6.1. $p_n(f) = \sup_{x \in [-n, n]} |f(x)|$ gives a countable separating family of seminorms on $C_c(\mathbb{R})$.

Theorem 6.1. Let $\mathcal{P} = \{p_\lambda\}$ be a separating family of seminorms on X . For $k \in \mathbb{N}$ set $U_{\lambda,k} = \{x : p_\lambda(x) < 1/k\}$, and let $\mathcal{B}$ be all finite intersections of such sets. Let $\mathcal{T}$ be the translation-invariant topology having $\mathcal{B}$ as a local base at zero. Then:

- (1) $(X, \mathcal{T})$ is a TVS;

- (2) $\mathcal{B}$ is a convex balanced local base;
- (3) every p_λ is continuous;
- (4) $E \subset X$ is bounded if and only if every p_λ is bounded on E .

Proof. For the Hausdorff property, it suffices by translation to separate 0 and $x \neq 0$. Choose $p_\lambda(x) = \varepsilon > 0$ and k with $1/k < \varepsilon/2$. Then $U_{\lambda,k}$ and $x + U_{\lambda,k}$ are disjoint. Indeed, if $y = x + z$ were in their intersection, with $z \in U_{\lambda,k}$, then

$$p_\lambda(x) = p_\lambda(y - z) \leq p_\lambda(y) + p_\lambda(z) < 2/k < \varepsilon,$$

a contradiction.

To show addition is continuous, suppose a neighborhood U of $x + y$ contains

$$x + y + \bigcap_{i=1}^{\ell} U_{\lambda_i, k_i}.$$

The neighborhoods

$$U_1 = x + \bigcap_{i=1}^{\ell} U_{\lambda_i, 2k_i}, \quad U_2 = y + \bigcap_{i=1}^{\ell} U_{\lambda_i, 2k_i}$$

satisfy $U_1 + U_2 \subset U$ by subadditivity.

For continuity of scalar multiplication, let U be a neighborhood of αx containing $\alpha x + \bigcap_{i=1}^{\ell} U_{\lambda_i, k_i}$. Write $p_i = p_{\lambda_i}$. If $\alpha = 0$, choose $A > \max_i p_i(x)$ and

$$0 < \delta < \min\{1, (2Ak_1)^{-1}, \dots, (2Ak_\ell)^{-1}\}.$$

If $|\beta| < \delta$ and $z \in \bigcap_i U_{\lambda_i, 2k_i}$, then

$$p_i(\beta(x + z)) \leq |\beta|p_i(x) + |\beta|p_i(z) < \frac{1}{2k_i} + \frac{1}{2k_i} = \frac{1}{k_i}.$$

If $\alpha \neq 0$, choose

$$A > \max_i \left\{ p_i(x), \frac{1}{3k_i|\alpha|} \right\}, \quad 0 < \delta < \min_i \left\{ \frac{1}{3k_i A}, |\alpha| \right\}.$$

For $|\beta - \alpha| < \delta$ and $p_i(z) < 1/(3k_i|\alpha|)$,

$$\begin{aligned} p_i(\beta(x + z) - \alpha x) &\leq |\beta - \alpha|p_i(x) + |\beta|p_i(z) \\ &< \frac{1}{3k_i} + \frac{2}{3k_i} = \frac{1}{k_i}. \end{aligned}$$

Thus scalar multiplication is continuous. The finitely many constraints on z specify a neighborhood even when their radii are not reciprocal integers, since smaller reciprocal-integer radii may be chosen.

Editorial note. The scalar-continuity estimates are written out from the two cases in the source. For $\alpha \neq 0$, the choice $\delta < |\alpha|$ supplies the needed bound $|\beta| < 2|\alpha|$; the source's displayed choice only includes a bound by 1, which alone would not ensure that estimate for small $|\alpha|$.

For (2), each $U_{\lambda,k}$ is balanced by homogeneity, and convex because for $0 \leq t \leq 1$ and $x, y \in U_{\lambda,k}$,

$$p_\lambda(tx + (1-t)y) \leq tp_\lambda(x) + (1-t)p_\lambda(y) < 1/k.$$

Finite intersections preserve these properties.

For (3), the inequality

$$|p_\lambda(x + z) - p_\lambda(x)| \leq p_\lambda(z)$$

shows continuity at each x , using the neighborhoods $U_{\lambda,k}$.

For (4), if E is bounded, choose $t > 0$ with $E \subset tU_{\lambda,1}$. Then $p_\lambda(x) < t$ on E . Conversely, suppose every p_λ is bounded on E . For a basic neighborhood $U = \bigcap_{i=1}^\ell U_{\lambda_i, k_i}$, choose $t_i > 0$ with $p_{\lambda_i}(x) < t_i$ on E . For every $t > \max_i t_i k_i$, we have

$$p_{\lambda_i}(x/t) = \frac{1}{t} p_{\lambda_i}(x) < \frac{1}{k_i},$$

so $E \subset tU$. Hence E is bounded. $\square$

6.3. Fréchet spaces. When a topology is defined by countably many seminorms, its basic finite intersections form a countable local base, so it is metrizable. We can write down a compatible translation-invariant metric explicitly.

Proposition 6.2. *If a topology on X is defined by a separating sequence $\{p_i\}$ of seminorms, then for any positive sequence $c_i \rightarrow 0$,*

$$d(x, y) = \sup_{i \geq 1} \frac{c_i p_i(x - y)}{1 + p_i(x - y)}$$

is a compatible translation-invariant metric.

Proof. Exercise, as in the source. $\square$

Editorial note. The source uses a maximum. It exists for $x \neq y$ because the nonnegative terms tend to zero and at least one is positive; at $x = y$ all terms vanish. The supremum notation also covers both cases directly.

Definition 6.2 (Fréchet space). A *Fréchet space* is a TVS whose topology is induced by a countable separating family of seminorms, and which is complete for a compatible translation-invariant metric.

Remark 6.1. If d_1, d_2 are compatible translation-invariant metrics on the same TVS, then d_1 is complete if and only if d_2 is complete. As illustrated in Problem Set 2, Part 2, Problem 1, this need not hold without translation invariance.

Example 6.2. $C(\mathbb{R})$ is a Fréchet space for the seminorms

$$p_n(f) = \max_{-n \leq x \leq n} |f(x)|.$$

These form a countable separating family. Since $p_1 \leq p_2 \leq \dots$, the convex sets

$$V_n = \{f \in C(\mathbb{R}) : p_n(f) < 1/n\}$$

form a local base at zero. A compatible metric is

$$d(f, g) = \sup_{n \geq 1} \frac{1}{n} \frac{p_n(f - g)}{1 + p_n(f - g)}.$$

To see completeness, let $\{f_i\}$ be Cauchy for d . For each fixed n , $p_n(f_i - f_j) \rightarrow 0$ as $i, j \rightarrow \infty$. Thus the restrictions converge uniformly on $[-n, n]$ to a continuous function $f_{(n)}$. Uniqueness of uniform limits gives $f_{(m)} = f_{(n)}$ on $[-n, n]$ for $m > n$. These limits define $f \in C(\mathbb{R})$. For every n , $p_n(f_i - f) \rightarrow 0$, and hence $d(f_i, f) \rightarrow 0$.

The space $C_c(\mathbb{R})$ with the same seminorm topology is not Fréchet, since it is not complete. More generally, $C(\Omega)$ is Fréchet when $\Omega \subset \mathbb{R}^n$ is open (with the topology of uniform convergence on compact subsets).

7. CONVEXITY AND HILBERT SPACES

7.1. Review (September 22, 2015). A seminorm is a nonnegative, subadditive, absolutely homogeneous function. A separating family of seminorms induces a locally convex TVS topology making every seminorm continuous. A Fréchet space has a countable such family and a complete compatible translation-invariant metric, for example

$$d(x, y) = \sup_n \frac{c_n p_n(x - y)}{1 + p_n(x - y)}, \quad c_n > 0, \quad c_n \rightarrow 0.$$

Today's topic is convexity.

7.2. The Minkowski functional. Let X be a vector space and $A \subset X$ be convex and absorbing, with $0 \in A$.

Definition 7.1. The *Minkowski functional* associated with A is

$$\mu_A(x) = \inf\{t > 0 : x \in tA\}.$$

It is finite and nonnegative because A is absorbing.

This need not be a linear functional in the sense to be defined later. If $x \in tA$ and $s > t > 0$, write $x = ty$ with $y \in A$. Then

$$x = s \left(\frac{t}{s} y + \frac{s-t}{s} 0 \right) \in sA$$

by convexity. Consequently, $\{t > 0 : x \in tA\}$ is an upper interval, with endpoint $\mu_A(x)$. It follows that

$$\{x : \mu_A(x) < 1\} \subset A \subset \{x : \mu_A(x) \leq 1\}.$$

Indeed, if $\mu_A(x) < 1$, some $0 < t < 1$ has $x \in tA \subset A$; if $x \in A$, then $\mu_A(x) \leq 1$.

Proposition 7.1. For a convex absorbing set A containing zero,

- (1) $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$;
- (2) $\mu_A(tx) = t\mu_A(x)$ for every real $t \geq 0$;
- (3) if A is also balanced, then μ_A is a seminorm.

Proof. Given $\varepsilon > 0$, set $t = \mu_A(x) + \varepsilon$ and $s = \mu_A(y) + \varepsilon$. Then $x/t, y/s \in A$. By convexity,

$$\frac{x+y}{s+t} = \frac{t}{s+t} \frac{x}{t} + \frac{s}{s+t} \frac{y}{s} \in A.$$

Hence $\mu_A(x+y) \leq s+t = \mu_A(x) + \mu_A(y) + 2\varepsilon$. Let $\varepsilon \downarrow 0$.

For $t > 0$, the equivalence $x \in sA \iff tx \in tsA$ gives

$$\mu_A(tx) = \inf\{s > 0 : x \in (s/t)A\} = t\mu_A(x).$$

For $t = 0$, use $\mu_A(0) = 0$. If A is balanced, $\alpha A = A$ whenever $|\alpha| = 1$, since $\alpha A \subset A$ and $\alpha^{-1}A \subset A$. For $\alpha \neq 0$, multiplication by $\alpha/|\alpha|$ therefore leaves A unchanged, so

$$\mu_A(\alpha x) = \mu_A(|\alpha|x) = |\alpha|\mu_A(x).$$

Together with subadditivity, this proves the seminorm property. $\square$

Theorem 7.2. A locally convex TVS whose topology is induced by a complete translation-invariant metric is a Fréchet space.

Proof. Metrizability gives a countable local base $\{W_n\}$ at zero. Local convexity lets us choose convex neighborhoods $U_n \subset W_n$; then choose balanced convex open neighborhoods $V_n \subset U_n$. The V_n form a countable local base. Let $p_n = \mu_{V_n}$. These are seminorms. If $x \neq 0$, choose n with $x \notin V_n$. Since $\{p_n < 1\} \subset V_n$, $p_n(x) \geq 1$, so the family separates points. We claim

$$V_n = \{x : p_n(x) < 1\}.$$

One inclusion was proved above. If $x \in V_n$, openness and continuity of scalar multiplication give $x/t \in V_n$ for some $0 < t < 1$ sufficiently close to one. Thus $p_n(x) \leq t < 1$. Consequently, the V_n are open for the seminorm topology, so the original topology is contained in it.

Conversely, its basic neighborhoods are finite intersections of

$$\{x : p_n(x) < 1/k\} = \frac{1}{k}V_n,$$

which are open in the original topology. Thus the topologies agree. Completeness now proves that X is Fréchet. $\square$

This yields the alternative definition: a Fréchet space is a locally convex TVS with a complete compatible translation-invariant metric.

Theorem 7.3 (Normability criterion). *A TVS is normable if and only if zero has a convex bounded neighborhood.*

Proof. If a compatible norm exists, its open unit ball is convex and bounded. Conversely, let V be a convex bounded neighborhood of zero, and choose a balanced convex open neighborhood $U \subset V$. Then U is bounded. Set $\|x\| = \mu_U(x)$, a seminorm. The sets $\{n^{-1}U\}$ form a local base by boundedness. For $x \neq 0$, the Hausdorff property gives n with $x \notin n^{-1}U$, so $\mu_U(x) \geq 1/n > 0$. Thus this is a norm. Moreover, $\{x : \|x\| < 1/n\} = n^{-1}U$, so the norm topology coincides with the original topology. $\square$

7.3. Hilbert spaces. We conclude the first part of the course with a brief introduction to Hilbert spaces, whose theory will be studied in detail later. The additional structure, compared with the preceding spaces, is that of angles or inner products.

Definition 7.2 (Inner product). An inner product on a real vector space is a symmetric positive definite bilinear form. On a complex vector space it is a function $(\cdot, \cdot) : X \times X \rightarrow \mathbb{C}$ which is linear in the first variable, conjugate-linear in the second, and satisfies

$$(x, y) = \overline{(y, x)}, \quad (x, x) > 0 \quad (x \neq 0).$$

In particular, $(\alpha x, y) = \alpha(x, y)$ and $(x, \alpha y) = \overline{\alpha}(x, y)$. Such an X is an *inner product space*.

Editorial note. The complex symmetry condition is called conjugate symmetry, rather than “skew-symmetry” as written in the source.

Example 7.1. For complex-valued square-integrable functions on $[0, 1]$,

$$(f, g) = \int_0^1 f(x) \overline{g(x)} \, dx$$

is an inner product on $L^2([0, 1])$.

Editorial note. As usual, L^2 consists of equivalence classes modulo equality almost everywhere; this is needed for positive definiteness.

Proposition 7.4. *An inner product induces a norm $\|x\| = (x, x)^{1/2}$.*

Proof. Positive definiteness gives $\|x\| > 0$ for $x \neq 0$. Sesquilinearity gives

$$\|\alpha x\| = (|\alpha|^2(x, x))^{1/2} = |\alpha| \|x\|.$$

The Cauchy–Schwarz inequality $|(x, y)| \leq \|x\| \|y\|$ (proof: exercise) gives

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + 2\operatorname{Re}(x, y) + \|y\|^2 \\ &\leq \|x\|^2 + 2|(x, y)| + \|y\|^2 \leq (\|x\| + \|y\|)^2. \end{aligned}$$

Taking square roots proves the triangle inequality. $\square$

Definition 7.3 (Hilbert space). An inner product space is a *Hilbert space* if its induced norm is complete.

Every Hilbert space is a Banach space. Hilbert spaces have a rich structure closely related to Euclidean space, with many applications in mathematics and physics. The source summarizes the spaces by the following lettered classes:

- A* Vector spaces
- B* Topological vector spaces
- C* Normable TVSs
- D* Metrizable TVSs
- E* Spaces with a complete compatible translation-invariant metric
- F* Banach spaces
- G* Spaces of second category in themselves
- H* Locally convex TVSs
- I* Locally bounded TVSs
- J* TVSs with a countable local base at zero
- K* Fréchet spaces
- L* Hilbert spaces
- M* Inner product spaces

The inclusion diagram can be expressed by

$$\begin{aligned} L &\subset F \subset K \subset E \subset D = J \subset B \subset A, \\ L &\subset M \subset C \subset H \subset B, \quad C \subset I \subset D, \quad E \subset G, \\ K &= H \cap E, \quad C = H \cap I, \quad F = C \cap E = K \cap I, \quad L = E \cap M = F \cap M. \end{aligned}$$

Editorial note. The hand-drawn diagram mixes abstract metric spaces and TVSs. Here E means TVSs complete for a compatible translation-invariant metric (the F -spaces of Lecture 9). The source’s formula $C = H \cap I = K \cap I$ is corrected: $K \cap I$ consists of Banach spaces, since it includes completeness.

Theorem 7.5 (Minimum-norm point in a closed convex set). *Let $\mathcal{H}$ be a Hilbert space, and $E \subset \mathcal{H}$ be nonempty, closed, and convex. There is a unique $x \in E$ such that*

$$\|x\| = \inf\{\|y\| : y \in E\}.$$

Proof. The parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

is an exercise. Set $d = \inf_{y \in E} \|y\|$, and choose $x_n \in E$ with $\|x_n\| \rightarrow d$. Convexity gives $(x_n + x_m)/2 \in E$, so the parallelogram law yields

$$\frac{1}{4} \|x_n - x_m\|^2 + d^2 \leq \frac{1}{2} \|x_n\|^2 + \frac{1}{2} \|x_m\|^2 \rightarrow d^2.$$

Thus $\{x_n\}$ is Cauchy. Completeness and closedness give a limit $x \in E$ with $\|x\| = d$. If also $y \in E$ and $\|y\| = d$, then

$$\frac{1}{4} \|x - y\|^2 + d^2 \leq \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2 = d^2,$$

so $x = y$. □

8. LINEAR OPERATORS AND FINITE-DIMENSIONAL SPACES

8.1. Review (September 25, 2015). The Minkowski functional of a convex absorbing set is subadditive and positively homogeneous. It yields the characterization of Fréchet spaces as locally convex complete metrizable TVSs, and the normability criterion: local convexity together with local boundedness. A Hilbert space is a complete inner product space.

8.2. Linear operators.

Definition 8.1. For vector spaces X, Y over the same field, a map $L : X \rightarrow Y$ is a *linear operator* (linear map or linear transformation) if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

When Y is the scalar field, it is a *linear functional*.

Example 8.1. Multiplication $M_\alpha(x) = \alpha x$ is linear. Translation $T_a(x) = x + a$ is linear only when $a = 0$. The map $f \mapsto \int_0^1 f(x) dx$ on $C([0, 1])$ is a linear functional; $f \mapsto \int_0^1 f(x)^2 dx$ is not linear. In a complex Hilbert space, $x \mapsto (x, y)$ is linear for fixed y , whereas $y \mapsto (x, y)$ is conjugate-linear for fixed x . For fixed $f \in C([0, 1])$, the multiplication operator $M_f : g \mapsto fg$ on $C([0, 1])$ is linear. For $f \in C([0, 2\pi])$, its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z},$$

define a linear map into $\ell^\infty(\mathbb{Z})$, since $|\hat{f}(n)| \leq \|f\|_\infty$. For a vector space of scalar functions on a set S , evaluation $\delta_a(f) = f(a)$ is a linear functional for each $a \in S$. If $\Omega \subset \mathbb{R}^n$ is open and bounded and $V \in C^\infty(\Omega)$, then

$$\Delta + V : C^\infty(\Omega) \rightarrow C^\infty(\Omega), \quad f \mapsto \Delta f + Vf, \quad \Delta f = \sum_{i=1}^n \partial_{x_i}^2 f,$$

is linear. An integral kernel gives another example:

$$(Kf)(x) = \int_\Omega k(x, y) f(y) dy.$$

Editorial note. For the integral-kernel example, the source states $k \in C(\Omega \times \Omega)$ but does not specify a domain for f or hypotheses ensuring integrability. The formula is linear whenever the integrals are defined, for example for $k \in C(\overline{\Omega} \times \overline{\Omega})$ and $f \in C(\overline{\Omega})$ on bounded Ω .

For a linear map $L : X \rightarrow Y$ the following properties are easy to prove:

- (1) $L0 = 0$.
- (2) The image of a linear subspace is a linear subspace; in particular, $\text{Im } L = L(X)$ is a subspace of Y .
- (3) The inverse image of a linear subspace is a linear subspace; in particular, $\ker L = L^{-1}(\{0\})$ is a subspace of X .
- (4) Images and inverse images preserve convexity and balancedness.

The linear maps $X \rightarrow Y$ form a vector space under pointwise operations. Compositions of linear maps are linear. Invertibility and linear isomorphisms are defined as in linear algebra. Many finite-dimensional concepts extend to infinite dimensions; others, such as trace and determinant, require special classes of operators.

When X, Y carry TVS structures, a natural question is when a linear map is continuous.

Proposition 8.1. *A linear map $L : X \rightarrow Y$ between TVSs is continuous if it is continuous at zero.*

Proof. Let $L(x) = y$ and let V be a neighborhood of y . Then $-y + V$ is a neighborhood of zero. By continuity at zero, choose a neighborhood U of zero in X with $L(U) \subset -y + V$. Thus

$$L(x + U) = L(x) + L(U) \subset y + (-y + V) = V.$$

□

Remark 8.1. This argument also gives uniform continuity in the vector-space sense: for every neighborhood V of zero in Y , one neighborhood U of zero in X works at all points, since $x_1 - x_2 \in U$ implies $L(x_1) - L(x_2) \in V$.

Definition 8.2 (Bounded linear operator). A linear map between TVSs is *bounded* if it takes bounded sets to bounded sets. This does not mean a globally bounded scalar-valued function; a nonzero linear functional cannot have globally bounded absolute value.

Proposition 8.2. *Every continuous linear map between TVSs is bounded.*

Proof. Let $E \subset X$ be bounded and V be a neighborhood of zero in Y . Choose a neighborhood U of zero with $L(U) \subset V$. For some $s > 0$ and all $t > s$, $E \subset tU$. Hence $L(E) \subset L(tU) = tL(U) \subset tV$, so $L(E)$ is bounded. □

Theorem 8.3. *Let X be a metrizable TVS, Y any TVS, and $L : X \rightarrow Y$ linear. The following are equivalent:*

- (1) L is continuous;
- (2) L is bounded;
- (3) $x_n \rightarrow 0$ implies $\{Lx_n : n \geq 1\}$ is bounded;
- (4) $x_n \rightarrow 0$ implies $Lx_n \rightarrow 0$;
- (5) L is continuous at zero.

Proof. We have proved (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (1). For (2) $\Rightarrow$ (3), a convergent sequence in any TVS is bounded (prove this as an exercise), so its image is bounded. For (3) $\Rightarrow$ (4), take a compatible translation-invariant metric d on X . For $x_n \rightarrow 0$, write $d_n = d(x_n, 0)$ and choose the positive integer

$$\gamma_n = \left\lfloor \frac{1}{\sqrt{d_n + 1/n}} \right\rfloor + 1.$$

Then $\gamma_n \rightarrow \infty$, and the triangle inequality with translation invariance gives

$$d(\gamma_n x_n, 0) \leq \gamma_n d_n \leq \frac{d_n}{\sqrt{d_n + 1/n}} + d_n \leq \sqrt{d_n + 1/n} + d_n \rightarrow 0.$$

Thus $\{\gamma_n Lx_n\}$ is bounded by (3). Since $1/\gamma_n \rightarrow 0$, the bounded-set criterion (Problem Set 3, Part 1, Problem 3(2)) gives $Lx_n \rightarrow 0$. Finally, if (5) fails, some neighborhood V of zero in Y has no neighborhood of zero mapped into it. Choose a decreasing countable local base U_n at zero and points $x_n \in U_n$ with $Lx_n \notin V$. Then $x_n \rightarrow 0$ but $Lx_n \not\rightarrow 0$, contradicting (4). □

8.3. Finite-dimensional vector spaces. For simplicity use the scalar field $\mathbb{C}$; the same conclusions hold over $\mathbb{R}$. Give $\mathbb{C}^n$ its standard norm topology.

Proposition 8.4. *Every linear map $L : \mathbb{C}^n \rightarrow Y$ into a complex TVS is continuous.*

Proof. For a basis $e_1, \dots, e_n$, if $z_k = \sum_{i=1}^n z_k^{(i)} e_i \rightarrow 0$, then all coordinates tend to zero. Thus $L(z_k) = \sum_i z_k^{(i)} L(e_i) \rightarrow 0$ by continuity of the vector operations. The preceding theorem gives continuity. $\square$

Theorem 8.5. *Let Y be an n -dimensional subspace of a complex TVS X . Every linear isomorphism $L : \mathbb{C}^n \rightarrow Y$ is a homeomorphism.*

Proof. L is continuous by the proposition. To show that L^{-1} is continuous, it suffices to check zero. The unit sphere $S \subset \mathbb{C}^n$ is compact, so $K = L(S)$ is compact. Since L is injective, $0 \notin K$. In the Hausdorff space Y , compact sets are closed, so choose a balanced open neighborhood V of zero disjoint from K . Then $E = L^{-1}(V)$ is balanced, contains zero, and avoids S . Hence $E \subset B(0, 1)$; otherwise a vector of norm at least one could be scaled into S while staying in E . For $\varepsilon > 0$,

$$L^{-1}(\varepsilon V) = \varepsilon E \subset B(0, \varepsilon).$$

This proves continuity of L^{-1} at zero. $\square$

Corollary 8.1. *There is only one Hausdorff topology on $\mathbb{C}^n$ making its usual vector operations continuous.*

Proof. The identity from standard $\mathbb{C}^n$ to $\mathbb{C}^n$ with any TVS topology is a linear isomorphism, hence a homeomorphism by the theorem. Therefore that topology equals the standard one. $\square$

Corollary 8.2. *Any two norms on $\mathbb{C}^n$ are equivalent: there exist $c_1, c_2 > 0$ such that*

$$c_1 \|z\|_1 \leq \|z\|_2 \leq c_2 \|z\|_1.$$

Proof. The two norm topologies coincide. Continuity of the identity implies that some ball $\{z : \|z\|_2 < r\}$ is contained in $\{z : \|z\|_1 < 1\}$. Homogeneity gives $\|z\|_1 \leq r^{-1} \|z\|_2$. Reverse the roles of the two norms for the other inequality. $\square$

Corollary 8.3. *Every norm on $\mathbb{C}^n$ is complete.*

Proof. It is equivalent to the complete standard norm. $\square$

Corollary 8.4. *Every finite-dimensional subspace Y of a TVS X is closed.*

Proof. Let $L : \mathbb{C}^n \rightarrow Y$ be a linear isomorphism and homeomorphism. For a convergent sequence $y_k \in Y$, $y_k \rightarrow y \in X$, take any neighborhood U of zero in $\mathbb{C}^n$. Choose a symmetric neighborhood V of zero in X such that $(V + V) \cap Y \subset L(U)$. Eventually $y_k \in y + V$, so $y_k - y_l \in (V + V) \cap Y$ and $L^{-1}(y_k) - L^{-1}(y_l) \in U$. Thus $L^{-1}(y_k)$ is Cauchy in $\mathbb{C}^n$, with limit z . Continuity gives $y_k \rightarrow L(z)$; uniqueness implies $y = L(z) \in Y$. For general X , apply the same argument to any net (y_λ) in Y converging in X . Its inverse image is a Cauchy net in the complete finite-dimensional Euclidean space and hence converges. This proves that all closure points of Y lie in Y , so Y is closed. $\square$

Editorial note. The source gives the sequence argument alone. For a general TVS, sequential closedness does not suffice; the last paragraph extends the argument to nets. In a metrizable X the sequence argument already proves the result.

9. THE BANACH–STEINHAUS THEOREM

9.1. Review (September 29, 2015). For a linear map between TVSs, continuity at zero implies continuity, and continuity implies boundedness. If the domain is metrizable, boundedness also implies continuity. Finite-dimensional vector spaces have a unique compatible Hausdorff vector-space topology; all their norms are equivalent and complete, and they are closed in every TVS containing them.

9.2. Equicontinuity. Recall the uniform form of equicontinuity for a family $\Lambda \subset C(\Omega)$:

$$\forall \varepsilon > 0 \exists \delta > 0 : |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon \quad (f \in \Lambda).$$

The analogous notion for linear maps is particularly natural because continuity at zero implies uniform continuity.

Definition 9.1. A family Λ of linear maps from a TVS X to a TVS Y is *equicontinuous* if for every neighborhood V of zero in Y , there is a neighborhood U of zero in X such that $L(U) \subset V$ for every $L \in \Lambda$.

By linearity, $x_1 - x_2 \in U$ then implies $L(x_1) - L(x_2) \in V$ simultaneously for all $L \in \Lambda$. For a one-element family, equicontinuity is continuity. Every member of an equicontinuous family is continuous and hence bounded.

Proposition 9.1. *If Λ is an equicontinuous family of linear maps $X \rightarrow Y$, then for each bounded set $E \subset X$, there is a bounded set $F \subset Y$ with $L(E) \subset F$ for every $L \in \Lambda$.*

Proof. Put $F = \bigcup_{L \in \Lambda} L(E)$. Given a neighborhood V of zero in Y , choose a neighborhood U of zero in X such that $L(U) \subset V$ for all L . For some $s > 0$ and every $t > s$, $E \subset tU$. Thus $L(E) \subset tL(U) \subset tV$ for every L , so $F \subset tV$. This proves boundedness. $\square$

Three fundamental theorems (in fact four, as usually grouped) form “the big three” of functional analysis. They are applications of the Baire category theorem. Recall that a set is of first category if it is a countable union of nowhere dense sets, and of second category otherwise. Nonempty complete metric spaces are of second category; countable intersections of dense open subsets remain dense.

Definition 9.2 (F-space). An *F-space* is a metrizable TVS complete for a compatible translation-invariant metric. Equivalently, every such metric is complete.

A Fréchet space is precisely a locally convex F-space. Thus

$$\text{Hilbert} \subset \text{Banach} \subset \text{Fréchet} \subset \text{F-space} \subset \text{metrizable TVS}.$$

Theorem 9.2 (Banach–Steinhaus). *Let X be an F-space, Y any TVS, and Λ a family of continuous linear maps $X \rightarrow Y$. If Λ is pointwise bounded, meaning*

$$\Lambda_x = \{L(x) : L \in \Lambda\} \text{ is bounded in } Y \text{ for every } x \in X,$$

then Λ is equicontinuous.

By Proposition 9.1, the conclusion also implies uniform boundedness on each bounded subset of X . Conversely, equicontinuity implies pointwise boundedness, since singletons are bounded in every TVS.

Proof of Theorem 9.2. Given a neighborhood V of zero in Y , choose a closed balanced neighborhood W with $W + W \subset V$. To justify this choice, observe that for any subset S and neighborhood U of zero,

$$\overline{S} \subset S + U.$$

Indeed, if $y \notin S + U$, the neighborhood $y - U$ contains no point of S , so $y \notin \overline{S}$. Choose a balanced open neighborhood O with $O + O + O + O \subset V$, and set $W = \overline{O}$. Then $W \subset O + O$, so $W + W \subset V$; closure preserves balancedness.

Editorial note. This spells out the closed-neighborhood construction in the source margin.

Pointwise boundedness implies that for each x , some integer n has $L(x) \in nW$ for every $L \in \Lambda$. Define

$$X_n = \{x \in X : L(x) \in nW \text{ for every } L \in \Lambda\} = \bigcap_{L \in \Lambda} L^{-1}(nW).$$

Then $X = \bigcup_n X_n$ and every X_n is closed, by continuity of the maps and closedness of nW . Baire's theorem gives an n with $\text{Int } X_n \neq \emptyset$. Set

$$U_0 = \frac{1}{n} \text{Int } X_n, \quad U = U_0 - U_0.$$

The set U is an open neighborhood of zero, and $L(U_0) \subset W$ for all L . Since W is symmetric,

$$L(U) \subset L(U_0) - L(U_0) \subset W + W \subset V.$$

Thus Λ is equicontinuous. □

The same argument gives a more general statement.

Theorem 9.3 (General Banach–Steinhaus theorem). *Let X, Y be TVSs and Λ a family of continuous linear maps $X \rightarrow Y$. If*

$$B = \{x \in X : \Lambda_x \text{ is bounded in } Y\}$$

is of second category in X , then Λ is equicontinuous and $B = X$.

Corollary 9.1 (Pointwise limits). *Let X be an F-space, Y a TVS, and $L_n : X \rightarrow Y$ continuous and linear. If $L(x) = \lim_{n \rightarrow \infty} L_n(x)$ exists for every $x \in X$, then $L : X \rightarrow Y$ is continuous and linear.*

Proof. Linearity follows by taking limits in the vector operations. Convergent sequences in a TVS are bounded, so Banach–Steinhaus implies that $\{L_n\}$ is equicontinuous. Given a neighborhood V of zero in Y , choose a closed neighborhood $W \subset V$. Equicontinuity gives a neighborhood U of zero in X with $L_n(U) \subset W$ for every n . Taking pointwise limits gives $L(U) \subset W \subset V$. Thus L is continuous at zero, and hence everywhere. □

Editorial note. The source obtains $L(U) \subset \overline{V}$ after passing to the limit. Choosing a smaller closed neighborhood first makes the continuity conclusion explicit.

Remark 9.1. More generally, if $B = \{x : \lim_n L_n(x) \text{ exists}\}$ is of second category in X and Y is an F-space, then $B = X$ and the resulting L is continuous.

9.3. The case of normed spaces. A subset F of a normed space Y is bounded exactly when $\sup_{y \in F} \|y\| < \infty$. For normed spaces X, Y , a linear map $L : X \rightarrow Y$ is bounded if and only if for each $c_1 > 0$, some $c_2 > 0$ satisfies

$$L(B_X(0, c_1)) \subset B_Y(0, c_2).$$

By linearity it suffices to check the unit ball. Bounded linear maps are precisely continuous linear maps. Write

$$\mathcal{L}(X, Y) = \{L : X \rightarrow Y : L \text{ is continuous and linear}\}.$$

This is a vector space, with the *operator norm*

$$\|L\| = \sup_{\|x\|_X=1} \|Lx\|_Y = \sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X}.$$

For $X = \{0\}$ the operator norm is zero.

Proposition 9.4. *The operator norm is a norm on $\mathcal{L}(X, Y)$.*

Proof. Exercise, as in the source. □

Theorem 9.5 (Uniform boundedness principle). *Let X be a Banach space, Y a normed space, and $\Lambda \subset \mathcal{L}(X, Y)$. If for each $x \in X$,*

$$\sup_{L \in \Lambda} \|Lx\|_Y < \infty,$$

then $\sup_{L \in \Lambda} \|L\| < \infty$.

Proof. Banach–Steinhaus gives equicontinuity. Apply Proposition 9.1 to the unit sphere of X : its images under all $L \in \Lambda$ lie in one bounded subset of Y . Thus $\|L\|$ has a common finite upper bound. □

Remark 9.2. This is also called the *resonance theorem*, since its contrapositive says

$$\sup_{L \in \Lambda} \|L\| = \infty \implies \exists x_0 \in X : \sup_{L \in \Lambda} \|Lx_0\|_Y = \infty.$$

10. THE OPEN MAPPING THEOREM

10.1. Review (October 9, 2015). The Banach–Steinhaus theorem says that a pointwise bounded family of continuous linear maps from an F-space to a TVS is equicontinuous and therefore uniformly bounded on bounded sets. For a Banach domain and a normed codomain, this becomes the uniform boundedness principle: pointwise boundedness implies a common bound for the operator norms.

10.2. Open maps. Recall that a map is continuous if inverse images of open sets are open.

Definition 10.1. A map $f : X \rightarrow Y$ between topological spaces is an *open map* if $f(U)$ is open in Y whenever U is open in X .

Example 10.1. A homeomorphism and its inverse are both open maps. For topological spaces X, Y , the coordinate projections $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are open, since open rectangles form a base of the product topology.

Proposition 10.1. *An open linear map $L : X \rightarrow Y$ between TVSs is surjective.*

Proof. $L(X)$ is an open vector subspace of Y and contains zero. It is therefore absorbing, so

$$Y = \bigcup_{k=1}^{\infty} kL(X) = L(X).$$

□

Lemma 10.1. *A linear map $L : X \rightarrow Y$ between TVSs is open if and only if it is open at zero: for each neighborhood U of zero in X , the set $L(U)$ contains a neighborhood of zero in Y .*

Proof. Only sufficiency needs proof. If U is open and $y \in L(U)$, choose $x \in U$ with $Lx = y$. The set $-x + U$ is a neighborhood of zero, so some neighborhood V of zero in Y satisfies

$$V \subset L(-x + U) = -y + L(U).$$

Thus $y + V \subset L(U)$ and $L(U)$ is open. □

Theorem 10.2 (Open mapping theorem). *Let X be an F-space, Y a TVS, and $L : X \rightarrow Y$ a continuous linear map. If $L(X)$ is of second category in Y , then L is open and $L(X) = Y$.*

Proof. Fix a complete compatible translation-invariant metric d on X . It suffices to show that $L(U)$ contains a neighborhood of zero for each $U = B_d(0, r)$. Since $X = \bigcup_{k \geq 1} kU$,

$$L(X) = \bigcup_{k \geq 1} kL(U).$$

The second-category hypothesis implies that some $\overline{kL(U)} = \overline{kL(U)}$ has nonempty interior. Hence $\overline{L(U)}$ has nonempty interior. Put $U_n = B_d(0, 2^{-n}r)$. By the same argument, each $\overline{L(U_n)}$ has nonempty interior. Translation invariance and the triangle inequality give $U_{n+1} - U_{n+1} \subset U_n$. If W_n is a nonempty open set contained in $\overline{L(U_{n+1})}$, then

$$\overline{L(U_n)} \supset \overline{L(U_{n+1}) - L(U_{n+1})} \supset \overline{L(U_{n+1})} - \overline{L(U_{n+1})} \supset W_n - W_n.$$

Thus every $\overline{L(U_n)}$ contains a neighborhood of zero.

We show that $\overline{L(U_1)} \subset L(U)$. Take $y_1 \in \overline{L(U_1)}$. Because $\overline{L(U_2)}$ contains a neighborhood of zero, choose $x_1 \in U_1$ such that $y_2 = y_1 - Lx_1 \in \overline{L(U_2)}$. Inductively choose $x_n \in U_n$ and $y_n \in \overline{L(U_n)}$ with

$$y_{n+1} = y_n - Lx_n.$$

Since $d(x_n, 0) < 2^{-n}r$, the partial sums $s_n = \sum_{k=1}^n x_k$ are Cauchy. Completeness gives $s_n \rightarrow x$. Moreover,

$$d(x, 0) \leq d(x_1, 0) + \sum_{k=2}^{\infty} 2^{-k}r < \frac{r}{2} + \frac{r}{2} = r,$$

so $x \in U$. We also have $y_n \rightarrow 0$. Indeed, for any neighborhood V of zero in Y , choose an open neighborhood W with $W + W \subset V$. Continuity of L gives $L(U_n) \subset W$ for large n , and hence $\overline{L(U_n)} \subset \overline{W} \subset W + W \subset V$. Finally,

$$Ls_n = \sum_{k=1}^n Lx_k = y_1 - y_{n+1} \rightarrow y_1.$$

By continuity $Ls_n \rightarrow Lx$, so $y_1 = Lx \in L(U)$. Since $\overline{L(U_1)}$ contains a neighborhood of zero, so does $L(U)$. The lemma proves openness, and the preceding proposition gives surjectivity. $\square$

Corollary 10.1. *Every continuous surjective linear map between F -spaces is open.*

Proof. Its image is the complete metric space Y , which is of second category. $\square$

Corollary 10.2 (Bounded inverse theorem). *A continuous bijective linear map between F -spaces has continuous inverse.*

Proof. It is bijective and open. $\square$

Corollary 10.3. *If X, Y are Banach spaces and $L : X \rightarrow Y$ is continuous, linear, and bijective, there are $c_1, c_2 > 0$ such that*

$$c_1 \|x\|_X \leq \|Lx\|_Y \leq c_2 \|x\|_X.$$

Proof. L is bounded, giving the upper estimate. The inverse is continuous and bounded, so $\|x\|_X \leq C \|Lx\|_Y$, giving the lower estimate. $\square$

Corollary 10.4. *Suppose two norms $\|\cdot\|_1, \|\cdot\|_2$ make X Banach and $\|x\|_1 \leq C_1 \|x\|_2$ for all x . Then there is $C_2 > 0$ with $C_2 \|x\|_2 \leq \|x\|_1$ for all x .*

Proof. Apply the bounded inverse theorem to the identity map from $(X, \|\cdot\|_2)$ to $(X, \|\cdot\|_1)$. $\square$

11. CLOSED GRAPHS AND CLOSED OPERATORS

11.1. Review (October 16, 2015). The open mapping theorem says that a continuous linear map from an F-space whose image is of second category in the target TVS is open, hence surjective. In particular, a continuous bijective linear map between F-spaces has continuous inverse. (Problem Set 5, Part 1 also shows that the target in the open mapping theorem is an F-space.) Today we study applications.

11.2. The closed graph theorem.

Definition 11.1 (Graph). For a map $f : X \rightarrow Y$, its graph is

$$\Gamma_f = \{(x, f(x)) : x \in X\} \subset X \times Y.$$

It is a *closed graph* if it is closed in the product topology.

Proposition 11.1. *If X is a topological space, Y is Hausdorff, and $f : X \rightarrow Y$ is continuous, then Γ_f is closed.*

Proof. We show that $\Omega = (X \times Y) \setminus \Gamma_f$ is open. If $(x_0, y_0) \in \Omega$, then $y_0 \neq f(x_0)$. Choose disjoint open neighborhoods V of y_0 and W of $f(x_0)$. By continuity choose a neighborhood U of x_0 with $f(U) \subset W$. Then $U \times V$ contains (x_0, y_0) and is disjoint from the graph: if $x \in U$ and $y = f(x)$, then $y \in W$, so $y \notin V$. $\square$

Remark 11.1. The Hausdorff hypothesis is needed. The identity map on any space is continuous, whereas its graph, the diagonal $\Delta = \{(x, x) : x \in X\}$, is closed only if X is Hausdorff. Why? Conversely, if $K \subset \mathbb{R}^n$ is compact, a map $f : K \rightarrow K$ with closed graph is continuous.

Theorem 11.2 (Closed graph theorem). *Let X, Y be F-spaces and $L : X \rightarrow Y$ linear. If Γ_L is closed in $X \times Y$, then L is continuous.*

Proof. Let d_X, d_Y be complete compatible translation-invariant metrics. Then

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

is a complete translation-invariant metric compatible with the product topology (exercise). Thus $X \times Y$ is an F-space. The graph Γ_L is a vector subspace, since L is linear, and is closed. A closed subset of a complete metric space is complete (Problem Set 2, Part 1, Problem 1(2)), so Γ_L is an F-space. The projection

$$\pi_1 : \Gamma_L \rightarrow X, \quad (x, Lx) \mapsto x$$

is continuous, linear, and bijective. The inverse mapping theorem makes π_1^{-1} continuous. Since the coordinate projection π_Y is continuous and $L = \pi_Y \circ \pi_1^{-1}$, L is continuous. $\square$

Remark 11.2. In this metrizable setting, graph closedness is equivalent to: if $x_n \rightarrow x$ and $Lx_n \rightarrow y$, then $y = Lx$. We used the inverse mapping theorem to prove the closed graph theorem. Conversely, if L is continuous and bijective, its graph is closed; swapping coordinates shows that $\Gamma_{L^{-1}}$ is closed, so the closed graph theorem gives continuity of L^{-1} . One can also prove the closed graph theorem directly, and use it to prove the version of the open mapping theorem for continuous surjective linear maps between F-spaces. Thus these formulations are equivalent.

11.3. Closed operators. Let X, Y be Banach spaces. In applications, a linear operator may only be defined on a vector subspace $\text{Dom}(L) \subset X$. Its graph is then

$$\Gamma_L = \{(x, Lx) : x \in \text{Dom}(L)\} \subset X \times Y.$$

Definition 11.2 (Closed operator). L is a *closed operator* (a closed graph operator in the source) if Γ_L is closed in $X \times Y$. Equivalently, if $x_n \in \text{Dom}(L)$, $x_n \rightarrow x$ in X , and $Lx_n \rightarrow y$ in Y , then $x \in \text{Dom}(L)$ and $Lx = y$.

Example 11.1. If $L : X \rightarrow Y$ is injective, continuous, and linear, then $L^{-1} : \text{Im } L \rightarrow X$ is a closed operator, since its graph is obtained from the closed graph of L by swapping coordinates.

Example 11.2. On $X = C([0, 1])$ with its uniform norm, differentiation $D = d/dx$ with domain $C^1([0, 1])$ is closed: uniform convergence $f_n \rightarrow f$ and $f'_n \rightarrow g$ implies $f \in C^1$ and $f' = g$. It is unbounded for the uniform norm, since

$$\|D(x^n)\|_\infty = n \|x^n\|_\infty = n.$$

In particular, it is not continuous on its domain equipped with the norm inherited from X .

The domain of an operator need not be closed in X .

Proposition 11.3. Let X be a normed space, Y a Banach space, and $L : \text{Dom}(L) \rightarrow Y$ continuous and linear for the inherited norm. There is a continuous linear extension

$$\bar{L} : \overline{\text{Dom}(L)} \rightarrow Y, \quad \|\bar{L}\| = \|L\|.$$

Proof. For $x \in \overline{\text{Dom}(L)}$, choose $x_n \in \text{Dom}(L)$ with $x_n \rightarrow x$. Boundedness gives $\|Lx_n - Lx_m\| \leq C \|x_n - x_m\|$, so Lx_n is Cauchy and has a limit in Y . Define $\bar{L}x = \lim_n Lx_n$. The limit is independent of the approximating sequence: interlace any two choices, or use the same norm estimate. Passing to limits proves linearity and $\|\bar{L}x\| \leq \|L\| \|x\|$. Thus $\|\bar{L}\| \leq \|L\|$; restriction gives the reverse inequality. $\square$

Definition 11.3 (Closable operator). An operator L *admits a closure*, or is *closable*, if $\overline{\Gamma_L}$ is the graph of an operator. Every continuous linear operator on a normed domain with Banach target is closable.

Theorem 11.4 (Hörmander). Let X, Y, Z be Banach spaces and $L_1 : \text{Dom}(L_1) \subset X \rightarrow Y$, $L_2 : \text{Dom}(L_2) \subset X \rightarrow Z$ be linear operators with $\text{Dom}(L_1) \subset \text{Dom}(L_2)$. Suppose L_1 is closed and L_2 is closable. Then there is $C > 0$ such that

$$\|L_2x\|_Z \leq C(\|L_1x\|_Y + \|x\|_X) \quad (x \in \text{Dom}(L_1)).$$

Proof. Γ_{L_1} is a closed subspace of the Banach space $X \times Y$, with norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$, so it is Banach. Define

$$L_3 : \Gamma_{L_1} \rightarrow Z, \quad L_3(x, L_1x) = L_2x.$$

This is linear and everywhere defined. We show its graph is closed. Suppose $(x_n, L_1x_n) \rightarrow (x, y)$ and $L_2x_n \rightarrow z$. Closedness of L_1 gives $x \in \text{Dom}(L_1)$ and $y = L_1x$. The closure of L_2 then satisfies $\bar{L}_2x = z$. Since $x \in \text{Dom}(L_2)$, $z = L_2x = L_3(x, y)$. The closed graph theorem gives boundedness of L_3 , which is exactly

$$\|L_2x\|_Z \leq C(\|x\|_X + \|L_1x\|_Y).$$

$\square$

Example 11.3. Take $X = Y = Z = C([0, 1])$, $L_1 = \mathbf{d}^2/\mathbf{d}x^2$ on $C^2([0, 1])$, and $L_2 = \mathbf{d}/\mathbf{d}x$ on $C^1([0, 1])$. The theorem yields

$$\|f'\|_\infty \leq C(\|f''\|_\infty + \|f\|_\infty) \quad (f \in C^2([0, 1])).$$

12. THE HAHN–BANACH EXTENSION THEOREM

12.1. Review (October 20, 2015). For F -spaces X, Y , a linear map $L : X \rightarrow Y$ is continuous if and only if its graph is closed. Equivalently, $x_n \rightarrow x$ and $Lx_n \rightarrow y$ must imply $y = Lx$. Today we prove Hahn–Banach.

12.2. Order and the axiom of choice. Unlike the Banach–Steinhaus and open mapping theorems, whose proofs rely on completeness and category, the Hahn–Banach theorem is proved using a form of the axiom of choice. We use Zorn’s lemma.

Definition 12.1 (Partial order). A set P with a relation $\preceq$ is partially ordered if the relation is transitive, reflexive, and antisymmetric:

$$a \preceq b, b \preceq c \Rightarrow a \preceq c, \quad a \preceq a, \quad a \preceq b, b \preceq a \Rightarrow a = b.$$

A subset $Q \subset P$ is *totally ordered*, or a *chain*, if any two of its elements are comparable. An upper bound for Q is an element $a \in P$ such that $b \preceq a$ for all $b \in Q$. An element $a \in P$ is *maximal* if $a \preceq b$ implies $a = b$.

Lemma (Zorn’s lemma). *If a nonempty partially ordered set has an upper bound for every chain, then it has a maximal element.*

An equivalent statement is the Hausdorff maximality theorem: every nonempty partially ordered set contains a chain maximal among chains.

12.3. The real Hahn–Banach theorem.

Theorem 12.1 (Hahn–Banach, real version). *Let X be a real vector space and $p : X \rightarrow \mathbb{R}$ satisfy*

$$p(x + y) \leq p(x) + p(y), \quad p(tx) = tp(x) \quad (t \geq 0).$$

Let $Y \subset X$ be a vector subspace and $\ell : Y \rightarrow \mathbb{R}$ linear with $\ell(y) \leq p(y)$ for all $y \in Y$. Then ℓ has a linear extension $\hat{\ell} : X \rightarrow \mathbb{R}$ satisfying $\hat{\ell}(x) \leq p(x)$ for all x .

The hypotheses on p are weaker than being a seminorm.

Proof. First extend by one dimension. If $Y \neq X$, choose $x_0 \notin Y$ and set $Z = Y + \mathbb{R}x_0$. For $y_1, y_2 \in Y$,

$$\ell(y_1) + \ell(y_2) = \ell(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x_0) + p(x_0 + y_2).$$

Thus

$$\ell(y_1) - p(y_1 - x_0) \leq p(x_0 + y_2) - \ell(y_2).$$

The left-hand expressions have a finite supremum a_0 , satisfying

$$\ell(y_1) - p(y_1 - x_0) \leq a_0 \leq p(x_0 + y_2) - \ell(y_2).$$

Define $\tilde{\ell}(y + \alpha x_0) = \ell(y) + \alpha a_0$. The representation is unique, so this defines a linear extension. If $\alpha > 0$, then

$$\tilde{\ell}(y + \alpha x_0) = \alpha(\ell(y/\alpha) + a_0) \leq \alpha p(y/\alpha + x_0) = p(y + \alpha x_0).$$

If $\alpha < 0$, put $b = -\alpha > 0$. Then

$$\tilde{\ell}(y + \alpha x_0) = b(\ell(y/b) - a_0) \leq bp(y/b - x_0) = p(y + \alpha x_0).$$

The case $\alpha = 0$ is the original domination.

Now let P be all pairs $(Z, \tilde{\ell})$ where Z is a subspace containing Y and $\tilde{\ell} : Z \rightarrow \mathbb{R}$ extends ℓ and is dominated by p . This set is nonempty, since $(Y, \ell) \in P$. Order it by extension:

$$(Z_1, \ell_1) \preceq (Z_2, \ell_2) \iff Z_1 \subset Z_2 \text{ and } \ell_2|_{Z_1} = \ell_1.$$

For a chain $\{(Z_\nu, \ell_\nu)\}$, put $Z = \bigcup_\nu Z_\nu$. This is a vector subspace: any two vectors lie together in one member of the chain. The formula $\tilde{\ell}(z) = \ell_\nu(z)$ for $z \in Z_\nu$ is well-defined by compatibility of the chain, linear, and dominated by p . Hence $(Z, \tilde{\ell})$ is an upper bound. Zorn's lemma gives a maximal element. The one-dimensional extension argument shows that its domain cannot be a proper subspace of X . It therefore provides the required extension to all of X . $\square$

Theorem 12.2 (Hahn–Banach, complex version). *Let X be a complex vector space, p a seminorm, $Y \subset X$ a complex subspace, and $\ell : Y \rightarrow \mathbb{C}$ complex-linear with $|\ell(y)| \leq p(y)$. There is a complex-linear extension $\hat{\ell} : X \rightarrow \mathbb{C}$ with $|\hat{\ell}(x)| \leq p(x)$ for every $x \in X$.*

Proof. Exercise, using the real version, as in the source. $\square$

12.4. Immediate applications. Hahn–Banach is an existence theorem rather than a constructive procedure; its extension is generally not unique. The set of extensions satisfying the same domination is convex.

Example 12.1. Let X be all bounded real-valued functions on $[0, 1]$, with no measurability requirement, and let $Y = C([0, 1])$. Set

$$p(f) = \sup_{x \in [0, 1]} |f(x)|, \quad \ell(f) = \int_0^1 f(x) \, dx \quad (f \in Y).$$

The Hahn–Banach hypotheses hold, so integration on continuous functions extends to a dominated linear functional on all bounded functions.

Editorial note. This extends the integration functional; it does not assert that every bounded function becomes Lebesgue measurable or integrable, nor that the extension has countable-additivity properties.

For normed spaces X, Y , recall that $\mathcal{L}(X, Y)$ with the operator norm is normed, and is Banach if Y is Banach.

Definition 12.2 (Dual space). The *dual space* of a normed space X over $\mathbb{F}$ is $X^* = \mathcal{L}(X, \mathbb{F})$, with dual norm

$$\|L\|_{X^*} = \sup_{\|x\|=1} |Lx|.$$

In particular, X^* is Banach even if X is incomplete.

Corollary 12.1 (Norm-preserving extension). *Let $Y \subset X$ be a subspace of a normed space and $\ell \in Y^*$. There is $L \in X^*$ extending ℓ with $\|L\|_{X^*} = \|\ell\|_{Y^*}$.*

Proof. Use $p(x) = \|\ell\|_{Y^*} \|x\|_X$, which is a seminorm and dominates $|\ell|$ on Y . Hahn–Banach gives an extension with norm at most $\|\ell\|$. Restriction gives the reverse inequality. $\square$

Corollary 12.2 (Norming functional). *For every nonzero x_0 in a normed space X , there is $L \in X^*$ with $\|L\| = 1$ and $Lx_0 = \|x_0\|$.*

Proof. On $Y = \mathbb{F}x_0$, define $\ell(\alpha x_0) = \alpha \|x_0\|$. Then $\|\ell\| = 1$; extend it by the preceding corollary. $\square$

Editorial note. The nonzero assumption makes the one-dimensional proof valid. For $x_0 = 0$, the conclusion still holds when $X \neq \{0\}$ by choosing a norming functional at any nonzero vector; it fails for $X = \{0\}$.

13. THE GEOMETRIC HAHN–BANACH THEOREM

13.1. Review (October 23, 2015). The real Hahn–Banach theorem concerns a real vector space X , a sublinear function $p : X \rightarrow \mathbb{R}$ (subadditive and positively homogeneous), a subspace Y , and a linear functional $\ell : Y \rightarrow \mathbb{R}$ with $\ell \leq p$ on Y . It gives an extension $L : X \rightarrow \mathbb{R}$ with $L|_Y = \ell$ and $L \leq p$. There is no topology in this statement and no assertion of continuity. For a normed space, every $\ell \in Y^*$ extends to $L \in X^*$ with $\|L\|_{X^*} = \|\ell\|_{Y^*}$. In particular, for $x_0 \neq 0$ there is $L \in X^*$ of norm one with $Lx_0 = \|x_0\|$. Consequences (quiz) are that X^* separates points, and separates a point x_0 from a closed subspace Y not containing it: there is $L \in X^*$ with $Lx_0 = 1$ and $L|_Y = 0$.

13.2. Separating convex sets in a TVS. Let X be real and $\ell : X \rightarrow \mathbb{R}$ be nonzero and linear. Its image is a nonzero subspace of $\mathbb{R}$, hence it is surjective. For $c \in \mathbb{R}$, the set

$$H_{\ell,c} = \{x \in X : \ell(x) = c\}$$

is a *hyperplane*. Different level sets differ by translation: if $\ell(x_0) = c - c'$, then $H_{\ell,c} = H_{\ell,c'} + x_0$.

Editorial note. The translation sign in the handwritten formula is corrected so that applying ℓ to the translated points gives c .

Continuity is not part of the definition of a hyperplane.

Proposition 13.1. *In a real TVS, a hyperplane $H_{\ell,c}$ is closed if and only if ℓ is continuous.*

Proof. If ℓ is continuous, $H_{\ell,c} = \ell^{-1}(\{c\})$ is closed. Conversely, if one level set is closed, all are closed by translation. For $\varepsilon > 0$, choose a balanced neighborhood U of zero disjoint from $H_{\ell,\varepsilon}$. The balanced set $\ell(U) \subset \mathbb{R}$ contains zero but not ε . Since a balanced real set is an interval symmetric about zero, $\ell(U) \subset (-\varepsilon, \varepsilon)$. Thus ℓ is continuous at zero, and therefore everywhere. $\square$

A hyperplane $H_{\ell,c}$ *separates* $A, B \subset X$ if

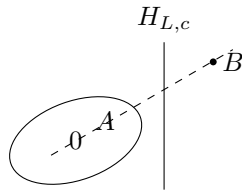
$$\ell(x) \leq c \leq \ell(y) \quad (x \in A, y \in B).$$

It *strictly separates* them if both inequalities are strict.

Theorem 13.2 (Geometric Hahn–Banach theorem). *Let X be a real TVS and A, B disjoint nonempty convex subsets. If A is open, there are $L \in X^*$ and $c \in \mathbb{R}$ such that*

$$Lx < c \leq Ly \quad (x \in A, y \in B).$$

The idea is first to separate an open convex neighborhood of zero from a point outside it, by extending a functional on the line joining zero to that point. In general, $0 \notin A - B$; translate $A - B$ to an open convex neighborhood of zero, and apply this idea to the old zero.



Proof. Fix $x_0 \in A$, $y_0 \in B$, and let

$$C = A - B + y_0 - x_0 = \bigcup_{y \in B} (A - y + y_0 - x_0).$$

This is convex, open, and contains zero. Its Minkowski functional $\mu_C(x) = \inf\{t > 0 : x \in tC\}$ is sublinear. Since $A \cap B$ is empty, $y_0 - x_0 \notin C$, so $\mu_C(y_0 - x_0) \geq 1$. On $Y = \mathbb{R}(y_0 - x_0)$ define $\ell(t(y_0 - x_0)) = t$. For $t \geq 0$, $t \leq t\mu_C(y_0 - x_0) = \mu_C(t(y_0 - x_0))$; for $t < 0$, $t < 0 \leq \mu_C(t(y_0 - x_0))$. Thus $\ell \leq \mu_C$, and Hahn–Banach gives a linear extension $L : X \rightarrow \mathbb{R}$ dominated by μ_C . On C , $L < 1$; by linearity, on $-C$, $L > -1$. Hence $|L(x)| < 1$ on the neighborhood $C \cap (-C)$. Scaling this neighborhood proves continuity of L , so $L \in X^*$. For $x \in A$ and $y \in B$,

$$Lx - Ly + 1 = L(x - y + y_0 - x_0) \leq \mu_C(x - y + y_0 - x_0) < 1.$$

Here $L(y_0 - x_0) = 1$, and openness of C gives the strict final bound (compare Quiz 3). Thus $Lx < Ly$ for every pair. Put $c = \sup_{x \in A} Lx$, which is finite because any Ly , $y \in B$, is an upper bound. Then $Lx \leq c \leq Ly$. Finally, the nonzero continuous linear functional L is an open map (Problem Set 5, Part 2, Problem 2(1)), so $L(A)$ is open in $\mathbb{R}$. No point of $L(A)$ equals its supremum; therefore $Lx < c$ on A . $\square$

Every real-linear functional on a complex vector space is uniquely the real part of a complex-linear functional (Problem Set 7, Part 1, Problem 1); one is continuous if and only if the other is.

Theorem 13.3 (Complex version). *For disjoint nonempty convex subsets A, B of a complex TVS, with A open, there are $L \in X^*$ and $c \in \mathbb{R}$ such that*

$$\operatorname{Re} Lx < c \leq \operatorname{Re} Ly \quad (x \in A, y \in B).$$

13.3. Separation in locally convex spaces. We extend the normed-space consequences in the review to locally convex TVSs, beginning with a variant of geometric Hahn–Banach.

Theorem 13.4. *Let X be a real locally convex TVS and A, B disjoint nonempty convex sets. If A is compact and B is closed, then there are $L \in X^*$ and $c_1 < c_2$ such that*

$$Lx \leq c_1 < c_2 \leq Ly \quad (x \in A, y \in B).$$

Recall that compactness means every open covering has a finite subcover.

Proof. For each $x \in A$, choose a neighborhood V_x of zero such that $(x + V_x + V_x) \cap B = \emptyset$; this is possible since B is closed. The sets $x + V_x$ cover A , so choose a finite subcover with centers $x_1, \dots, x_n$. Choose a convex neighborhood $V \subset \bigcap_i V_{x_i}$ of zero. Then $(A + V) \cap B = \emptyset$. Apply Theorem 13.2 to the open convex set $A + V$ and B , obtaining $L \in X^*$ separating their images. Set $c_1 = \sup L(A)$ and $c_2 = \sup L(A + V)$. The set $L(A)$ is compact and $L(A + V)$ is open; since the former is contained in the latter, $c_1 < c_2$. Separation gives $c_2 \leq Ly$ on B , completing the proof. $\square$

There is a complex version with $\operatorname{Re} L$ in the inequalities. Since singletons are compact, the following corollaries are immediate.

Corollary 13.1. *The continuous dual of a locally convex TVS separates points: if $x \neq y$, there exists $L \in X^*$ with $Lx \neq Ly$.*

Corollary 13.2. *If Y is a vector subspace of a locally convex TVS X and $x_0 \notin \overline{Y}$, there exists $L \in X^*$ such that $Lx_0 = 1$ and $Ly = 0$ for all $y \in Y$.*

Corollary 13.3. *If Y is a vector subspace of a locally convex TVS X , then every $\ell \in Y^*$ extends to some $L \in X^*$.*

14. WEAK TOPOLOGIES

14.1. Review (October 27, 2015). Geometric Hahn–Banach separates disjoint nonempty convex sets A, B in a real TVS when A is open: $Lx < c \leq Ly$ for $L \in X^*$. In a locally convex TVS, compact A and closed B admit a uniform gap $Lx \leq c_1 < c_2 \leq Ly$. The continuous dual then separates points, separates a point outside the closure of a subspace from that subspace, and extends continuous linear functionals from subspaces. Recall that every Fréchet space is locally convex.

14.2. Weak topologies induced by maps. A topology $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ when $\mathcal{T}_1 \subset \mathcal{T}_2$: every $\mathcal{T}_1$ -open set is $\mathcal{T}_2$ -open. The topology generated by a family $\mathcal{S}$ of subsets is the intersection of all topologies containing $\mathcal{S}$; finite intersections of members of $\mathcal{S}$ form a base for it (including the empty intersection X).

Example 14.1. For a TVS X and a closed subspace N , the quotient topology on X/N is characterized by E open if and only if $\pi^{-1}(E)$ is open in X . It is the strongest topology making $\pi : X \rightarrow X/N$ continuous, and the weakest making π open. The product topology on $X \times Y$, generated by open rectangles $U \times V$, is the weakest making both coordinate projections continuous.

Definition 14.1. Let $\mathcal{E} = \{f_\alpha : X \rightarrow Y_\alpha\}$ be maps from a set X into topological spaces. The weakest topology making every f_α continuous is the $\mathcal{E}$ -topology, or weak topology induced by $\mathcal{E}$. It is generated by all sets $f_\alpha^{-1}(U_\alpha)$ with U_α open in Y_α .

Example 14.2. On $C([0, 1])$, evaluation $\text{Ev}_x(f) = f(x)$ for $x \in [0, 1]$ induces the topology of pointwise convergence: $f_i \rightarrow f$ means $f_i(x) \rightarrow f(x)$ for every x . This differs from the topology of uniform convergence induced by $\|f\|_\infty = \sup_{[0,1]} |f|$. For any family of topological spaces X_α , the product topology on

$$\prod_{\alpha} X_{\alpha} = \{(x_{\alpha}) : x_{\alpha} \in X_{\alpha}\}$$

is the weak topology induced by all coordinate projections.

Theorem 14.1 (Tychonoff). *A product of compact topological spaces is compact in the product topology.*

This standard and difficult theorem is equivalent to the axiom of choice (and to Zorn's lemma). The product topology is weaker than the box topology, generated by all products $\prod_{\alpha} U_{\alpha}$ with every U_{α} open. Such a box product need not be compact even when all factors are compact.

Proposition 14.2. *If $\mathcal{E} = \{f_{\alpha} : X \rightarrow Y_{\alpha}\}$ separates points of X and each Y_{α} is Hausdorff, its induced weak topology is Hausdorff.*

Proof. For $x \neq y$, choose α with $f_{\alpha}(x) \neq f_{\alpha}(y)$. Disjoint open neighborhoods U_1, U_2 of these images have disjoint inverse images, which are neighborhoods of x, y in the weak topology. $\square$

14.3. The weak topology of a TVS.

Theorem 14.3. *Let X' be a separating family of linear functionals on a vector space X . The X' -topology makes X a locally convex TVS. Its continuous dual is exactly the linear span of X' .*

The proof is omitted in the source; see Rudin, pp. 64–65. A local base at zero consists of the convex balanced sets

$$V_{L_1, \dots, L_n; r_1, \dots, r_n} = \{x \in X : |L_i x| < r_i, \ 1 \leq i \leq n\}, \quad L_i \in X', \quad r_i > 0.$$

Now suppose X is a TVS whose continuous dual X^* separates points, as is true for every locally convex TVS.

Definition 14.2. The weak topology of X is the X^* -topology, denoted $\sigma(X, X^*)$. It is the weakest topology keeping all originally continuous linear functionals continuous.

It is weaker than the original topology, usually strictly so. Even a TVS which is not locally convex may have a separating dual; its weak topology then makes it locally convex. Basic weak neighborhoods of zero have the preceding form with $L_i \in X^*$.

Definition 14.3 (Weak convergence). We write $x_n \rightharpoonup x$ if $L(x_n) \rightarrow L(x)$ for every $L \in X^*$.

If X is infinite-dimensional, every weak neighborhood of zero contains an infinite-dimensional subspace, namely the common kernel of the finitely many functionals defining a contained basic neighborhood. In particular, the weak topology is not locally bounded. This is one reason to study general topologies beyond metric topologies.

Example 14.3. Let $X = \ell^2$. As will be shown later (or in Problem Set 8, Part 1), X^* is represented by ℓ^2 : every functional is $Lx = (x, y) = \sum_i a_i \bar{b}_i$ for some $y = (b_i) \in \ell^2$. For the standard unit vectors e_n , $\|e_n\| = 1$, so e_n does not converge in norm to zero. But $(e_n, y) = \bar{b}_n \rightarrow 0$ for every $y \in \ell^2$, hence $e_n \rightharpoonup 0$.

Editorial note. The handwritten example uses real coordinates; conjugates are included here so the same displayed formula applies over $\mathbb{C}$.

Proposition 14.4. If E is a convex subset of a locally convex TVS, its weak closure $\overline{E}^w$ equals its closure $\overline{E}$ in the original topology.

Proof. Every weakly closed set is originally closed, so $\overline{E} \subset \overline{E}^w$. Conversely, if $x_0 \notin \overline{E}$, separate the compact set $\{x_0\}$ and the closed convex set $\overline{E}$ by geometric Hahn–Banach. There are $L \in X^*$ and c with $Lx_0 < c < Ly$ for every $y \in \overline{E}$ in the real case (use $\operatorname{Re} L$ in the complex case). The weak neighborhood $\{x : Lx < c\}$ of x_0 misses E , so $x_0 \notin \overline{E}^w$. Thus $\overline{E}^w \subset \overline{E}$. $\square$

Corollary 14.1. For a convex subset E of a locally convex TVS, E is closed if and only if it is weakly closed, and E is dense if and only if it is weakly dense.

Editorial note. The final inclusion in the handwritten proof is corrected to $\overline{E}^w \subset \overline{E}$, which is what the argument proves.

15. THE WEAK-STAR TOPOLOGY AND BANACH-ALAOGLU

15.1. Review (October 30, 2015). A separating family of linear functionals induces a locally convex TVS topology. The weak topology of X is the X^* -topology. It is weaker than the original topology; in infinite dimensions it is not normable and often not even metrizable. In a locally convex TVS, convex sets have the same closure for the weak and original topologies.

15.2. The weak-star topology on the dual. For a normed space, the dual is naturally a Banach space with $\|L\|_{X^*} = \sup_{\|x\| \leq 1} |Lx|$. For a general TVS this construction does not apply, but a weak topology still does. For any TVS X (without requiring its dual to separate points), each $x \in X$ defines a linear evaluation functional

$$\text{ev}_x : X^* \rightarrow \mathbb{F}, \quad \text{ev}_x(L) = Lx.$$

The family $\mathcal{X} = \{\text{ev}_x : x \in X\}$ separates points of X^* .

Definition 15.1. The $\mathcal{X}$ -topology on X^* is called its *weak-star topology*. We denote it by $\sigma(X^*, X)$.

It makes X^* a locally convex TVS. Basic neighborhoods of zero are

$$\{L \in X^* : |Lx_i| < r_i, \ 1 \leq i \leq n\}, \quad r_i > 0.$$

For normed spaces, this generally differs from the dual norm topology. For an infinite-dimensional Banach space X , the weak-star topology on all of X^* is not metrizable. Although unit balls in infinite-dimensional normed spaces need not be compact, weak-star topology has the following compactness property.

Theorem 15.1 (Banach–Alaoglu). *Let X be a TVS and V a neighborhood of zero. Then its polar*

$$K = \{L \in X^* : |Lx| \leq 1 \text{ for every } x \in V\}$$

is weak-star compact.

The idea is to realize K as a closed subset of a compact product.

Proof. Since V is absorbing, for each $x \in X$ choose a finite $C(x) > 0$ with $x \in C(x)V$. For $L \in K$, linearity gives $|Lx| \leq C(x)$. Let

$$D_x = \{\alpha \in \mathbb{F} : |\alpha| \leq C(x)\}, \quad P = \prod_{x \in X} D_x.$$

Each D_x is compact; Tychonoff's theorem makes P compact. An element of P is a scalar function f on X with $|f(x)| \leq C(x)$ for every x . Thus $K \subset P$.

First, the weak-star topology and the topology inherited from P agree on K . A neighborhood base at L_0 for the former consists of

$$\{L \in X^* : |Lx_i - L_0x_i| < \varepsilon, \ 1 \leq i \leq n\},$$

and a base for the latter consists of the corresponding conditions $|f(x_i) - L_0x_i| < \varepsilon$ on $f \in P$. Restricting both to K gives the same sets.

Second, K is closed in P . Let f_0 lie in its product closure. For any $x, y \in X$, scalars α, β , and $\varepsilon > 0$, a product neighborhood of f_0 controls the three values at $x, y, \alpha x + \beta y$. Choose $f \in K$ in this neighborhood. Its linearity gives

$$\begin{aligned} & |f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| \\ & \leq (1 + |\alpha| + |\beta|)\varepsilon. \end{aligned}$$

Since ε is arbitrary, f_0 is linear. Similarly, for $x \in V$, approximating its single value by $f \in K$ gives $|f_0(x)| \leq 1 + \varepsilon$, hence $|f_0(x)| \leq 1$. This bound on a neighborhood of zero proves continuity: for any $\varepsilon > 0$, $|f_0(x)| < \varepsilon$ on $(\varepsilon/2)V$. Thus $f_0 \in K$. So K is closed in compact P , and the equality of the two induced topologies proves weak-star compactness. $\square$

15.3. Applications.

Corollary 15.1. *For any normed space X , the closed unit ball of X^* is weak-star compact.*

Proof. Take $V = \{x : \|x\| \leq 1\}$. Its polar is $\{L : \|L\|_{X^*} \leq 1\}$. Indeed, the operator norm inequality gives one inclusion; if $\|L\| > 1$, some unit vector has $|Lx| > 1$, giving the reverse inclusion. $\square$

Corollary 15.2. *Every Banach space can be realized isometrically as a closed vector subspace of $C(\Omega)$ with its uniform norm, for some compact Hausdorff space Ω .*

Proof. Let Ω be the closed dual unit ball with its weak-star topology. It is compact and Hausdorff. Define

$$\Phi : X \rightarrow C(\Omega), \quad (\Phi x)(L) = Lx.$$

This is linear, since $(\Phi(\alpha x_1 + \beta x_2))(L) = \alpha Lx_1 + \beta Lx_2$. Each Φx is continuous by the definition of weak-star topology. Hahn–Banach gives

$$\|\Phi x\|_\infty = \sup_{\|L\| \leq 1} |Lx| = \|x\|.$$

Thus Φ is a linear isometry. Completeness of X implies that $\Phi(X)$ is closed. $\square$

Theorem 15.2. *A subset E of a locally convex TVS is bounded if and only if it is weakly bounded.*

Proof. Each weak neighborhood of zero is an original neighborhood; therefore original boundedness implies weak boundedness.

Conversely, suppose E is weakly bounded. Given an original neighborhood U of zero, choose a convex balanced open neighborhood V with $\bar{V} \subset U$. Set

$$K = \{L \in X^* : |Lx| \leq 1 \text{ for every } x \in V\}.$$

We claim

$$\bar{V} = \{x \in X : |Lx| \leq 1 \text{ for every } L \in K\}.$$

One inclusion follows because the right-hand side is a closed set containing V . If $x_0 \notin \bar{V}$, geometric Hahn–Banach separates x_0 from $\bar{V}$. Because V is balanced, one can normalize the separating functional to obtain $L \in K$ with $|Lx_0| > 1$, proving the other inclusion.

Weak boundedness says that for each $L \in X^*$, $\sup_{x \in E} |Lx| < \infty$. The set K is convex and weak-star compact by Banach–Alaoglu. Apply the following variation of Banach–Steinhaus (the source refers to Rudin, p. 47): if K is compact and convex in a TVS and Γ is a family of continuous linear maps into another TVS, pointwise boundedness on K implies that all $T(K)$, $T \in \Gamma$, lie in one bounded set. Here use the weak-star TVS X^* , the scalar target, and the evaluations ev_x , $x \in E$. We obtain a finite $a > 0$ with

$$|Lx| \leq a \quad (x \in E, L \in K).$$

The claim gives $x/a \in \overline{V}$ for every $x \in E$. For $t > a$, balancedness yields $E \subset t\overline{V} \subset tU$. Thus E is originally bounded. $\square$

Corollary 15.3. *If X is normed and $E \subset X$ satisfies $\sup_{x \in E} |Lx| < \infty$ for every $L \in X^*$, then $\sup_{x \in E} \|x\| < \infty$.*

15.4. Separability and metrizable compact sets.

Definition 15.2. A space is *separable* if it has a countable dense subset.

For example, $\mathbb{R}^n$ is separable since $\mathbb{Q}^n$ is dense. The space ℓ^∞ is not separable: the uncountable set of all sequences with entries zero or one has pairwise distance one, so balls of radius less than $1/2$ centered at its points are disjoint and a countable dense set cannot meet all of them.

Theorem 15.3. *If X is a separable TVS and $K \subset X^*$ is weak-star compact, then K is metrizable in its weak-star topology.*

Proof. Choose a countable dense subset $\{x_n\}$ of X , and set $f_n(L) = Lx_n$. Each f_n is weak-star continuous. They separate points of X^* : equality on the dense set implies equality everywhere by continuity of the functionals. Compactness gives positive constants C_n with $|f_n(L)| \leq C_n$ on K . Then

$$d(L_1, L_2) = \sum_{n=1}^{\infty} \frac{|f_n(L_1) - f_n(L_2)|}{2^n C_n}$$

defines a metric on K . The series converges uniformly, so this metric topology is weaker than the weak-star topology. The identity from the compact weak-star space to the Hausdorff metric space is a continuous bijection, hence a homeomorphism. $\square$

Editorial note. The constants C_n are chosen strictly positive, and the compact-to-Hausdorff argument completes the topology verification left implicit after the metric formula in the source.

Corollary 15.4. *Let X be a separable TVS and V a neighborhood of zero. If $L_n \in X^*$ and $|L_n x| \leq 1$ for all $x \in V$ and all n , then some subsequence L_{n_k} converges pointwise on X to a functional $L \in X^*$.*

Proof. All L_n lie in the polar of V , which is compact and metrizable, hence sequentially compact. $\square$

16. COMPACT CONVEX SETS AND EXTREME POINTS

16.1. Review (November 6, 2015). The weak-star topology on X^* is the weakest making every evaluation $L \mapsto Lx$ continuous. Its basic neighborhoods impose finitely many bounds $|Lx_i| < \varepsilon$. Banach–Alaoglu makes polars of neighborhoods weak-star compact, in particular the closed dual unit ball. Every Banach space embeds as a closed subspace of some $C(\Omega)$. In a locally convex TVS, boundedness equals weak boundedness. For separable X , weak-star compact subsets of X^* are metrizable.

16.2. Convex hulls.

Definition 16.1. The convex hull of E in a vector space is

$$\text{co}(E) = \left\{ \sum_{i=1}^k t_i x_i : x_i \in E, t_i \geq 0, \sum_{i=1}^k t_i = 1, k \in \mathbb{N} \right\}.$$

Such sums are *convex combinations*.

This hull is convex and contains E ; any convex set containing E contains its hull. Consequently,

$$\text{co}(E) = \bigcap \{A : E \subset A, A \text{ convex}\}.$$

Indeed, one inclusion follows from minimality, and the intersection is convex and contains E , giving the other. In a TVS the closure $\overline{\text{co}(E)}$ is the *closed convex hull*.

Example 16.1. For $L^p([0, 1])$, $0 < p < 1$, every set E with an interior point has $\text{co}(E) = L^p([0, 1])$; see Problem Set 2, Part 2, Problem 4(2), or Rudin, p. 37.

Proposition 16.1. In a locally convex TVS, $x_n \rightharpoonup x_0$ implies $x_0 \in \overline{\text{co}\{x_n : n \geq 1\}}$.

Proof. The weak and original closures of the convex hull coincide, and the weak limit belongs to its weak closure. $\square$

Proposition 16.2. If $A_1, \dots, A_n$ are compact convex subsets of a TVS, then $\text{co}(A_1 \cup \dots \cup A_n)$ is compact.

Proof. Let $S = \{(t_1, \dots, t_n) : t_i \geq 0, \sum_i t_i = 1\}$, the standard simplex. The continuous map

$$S \times A_1 \times \dots \times A_n \rightarrow X, \quad ((t_i), (x_i)) \mapsto \sum_i t_i x_i$$

has compact image K , contained in the indicated convex hull. The image contains each A_i and is convex. In fact, in a convex combination of two such sums, combine the terms belonging to A_i using the coefficient $s_i = tt_i + (1-t)t'_i$. If $s_i > 0$, their new point is

$$\frac{tt_i}{s_i} x_i + \frac{(1-t)t'_i}{s_i} x'_i \in A_i;$$

terms with $s_i = 0$ vanish. Thus K equals the convex hull and is compact. $\square$

In particular, the convex hull of finitely many points is compact. For general compact E , its convex hull need not be closed or compact.

Definition 16.2. A subset E of a TVS is *totally bounded* if for every neighborhood U of zero there are $x_1, \dots, x_n$ with $E \subset \bigcup_{i=1}^n (x_i + U)$.

For a translation-invariant compatible metric this means that every $\varepsilon > 0$ admits a finite covering by ε -balls. Every compact subset is totally bounded.

Proposition 16.3. *The convex hull of a totally bounded subset of a locally convex TVS is totally bounded.*

Proof. Given a neighborhood U of zero, choose a convex neighborhood V with $V + V \subset U$. Cover E by finitely many $x_i + V$. Then $\text{co}(E) \subset \text{co}\{x_1, \dots, x_n\} + V$, since the latter is convex and contains E . The finite convex hull is compact, so finitely many translates $y_j + V$ cover it. Therefore

$$\text{co}(E) \subset \bigcup_j (y_j + V + V) \subset \bigcup_j (y_j + U).$$

□

Corollary 16.1. *For a compact subset K of a Fréchet space, $\overline{\text{co}(K)}$ is compact.*

Proof. K and hence its convex hull are totally bounded. In a complete metric space the closure of a totally bounded set is compact (exercise). □

16.3. Extreme points.

Definition 16.3. A nonempty subset $S \subset K$ is an *extreme set* of K if

$$x \in S, \quad x = tx_1 + (1-t)x_2, \quad 0 < t < 1, \quad x_1, x_2 \in K \implies x_1, x_2 \in S.$$

A point x is *extreme* if $\{x\}$ is an extreme set. Write $\text{Ex}(K)$ for all extreme points. The set K itself is an extreme set of K .

Example 16.2. For real $C([0, 1])$ with its uniform norm, the extreme points of the closed unit ball are the constant functions 1 and -1 . If $|f(t_0)| < 1$ somewhere, a sufficiently small continuous perturbation near t_0 expresses f as the midpoint of two distinct elements of the unit ball. For a closed Euclidean unit ball in $\mathbb{R}^n$, the extreme points form its unit sphere. An open ball has no extreme point (exercise).

Theorem 16.4 (Krein–Milman). *Let X be a TVS whose dual separates points. If K is a nonempty compact convex subset, then*

$$K = \overline{\text{co}(\text{Ex}(K))}.$$

In particular, K has extreme points.

We first establish the following proposition.

Proposition 16.5. *If X^* separates points and $K \subset X$ is compact, then every compact extreme set of K contains an extreme point of K .*

Proof. Fix such a set K_0 . Let $\mathcal{P}$ be all compact extreme sets of K contained in K_0 , ordered by inclusion. It is nonempty since $K_0 \in \mathcal{P}$. For a chain $\mathcal{Q} \subset \mathcal{P}$, let $S_{\mathcal{Q}} = \bigcap_{S \in \mathcal{Q}} S$.

This is compact. It is nonempty: otherwise the open sets $K_0 \setminus S$ would cover K_0 , and a finite subcover would give a finite intersection of chain members equal to the empty set. Such an intersection is one of its nonempty members, a contradiction. It is extreme as well: a nontrivial convex decomposition of a point in every $S \in \mathcal{Q}$ has both endpoints in every S . Thus each chain has a lower bound. Zorn's lemma gives a minimal member M of $\mathcal{P}$. If M had two points, a continuous linear functional separating them would give a proper compact extreme subset by maximizing its

real part on M , contradicting minimality. Hence M is a singleton, proving the proposition. $\square$

The source uses the following two lemmas, whose proofs are exercises.

Lemma 16.1. *If X^* separates points and A, B are disjoint nonempty compact convex subsets, there is $L \in X^*$ with $\sup_{x \in A} \operatorname{Re} Lx < \inf_{y \in B} \operatorname{Re} Ly$.*

Lemma 16.2. *If S is a compact extreme set of compact K and $L \in X^*$, then*

$$S_L = \{x \in S : \operatorname{Re} Lx = \sup_{y \in S} \operatorname{Re} Ly\}$$

is a compact extreme set of K .

Proof of Theorem 16.4. Let $C = \overline{\operatorname{co}(\operatorname{Ex}(K))}$. Since K is closed and convex, $C \subset K$ and C is compact. If $x_0 \in K \setminus C$, Lemma 16.1 gives $L \in X^*$ with $\sup_{x \in C} \operatorname{Re} Lx < \operatorname{Re} Lx_0$. The maximizing set

$$K_L = \{x \in K : \operatorname{Re} Lx = \sup_{y \in K} \operatorname{Re} Ly\}$$

is compact and extreme by Lemma 16.2, and disjoint from C . But Proposition 16.5 gives an extreme point of K in K_L , contradicting $\operatorname{Ex}(K) \subset C$. $\square$

Remark 16.1. Using compact-versus-closed geometric separation in a locally convex TVS in place of Lemma 16.1, the same argument shows that any compact subset K , even without assuming convexity, satisfies $K \subset \overline{\operatorname{co}(\operatorname{Ex}(K))}$. In general the closed convex hull may have extreme points outside K . If that hull is compact, however, the following converse holds.

Theorem 16.6. *If K is compact in a locally convex TVS and $\overline{\operatorname{co}(K)}$ is compact, then $\operatorname{Ex}(\overline{\operatorname{co}(K)}) \subset K$.*

See Rudin, pp. 76–77, for the proof.

Applications of Krein–Milman include the Stone–Weierstrass theorem (the source cites a Hua seminar presentation), and the following.

Corollary 16.2. *The closed unit ball B of the dual of a Banach space satisfies $B = \overline{\operatorname{co}(\operatorname{Ex}(B))}^{w^*}$.*

Proof. Combine Banach–Alaoglu and Krein–Milman. $\square$

Corollary 16.3. *The space $c_0 = \{(a_i) : a_i \rightarrow 0\}$, with its usual supremum norm, is not isometrically the dual of a Banach space.*

Proof. Its closed unit ball has no extreme points. $\square$

Editorial note. The source says “not the dual of any Banach space”. The displayed extreme-point argument directly rules out an isometric dual realization with this norm; that is the precise conclusion used here.

Another application is Birkhoff’s theorem on doubly stochastic matrices. Let

$$K = \{(a_{ij}) \in \mathbb{R}^{n \times n} : a_{ij} \geq 0, \sum_i a_{ij} = 1, \sum_j a_{ij} = 1\}.$$

Theorem 16.7 (Birkhoff). *Every doubly stochastic matrix is a convex combination of permutation matrices.*

The idea is that K is compact and convex, and its extreme points are exactly the permutation matrices, namely its zero-one matrices.

17. DUAL SPACES AND REFLEXIVITY

17.1. Review (November 10, 2015). We studied convex hulls and extreme points. Krein–Milman says that a nonempty compact convex set in a TVS with separating dual is the closed convex hull of its extreme points. For applications see Lax, Chapters 13 and 14.

17.2. Examples of dual spaces. For a Banach space X , $X^* = \mathcal{L}(X, \mathbb{F})$ is a Banach space. Examples are:

- (1) $c_0^* = \ell^1$ (quiz), hence $c_0^{**} = \ell^\infty \supsetneq c_0$.
- (2) $(\ell^1)^* = \ell^\infty$ (Problem Set 9, Part 1, Problem 2).
- (3) For $1 < p < \infty$ and $1/p + 1/q = 1$, $(\ell^p)^* = \ell^q$; thus $(\ell^p)^{**} = \ell^p$ (Problem Set 8, Part 1, Problem 4).
- (4) Similarly, $(L^1)^* = L^\infty$ and $(L^p)^* = L^q$ for $1 < p < \infty$, $1/p + 1/q = 1$, in the usual measure-space setting.
- (5) For a Hilbert space H , its dual is represented by H .

For the last example, each $y \in H$ defines $L_y(x) = (x, y)$. Cauchy–Schwarz gives $\|L_y\| \leq \|y\|$; testing at $x = y$ gives the reverse inequality, so $\|L_y\| = \|y\|$.

Theorem 17.1 (Riesz representation for Hilbert spaces). *For every $L \in H^*$ there is a unique $y \in H$ with $L = L_y$.*

We use the orthogonal decomposition lemma.

Lemma 17.1. *For a closed subspace M of a Hilbert space, $H = M \oplus M^\perp$, where $M^\perp = \{y : (x, y) = 0 \text{ for all } x \in M\}$.*

Proof. $M^\perp$ is a closed vector subspace, as an intersection of kernels of continuous (possibly conjugate-linear) functionals. Also $M \cap M^\perp = \{0\}$ by positive definiteness. For $x \in H$, the closed convex set $x - M$ has a minimum-norm point by Lecture 7. Thus there is $x_1 \in M$ with $\|x - x_1\| = \inf_{y \in M} \|x - y\|$.

Put $x_2 = x - x_1$. If $x_2 \notin M^\perp$, choose and scale $y \in M$ so that $(x_2, y) = 1$. For sufficiently small positive ε ,

$$\|x - x_1 - \varepsilon y\|^2 = \|x_2\|^2 - 2\varepsilon + \varepsilon^2 \|y\|^2 < \|x_2\|^2,$$

contradicting minimality. Hence $x = x_1 + x_2$ with $x_2 \in M^\perp$. $\square$

Proof of Theorem 17.1. The case $L = 0$ is immediate. Otherwise $\ker L$ is a proper closed subspace, and $H = \ker L \oplus (\ker L)^\perp$. Choose a unit vector $z \in (\ker L)^\perp$ and put $y = \overline{Lz} z$. For every x ,

$$x = \left(x - \frac{Lx}{Lz} z \right) + \frac{Lx}{Lz} z$$

is its orthogonal decomposition, the first term lying in $\ker L$. Thus

$$(x, y) = \left(\frac{Lx}{Lz} z, \overline{Lz} z \right) = Lx.$$

If $(x, y_1) = (x, y_2)$ for all x , taking $x = y_1 - y_2$ gives $\|y_1 - y_2\|^2 = 0$, proving uniqueness. $\square$

Theorem 17.2 (Riesz representation for continuous functions). *For a compact Hausdorff space Ω , the dual of real $C(\Omega)$ is the space of finite signed regular Borel measures. For complex scalars one uses finite complex regular Borel measures.*

For proofs see Rudin's *Real and Complex Analysis*, Reed–Simon, Volume I, or the appendix of Lax. The dual of ℓ^∞ is not ℓ^1 ; see Problem Set 10, Part 1.

Elements of X^* are linear functionals on X , while elements of X act by evaluation on X^* . To treat them symmetrically, write x^* instead of L or ℓ , and write $\langle x, x^* \rangle = x^*(x)$.

For any normed space,

$$\|x^*\| = \sup_{\|x\| \leq 1} |\langle x, x^* \rangle|, \quad \|x\| = \sup_{\|x^*\| \leq 1} |\langle x, x^* \rangle|.$$

The latter is the norm-preservation fact in the embedding of Lecture 15.

Theorem 17.3. For normed spaces X, Y and $L \in \mathcal{L}(X, Y)$,

$$\|L\| = \sup\{|\langle Lx, y^* \rangle| : \|x\| \leq 1, \|y^*\| \leq 1\}.$$

Proof. Take the supremum of $\|Lx\| = \sup_{\|y^*\| \leq 1} |\langle Lx, y^* \rangle|$ over the unit ball of X . $\square$

17.3. Reflexive spaces. A Banach space has the canonical embedding

$$J : X \longrightarrow X^{**}, \quad (Jx)(x^*) = \langle x, x^* \rangle.$$

The norm formula shows that J is isometric. Its image is closed, so X may be regarded as a closed subspace of its bidual.

Definition 17.1. A normed space is *reflexive* if its canonical embedding onto X^{**} is surjective. A reflexive space is necessarily Banach.

Finite-dimensional spaces are reflexive, since the dimensions of X , X^* , and X^{**} agree. Hilbert spaces are reflexive, as are ℓ^p and L^p for $1 < p < \infty$. The spaces ℓ^1 , c_0 , and $C([0, 1])$ are not reflexive. We will prove the following characterization.

Theorem 17.4. A Banach space is reflexive if and only if its closed unit ball is weakly compact.

18. REFLEXIVITY AND WEAK COMPACTNESS

18.1. Review (November 13, 2015). Riesz representation identifies Hilbert-space functionals as $Lx = (x, y)$ for unique y . The orthogonal decomposition $H = M \oplus M^\perp$ for closed subspaces follows from the unique minimum-norm point in any nonempty closed convex subset of a Hilbert space. We use $\langle x, x^* \rangle = x^*(x)$, and call a Banach space reflexive when its canonical embedding into X^{**} is onto.

18.2. Kakutani's characterization.

Theorem 18.1 (Kakutani). *A Banach space is reflexive if and only if its closed unit ball is weakly compact.*

Proof. If X is reflexive, identify it with X^{**} . The closed unit ball is weak-star compact in X^{**} by Banach–Alaoglu. This weak-star topology is precisely the weak topology of X , since both are induced by X^* .

Conversely, let $B \subset X$ be weakly compact and let $J : X \rightarrow X^{**}$ be the canonical isometric embedding. By Problem Set 8, Part 1, Problem 3, J is continuous for the weak topologies. The weak-star topology on X^{**} is weaker than its weak topology, since X^{***} contains the canonical copy of X^* . Hence $J : X_{\text{weak}} \rightarrow X_{\text{weak-star}}^{**}$ is continuous. Thus $J(B)$ is weak-star compact and therefore closed in the Hausdorff weak-star bidual. By Problem Set 10, Part 1, Problem 3 (Goldstine's statement), $J(B)$ is weak-star dense in the closed bidual unit ball B^{**} . Consequently $J(B) = B^{**}$. Scaling now gives $J(X) = X^{**}$, proving reflexivity. $\square$

Proposition 18.2. *Every closed subspace of a reflexive Banach space is reflexive.*

Proof. Hahn–Banach says every element of M^* is a restriction of an element of X^* . Therefore the weak topology on M equals that inherited from weak X . Its closed unit ball is $B_M = B_X \cap M$. Both M and B_X are norm-closed and convex, hence weakly closed. Since B_X is weakly compact, B_M is weakly compact in M ; Theorem 18.1 applies. $\square$

Proposition 18.3. *A Banach space X is reflexive if and only if X^* is reflexive.*

Proof. If $X = X^{**}$, the weak and weak-star topologies on X^* agree. Banach–Alaoglu therefore makes its unit ball weakly compact, so X^* is reflexive. Conversely, if X^* is reflexive, the implication just proved makes X^{**} reflexive. Its closed subspace $J(X)$ is then reflexive, and hence so is X . $\square$

Corollary 18.1. *A bounded closed convex subset of a reflexive Banach space is weakly compact.*

Proof. It is weakly closed by convexity, and contained in a sufficiently large multiple of the weakly compact unit ball. $\square$

Proposition 18.4. *Every nonempty closed convex subset C of a reflexive Banach space contains a point c with $\|c\| = \min_{x \in C} \|x\|$. The minimizing point need not be unique.*

Proof. Let $t_0 = \inf_{x \in C} \|x\|$. For $t > t_0$, the sets

$$C_t = \{x \in C : \|x\| \leq t\} = C \cap \overline{B}(0, t)$$

are nonempty, bounded, closed, and convex, hence weakly compact. The nested family $C_{t_0+1/n}$ has nonempty intersection. A point c in the intersection has $\|c\| \leq t_0 + 1/n$ for every n , hence $\|c\| = t_0$. $\square$

The norm itself is not generally weakly continuous: in ℓ^2 , $e_n \rightharpoonup 0$ while $\|e_n\| = 1$.

18.3. Separability and weak subsequences. Recall that separability means having a countable dense set. Lecture 15 showed that weak-star compact subsets of the dual of a separable TVS are metrizable; sequences in a fixed neighborhood polar therefore have weak-star convergent subsequences (compare Problem Set 9, Part 1, Problem 3(3)).

Editorial note. The review in the source says “any weak-star bounded sequence” in the dual of a separable TVS. Without further assumptions, pointwise boundedness alone need not place a sequence in a compact polar. The preceding precise form is the result proved in Lecture 15; for Banach domains, uniform boundedness supplies the needed control.

Theorem 18.5. *If the dual of a normed space is norm-separable, the space is separable.*

Proof. Exercise. Hint: choose a countable dense family $\{x_n^*\}$ in X^* , and $x_n \in X$ with $\|x_n\| \leq 1$ and $|\langle x_n, x_n^* \rangle| \geq \frac{1}{2} \|x_n^*\|$. Use geometric Hahn–Banach to show that the rational linear combinations of the x_n (rational complex coefficients in the complex case) are dense in X . $\square$

Theorem 18.6. *Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence.*

Proof. Let $\{x_n\}$ be bounded and let Y be the norm-closed span of these vectors. It is a closed subspace of X , hence reflexive, and is separable by construction. Since $Y^{**} = Y$ is separable, the preceding theorem applied to Y^* shows that Y^* is separable. Regard the bounded sequence as a sequence in Y^{**} . A suitable multiple of its closed unit ball is weak-star compact and metrizable, since its predual Y^* is separable. Thus a subsequence x_{n_k} converges weak-star in Y^{**} to some $x_0 \in Y^{**} = Y$:

$$\langle x_{n_k}, y^* \rangle \rightarrow \langle x_0, y^* \rangle \quad (y^* \in Y^*).$$

For any $x^* \in X^*$, restrict to Y to obtain

$$\langle x_{n_k}, x^* \rangle = \langle x_{n_k}, x^*|_Y \rangle \longrightarrow \langle x_0, x^*|_Y \rangle = \langle x_0, x^* \rangle.$$

Thus $x_{n_k} \rightharpoonup x_0$ in X . $\square$

The conclusion fails with norm convergence in place of weak convergence. A deep converse says that a Banach space is reflexive if every bounded sequence has a weakly convergent subsequence.

19. ANNIHILATORS AND ADJOINT OPERATORS

19.1. Review (November 17, 2015). For a Banach space, reflexivity, reflexivity of the dual, and weak compactness of the closed unit ball are equivalent. Closed subspaces of reflexive spaces are reflexive. In a reflexive space, bounded closed convex sets are weakly compact, nonempty closed convex sets have minimum-norm elements, and bounded sequences have weakly convergent subsequences (with a converse characterization).

19.2. Duals of subspaces and quotient spaces. In a Hilbert space, $(M^\perp)^\perp = \overline{M}$ (Problem Set 10, Part 1, Problem 1), and $H = M \oplus M^\perp$ when M is closed. A general Banach space has no angle structure, but duality replaces orthogonality.

Definition 19.1 (Annihilators). For subspaces $M \subset X$ and $N \subset X^*$, define

$$M^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0 \ (x \in M)\},$$

$${}^\perp N = \{x \in X : \langle x, x^* \rangle = 0 \ (x^* \in N)\}.$$

These are intersections of kernels: $M^\perp$ is weak-star closed (and hence weakly and norm closed); ${}^\perp N$ is weakly closed (and hence norm closed). Clearly $M \subset {}^\perp(M^\perp)$ and $N \subset ({}^\perp N)^\perp$.

Proposition 19.1.

$${}^\perp(M^\perp) = \overline{M}, \quad ({}^\perp N)^\perp = \overline{N}^{w*}.$$

Here $\overline{M}$ is also the weak closure of M .

Proof. Closedness gives $\overline{M} \subset {}^\perp(M^\perp)$. If $x \notin \overline{M}$, geometric Hahn–Banach gives $x^* \in M^\perp$ with $\langle x, x^* \rangle \neq 0$, so $x \notin {}^\perp(M^\perp)$. For the second equality use the same argument in the weak-star TVS X^* , whose continuous dual consists of evaluations by elements of X . $\square$

In general the norm closure of N may be strictly smaller than $({}^\perp N)^\perp$. If X is reflexive, the two agree.

For a closed subspace M of a Banach space, X/M is Banach with the quotient norm $\|\pi(x)\|_{X/M} = \inf_{z \in M} \|x + z\|$.

Theorem 19.2. *There are canonical isometric linear isomorphisms*

$$M^* \cong X^*/M^\perp, \quad (X/M)^* \cong M^\perp.$$

Proof. For $m^* \in M^*$ choose a Hahn–Banach extension $x^* \in X^*$ and set $\Phi(m^*) = x^* + M^\perp$. This is well-defined since two extensions differ by an element of $M^\perp$, and is linear. It is surjective by restriction. Choose the extension with $\|x^*\| = \|m^*\|$. Every $x^* + n^*$, $n^* \in M^\perp$, restricts to m^* , so

$$\|m^*\| \leq \inf_{n^* \in M^\perp} \|x^* + n^*\| = \|\Phi(m^*)\| \leq \|x^*\| = \|m^*\|.$$

Thus Φ is isometric and injective.

Define $\Psi : (X/M)^* \rightarrow M^\perp$ by $\Psi(y^*) = y^* \circ \pi$. This vanishes on M and is linear. For $x^* \in M^\perp$, the formula $y^*(\pi x) = x^*(x)$ is well-defined and continuous, giving surjectivity. Equivalently, its kernel is the closed subspace $\pi(\ker x^*)$ of X/M , as

noted in the source. The quotient map takes the open unit ball B of X onto the open unit ball of X/M . Therefore

$$\begin{aligned}\|\Psi(y^*)\| &= \sup_{x \in B} |\langle \pi x, y^* \rangle| \\ &= \sup_{y \in \pi(B)} |\langle y, y^* \rangle| = \|y^*\|.\end{aligned}$$

Thus Ψ too is an isometric isomorphism. $\square$

19.3. Adjoints of linear operators. Problem Set 6, Part 2 considers a densely defined operator in a Hilbert space and its adjoint, characterized by $(Lx, y) = (x, L^*y)$ for $x \in \text{Dom } L$ and $y \in \text{Dom } L^*$, and shows that L^* has closed graph. We now give the Banach-space formulation. Let X, Y be Banach spaces and let $L : \text{Dom } L \subset X \rightarrow Y$ be linear with dense domain. Define

$$\text{Dom } L^* = \{y^* \in Y^* : \exists C > 0 \text{ } |\langle Lx, y^* \rangle| \leq C \|x\| \text{ } (x \in \text{Dom } L)\}.$$

The constant may depend on y^* ; this domain is a vector subspace, but need not be dense in Y^* . For such a y^* , the bounded functional $x \mapsto \langle Lx, y^* \rangle$ on $\text{Dom } L$ extends to $x^* \in X^*$ by Hahn–Banach. Density and continuity make the extension unique. Set $L^*y^* = x^*$. This defines the linear *adjoint*

$$L^* : \text{Dom } L^* \subset Y^* \rightarrow X^*, \quad \langle Lx, y^* \rangle = \langle x, L^*y^* \rangle.$$

If $L \in \mathcal{L}(X, Y)$, its adjoint domain is all of Y^* .

Proposition 19.3. *For $L \in \mathcal{L}(X, Y)$, $L^* \in \mathcal{L}(Y^*, X^*)$ and $\|L^*\| = \|L\|$.*

Proof. Using the dual norm formula from Lecture 17,

$$\begin{aligned}\|L^*\| &= \sup_{\|y^*\| \leq 1} \sup_{\|x\| \leq 1} |\langle x, L^*y^* \rangle| \\ &= \sup_{\|x\| \leq 1, \|y^*\| \leq 1} |\langle Lx, y^* \rangle| = \|L\|.\end{aligned}$$

$\square$

In finite dimensions the adjoint is represented by the corresponding adjoint matrix.

Editorial note. For the Banach dual pairing used here, the matrix in dual bases is the transpose. Under the Hilbert-space Riesz identification, the complex Hilbert adjoint has the conjugate-transpose matrix. This clarifies the source's use of the symbol A^* .

Proposition 19.4. *The adjoint of any densely defined linear operator between Banach spaces has closed graph.*

Proof. Suppose $y_n^* \in \text{Dom } L^*$, $y_n^* \rightarrow y^*$ and $L^*y_n^* \rightarrow x^*$ in norm. For $x \in \text{Dom } L$, pass to the limit in $\langle Lx, y_n^* \rangle = \langle x, L^*y_n^* \rangle$ to get $\langle Lx, y^* \rangle = \langle x, x^* \rangle$. Hence $|\langle Lx, y^* \rangle| \leq \|x^*\| \|x\|$, so $y^* \in \text{Dom } L^*$ and $L^*y^* = x^*$. $\square$

For the graphs $\Gamma_L \subset X \times Y$ and $\Gamma_{L^*} \subset Y^* \times X^*$, introduce

$$J : Y^* \times X^* \rightarrow X^* \times Y^*, \quad J(y^*, x^*) = (-x^*, y^*).$$

Lemma 19.1. $J(\Gamma_{L^*}) = \Gamma_L^\perp$.

Proof. $(y^*, x^*) \in \Gamma_{L^*}$ is equivalent to $\langle Lx, y^* \rangle = \langle x, x^* \rangle$ for all $x \in \text{Dom } L$, or $\langle x, -x^* \rangle + \langle Lx, y^* \rangle = 0$ for every such x . This is precisely $(-x^*, y^*) \in \Gamma_L^\perp$. $\square$

Proposition 19.5. *For a densely defined linear operator, $\ker L^* = (\operatorname{im} L)^\perp$.*

Proof. $L^*y^* = 0$ implies $\langle Lx, y^* \rangle = 0$ on the domain. Conversely this vanishing puts y^* in the adjoint domain, and $\langle x, L^*y^* \rangle = 0$ on the dense domain implies $L^*y^* = 0$. $\square$

Proposition 19.6. *If L is closed and densely defined, then $\ker L = {}^\perp(\operatorname{im} L^*)$.*

Proof. As an exercise, verify

$$\begin{aligned}\Gamma_L \cap (X \times \{0\}) &= \ker L \times \{0\}, \\ {}^\perp(\Gamma_L^\perp + (X \times \{0\})^\perp) &= {}^\perp(\operatorname{im} L^*) \times \{0\}.\end{aligned}$$

Since Γ_L is closed, the double-annihilator identity makes the left-hand side of the second equality equal to $\Gamma_L \cap (X \times \{0\})$. The conclusion follows. $\square$

Consequently,

$${}^\perp \ker L^* = \overline{\operatorname{im} L}.$$

If L is closed, also

$$(\ker L)^\perp = \overline{\operatorname{im} L^*}^{w^*} \supset \overline{\operatorname{im} L^*},$$

where the final closure is in norm.

20. DOUBLE ADJOINTS AND THE CLOSED RANGE THEOREM

20.1. Review (November 20, 2015). For a closed subspace M of a Banach space X , $M^* \cong X^*/M^\perp$ and $(X/M)^* \cong M^\perp$. A densely defined linear operator $L : \text{Dom}(L) \subset X \rightarrow Y$ has an adjoint $L^* : \text{Dom}(L^*) \subset Y^* \rightarrow X^*$ satisfying

$$\langle Lx, y^* \rangle = \langle x, L^*y^* \rangle.$$

The adjoint is always closed. If $L \in \mathcal{L}(X, Y)$, then $L^* \in \mathcal{L}(Y^*, X^*)$ and $\|L^*\| = \|L\|$. Here dual pairings mean evaluation of a functional on a vector.

20.2. The double adjoint. Let X, Y be Banach spaces, and L densely defined. Assume $\text{Dom}(L^*)$ is norm dense in Y^* , so the double adjoint

$$L^{**} : \text{Dom}(L^{**}) \subset X^{**} \rightarrow Y^{**}$$

is defined. It satisfies

$$\langle L^*y^*, x^{**} \rangle = \langle y^*, L^{**}x^{**} \rangle \quad (y^* \in \text{Dom}(L^*)),$$

and its domain consists of the x^{**} for which

$$|\langle L^*y^*, x^{**} \rangle| \leq C \|y^*\| \quad \text{for all } y^* \in \text{Dom}(L^*)$$

for some C . The canonical embedding identifies the Banach space X with a closed subspace of X^{**} .

Proposition 20.1. *Under the canonical embeddings, $\text{Dom}(L) \subset \text{Dom}(L^{**})$ and $L^{**}x = Lx$ for $x \in \text{Dom}(L)$.*

Proof. For such x and every $y^* \in \text{Dom}(L^*)$,

$$\langle L^*y^*, x \rangle = \langle x, L^*y^* \rangle = \langle Lx, y^* \rangle = \langle y^*, Lx \rangle.$$

The last expression is bounded by $\|Lx\| \|y^*\|$, proving the claim by the definition of L^{**} . $\square$

Corollary 20.1. *If $L \in \mathcal{L}(X, Y)$, then $L^{**}|_X = L$ under the canonical embeddings. If X, Y are also reflexive, then $L^{**} = L$.*

For closed operators the continuity assumption can often be weakened.

Theorem 20.2. *Suppose X, Y are reflexive Banach spaces and L is densely defined and closed. Then $\text{Dom}(L^*)$ is norm dense in Y^* , and $L^{**} = L$.*

Proof. First let $y \in Y = Y^{**}$ vanish on $\text{Dom}(L^*)$. It suffices to prove $y = 0$ by Hahn–Banach and reflexivity. If $y \neq 0$, then $(0, y) \notin \Gamma_L$. Since Γ_L is a closed subspace of $X \times Y$, separation gives $(x_0^*, y_0^*) \in X^* \times Y^*$ vanishing on Γ_L but with $\langle y, y_0^* \rangle \neq 0$. (Check $(X \times Y)^* \cong X^* \times Y^*$.) Vanishing on the graph means

$$\langle x, x_0^* \rangle + \langle Lx, y_0^* \rangle = 0 \quad (x \in \text{Dom}(L)),$$

so $y_0^* \in \text{Dom}(L^*)$ and $L^*y_0^* = -x_0^*$. This contradicts the choice of y . Thus the domain of L^* is dense.

To identify the double adjoint, recall $J(a, b) = (-b, a)$ and the graph-annihilator identity $J(\Gamma_{L^*}) = \Gamma_L^\perp$. By closedness and reflexivity,

$$\Gamma_L = {}^\perp(\Gamma_L^\perp) = {}^\perp J(\Gamma_{L^*}).$$

But $(x, y) \in \Gamma_{L^{**}}$ is equivalent to

$$\langle x, L^*y^* \rangle = \langle y, y^* \rangle \quad \text{for every } y^* \in \text{Dom}(L^*),$$

which is precisely $(x, y) \in {}^\perp J(\Gamma_{L^*})$. Therefore $\Gamma_{L^{**}} = \Gamma_L$. $\square$

20.3. The closed range theorem. Write $N(L) = \ker L$ and $R(L) = \operatorname{Im} L$. For a densely defined linear operator, $N(L^*) = R(L)^\perp$. If L is also closed, $N(L) = {}^\perp R(L^*)$. Consequently,

$${}^\perp N(L^*) = \overline{R(L)}, \quad N(L)^\perp = \overline{R(L^*)}^{w^*} \quad \text{when } L \text{ is closed.}$$

Closedness of the range is important in applications.

Theorem 20.3 (Closed range theorem). *Let X, Y be Banach spaces and $L : \operatorname{Dom}(L) \subset X \rightarrow Y$ a densely defined closed linear operator. The following are equivalent:*

- (1) $R(L)$ is norm closed in Y ;
- (2) $R(L^*)$ is norm closed in X^* ;
- (3) $R(L^*)$ is weak-star closed in X^* ;
- (4) $R(L) = {}^\perp N(L^*)$;
- (5) $R(L^*) = N(L)^\perp$.

Corollary 20.2. *Under the same hypotheses,*

- (1) $R(L) = Y$ if and only if L^* is injective and $R(L^*)$ is closed;
- (2) $R(L^*) = X^*$ if and only if L is injective and $R(L)$ is closed.

Proof. For (1), surjectivity gives $N(L^*) = R(L)^\perp = \{0\}$ and, by the theorem, closedness of $R(L^*)$. Conversely, injectivity of L^* makes $R(L)$ dense, and the theorem makes it closed. The proof of (2) is analogous using (5). $\square$

In general, surjectivity of L implies injectivity of L^* , and surjectivity of L^* implies injectivity of L , but the converses fail. For example,

$$L : \ell^2 \rightarrow \ell^2, \quad (a_1, a_2, \dots) \mapsto (a_1, a_2/2, a_3/3, \dots)$$

is self-adjoint and injective but not surjective.

Proof of Theorem 20.3. The identities give (1) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5). Every weak-star closed set is norm closed, so (3) $\Rightarrow$ (2).

For (1) $\Rightarrow$ (3), assume $R(L)$ is closed and take $x^* \in N(L)^\perp$. Define

$$\ell : R(L) \rightarrow \mathbb{F}, \quad \ell(Lx) = \langle x, x^* \rangle.$$

This is well-defined: two preimages differ by a vector of $N(L)$. We use the following form of the open mapping theorem, whose proof is an exercise: if X, Y are Banach, $T : \operatorname{Dom}(T) \subset X \rightarrow Y$ is closed, and $R(T)$ is of second category in Y , then T is open for the subspace topology on its domain. Apply it to $L : \operatorname{Dom}(L) \rightarrow R(L)$, whose target is Banach. For every $\varepsilon > 0$, some $\delta > 0$ satisfies

$$L(B_X(0, \varepsilon) \cap \operatorname{Dom}(L)) \supset B_Y(0, \delta) \cap R(L).$$

On this target ball, $|\ell| \leq \varepsilon \|x^*\|$, proving continuity. Hahn–Banach extends ℓ to $y^* \in Y^*$. Then

$$\langle Lx, y^* \rangle = \ell(Lx) = \langle x, x^* \rangle \quad (x \in \operatorname{Dom}(L)),$$

so $y^* \in \operatorname{Dom}(L^*)$ and $L^*y^* = x^*$. Thus $R(L^*) \supset N(L)^\perp$, proving (5) and therefore (3).

For (2) $\Rightarrow$ (1), suppose $R(L^*)$ is norm closed. *Step 1.* Put $Y_1 = \overline{R(L)}$ and regard L as $L_1 : \operatorname{Dom}(L) \rightarrow Y_1$. Then $L_1^* : \operatorname{Dom}(L_1^*) \subset Y_1^* \rightarrow X^*$ is injective, since $R(L_1)$ is dense in Y_1 . Every $y_1^* \in Y_1^*$ extends to $y^* \in Y^*$ by Hahn–Banach. The identities

$$\langle L_1x, y_1^* \rangle = \langle Lx, y^* \rangle$$

show that $y_1^* \in \text{Dom}(L_1^*)$ if and only if its extension lies in $\text{Dom}(L^*)$, and their images under the adjoints agree. Restricting a functional in the other direction gives the converse inclusion of ranges. Hence $R(L_1^*) = R(L^*)$, which is closed. The adjoint L_1^* is closed. Applying the preceding open mapping statement to L_1^* and its Banach range gives, after shrinking a constant $\delta > 0$ if necessary:

$$x^* \in R(L_1^*), \|x^*\| \leq 1 \implies \exists y^* \in \text{Dom}(L_1^*) : L_1^* y^* = x^*, \quad \|y^*\| \leq 1/\delta.$$

Step 2. We claim

$$\overline{L_1(B_X(0, 1) \cap \text{Dom}(L_1))} \supset B_Y(0, \delta) \cap Y_1.$$

Compare the argument (a) $\Rightarrow$ (b) on p. 100 of Rudin. Write C for the closed set on the left; it is convex and balanced. If $y_0 \in Y_1 \setminus C$, geometric Hahn–Banach gives $y_0^* \in Y_1^*$ with

$$|\langle L_1 x, y_0^* \rangle| \leq 1 \quad (x \in B_X(0, 1) \cap \text{Dom}(L_1)), \quad |\langle y_0, y_0^* \rangle| = a > 1.$$

Homogeneity gives $|\langle L_1 x, y_0^* \rangle| \leq \|x\|$ on the domain, so $y_0^* \in \text{Dom}(L_1^*)$.

Density of the domain and the last bound imply $\|L_1^* y_0^*\| \leq 1$. Step 1 gives $y^* \in \text{Dom}(L_1^*)$ with $\|y^*\| \leq 1/\delta$ and $L_1^* y^* = L_1^* y_0^*$. For $y_0 = L_1 x_0 \in R(L_1)$, it follows that

$$a = |\langle y_0, y_0^* \rangle| = |\langle x_0, L_1^* y_0^* \rangle| = |\langle x_0, L_1^* y^* \rangle| = |\langle y_0, y^* \rangle| \leq \|y_0\| \|\delta\|.$$

In fact injectivity of L_1^* gives $y^* = y_0^*$, so this last inequality holds for every $y_0 \in Y_1$. Thus $\|y_0\| \geq a\delta > \delta$, proving the claim.

Editorial note. The source's final estimate in Step 2 is written only for $y_0 \in R(L_1)$. Injectivity of L_1^* , already proved in Step 1, extends it to all $y_0 \in Y_1$, giving the claimed inclusion. The constant in Step 1 may be reduced to avoid an issue at a ball boundary.

Step 3. We prove

$$L_1(B_X(0, 1) \cap \text{Dom}(L_1)) \supset B_Y(0, \delta/2) \cap Y_1.$$

Compare (b) $\Rightarrow$ (c) on pp. 100–101 of Rudin. Fix y_1 in the target ball and positive ε_n with $\sum_n \varepsilon_n < \delta/2$. By Step 2 and rescaling, for any $y \in Y_1$ and any desired positive error, one can choose $x \in \text{Dom}(L_1)$ with $\|x\| \leq \|y\|/\delta$ and $L_1 x$ approximating y to that error. Here the closure in Step 2 also contains the closed ball of radius δ , by closedness. Inductively choose x_n and $y_{n+1} = y_n - L_1 x_n$ so that

$$\|x_n\| \leq \|y_n\|/\delta, \quad \|y_{n+1}\| < \varepsilon_n.$$

Then

$$\sum_n \|x_n\| \leq \frac{\|y_1\|}{\delta} + \frac{1}{\delta} \sum_n \varepsilon_n < 1.$$

The partial sums z_n lie in the domain and converge in X to some z with $\|z\| < 1$. Also $L_1 z_n = y_1 - y_{n+1} \rightarrow y_1$. Closedness gives $z \in \text{Dom}(L_1)$ and $L_1 z = y_1$. Thus Step 3 holds. Since the range is a vector subspace containing a neighborhood of zero in Y_1 , $R(L_1) = Y_1$, proving $R(L)$ closed. $\square$

21. COMPACT OPERATORS

21.1. Review (November 27, 2015). A densely defined operator L has an adjoint L^* ; when its domain is also dense, the double adjoint is defined. For a densely defined closed operator between reflexive Banach spaces, $\text{Dom}(L^*)$ is dense and $L^{**} = L$. For a densely defined closed operator between Banach spaces, the closed range theorem equates closedness of $R(L)$, norm and weak-star closedness of $R(L^*)$, and the identities $R(L) = {}^\perp N(L^*)$, $R(L^*) = N(L)^\perp$.

21.2. Compact operators. Let X, Y be Banach spaces and $U = \{x \in X : \|x\| < 1\}$.

Definition 21.1. A linear operator $L : X \rightarrow Y$ is *compact* if $\overline{L(U)}$ is compact in Y . Equivalently, the image of every bounded set has compact closure; equivalently, the image of every bounded sequence has a convergent subsequence.

Every compact subset of a Banach space is bounded (why?), so compact operators are bounded and hence continuous. Write $\mathcal{K}(X, Y)$ for all compact operators. A linear combination of two compact operators is compact, so $\mathcal{K}(X, Y)$ is a vector subspace of $\mathcal{L}(X, Y)$.

Theorem 21.1. $\mathcal{K}(X, Y)$ is closed in the operator norm.

Proof. Suppose L_n are compact and $\|L_n - L\| \rightarrow 0$. To prove $\overline{L(U)}$ compact, completeness of Y reduces the problem to total boundedness of $L(U)$ (Problem Set 9, Part 2, Problem 4(1)). Given $\varepsilon > 0$, choose n with $\|L_n - L\| < \varepsilon/2$. Compactness gives finitely many $y_1, \dots, y_m \in Y$ with

$$L_n(U) \subset \bigcup_{k=1}^m B(y_k, \varepsilon/2).$$

For $x \in U$, $\|Lx - L_n x\| < \varepsilon/2$, and $L_n x$ is within $\varepsilon/2$ of some y_k . Therefore $L(U) \subset \bigcup_k B(y_k, \varepsilon)$. $\square$

21.3. Finite-rank approximation. A bounded operator with finite-dimensional range is compact. Its unit-ball image is bounded, and its closure in the finite-dimensional range is compact. Such an operator is called *finite-rank*. Consequently, an operator-norm limit of finite-rank operators is compact. The converse need not hold for general Banach spaces, but it does hold with a Hilbert codomain.

Theorem 21.2. If Y is Hilbert, every $L \in \mathcal{K}(X, Y)$ is an operator-norm limit of finite-rank operators.

Proof. Given $\varepsilon > 0$, choose $y_1, \dots, y_n \in Y$ such that $\overline{L(U)} \subset \bigcup_k B(y_k, \varepsilon)$. Let $Y_1 = \text{span}\{y_1, \dots, y_n\}$, a closed subspace. The orthogonal decomposition $Y = Y_1 \oplus Y_1^\perp$ defines a linear projection $P : Y \rightarrow Y_1$. If $y = y' + y''$ is that decomposition,

$$\|Py\|^2 = \|y'\|^2 \leq \|y'\|^2 + \|y''\|^2 = \|y\|^2,$$

so $\|P\| \leq 1$. Set $L_\varepsilon = PL$, which has finite rank. For $x \in U$, choose y_k with $\|Lx - y_k\| < \varepsilon$. Since $Py_k = y_k$,

$$\|L_\varepsilon x - y_k\| = \|P(Lx - y_k)\| < \varepsilon, \quad \|L_\varepsilon x - Lx\| < 2\varepsilon.$$

Thus $\|L_\varepsilon - L\| \leq 2\varepsilon$, proving approximation. $\square$

Remark 21.1. If a compact operator $L : X \rightarrow Y$ has closed range, it has finite rank. Indeed, $L : X \rightarrow \overline{R(L)}$ is a continuous surjection between Banach spaces and hence open. Since $\overline{L(U)}$ is compact, the range is locally compact, so it is finite-dimensional.

21.4. Schauder's theorem.

Theorem 21.3 (Schauder). *For Banach spaces X, Y and $L \in \mathcal{L}(X, Y)$, L is compact if and only if L^* is compact.*

Proof. Suppose L is compact and let y_n^* be a sequence in the unit ball of Y^* . These functionals are equicontinuous, since

$$|y_n^*(y) - y_n^*(y')| \leq \|y - y'\|.$$

They are pointwise bounded on the compact set $\overline{L(U)}$. The Ascoli theorem (the source refers to p. 394) gives a subsequence $y_{n_k}^*$ converging uniformly on that compact set. Then

$$\begin{aligned} \|L^* y_{n_k}^* - L^* y_{n_l}^*\| &= \sup_{x \in U} |\langle x, L^*(y_{n_k}^* - y_{n_l}^*) \rangle| \\ &= \sup_{x \in U} |y_{n_k}^*(Lx) - y_{n_l}^*(Lx)| \longrightarrow 0. \end{aligned}$$

Completeness of X^* gives convergence of this image subsequence, so L^* is compact. Conversely, compactness of L^* gives compactness of L^{**} by the implication just proved. The canonical embedding identifies the unit ball U of X with a subset of the unit ball U^{**} of X^{**} , and $L^{**}|_X = L$ under the canonical embeddings. Thus $L(U) \subset L^{**}(U^{**})$ is totally bounded in Y^{**} and hence in the isometrically embedded copy of Y . Completeness of Y shows that $\overline{L(U)}$ is compact (again Problem Set 9, Part 2, Problem 4(1)). $\square$

Next time we will prove the following theorem.

Theorem 21.4. *Let X be Banach and $L \in \mathcal{K}(X)$. Then*

- (a) $\dim N(I - L) < \infty$;
- (b) $R(I - L) = {}^\perp N(I - L^*)$, and in particular the range is closed;
- (c) $N(I - L) = \{0\}$ if and only if $R(I - L) = X$;
- (d) $\dim N(I - L) = \dim N(I - L^*)$.

Editorial note. The annihilator in (b) is the preannihilator in X of a subspace of X^* ; the notation is made consistent with Lecture 20.

22. THE FREDHOLM ALTERNATIVE

22.1. Review (December 1, 2015). A linear operator $L : X \rightarrow Y$ is compact if the image of its unit ball has compact closure; equivalently, the image of every bounded sequence has a convergent subsequence. The compact operators $\mathcal{K}(X, Y)$ form a closed subspace of $\mathcal{L}(X, Y)$. Every finite-rank operator is compact. If Y is a Hilbert space, every compact operator can be approximated in operator norm by finite-rank operators. Schauder's theorem says that L is compact if and only if its dual operator L^* is compact.

22.2. The Fredholm alternative. Let X be a Banach space and I its identity operator.

Theorem 22.1. *If $L \in \mathcal{K}(X) = \mathcal{K}(X, X)$, then:*

- (a) $\dim \ker(I - L) < \infty$;
- (b) $\text{Ran}(I - L)$ is closed;
- (c) $\text{Ran}(I - L) = {}^\perp \ker(I - L^*)$;
- (d) $\text{Ran}(I - L) = X$ if and only if $\ker(I - L) = \{0\}$;
- (e) $\dim \ker(I - L) = \dim \ker(I - L^*)$.

Here ${}^\perp E = \{x \in X : f(x) = 0 \text{ for all } f \in E\}$ for $E \subset X^*$.

Corollary 22.1 (Fredholm alternative). *For a compact operator L on a Banach space X , exactly one of the following occurs:*

- For every $x_0 \in X$, the equation $x - Lx = x_0$ has a unique solution.
- The homogeneous equation $x - Lx = 0$ has a nonzero finite-dimensional solution space, of dimension $n = \dim \ker(I - L)$.

Proof. If $\ker(I - L) = \{0\}$, part (d) gives existence for every right-hand side. If $(I - L)x_1 = (I - L)x_2$, then $x_1 - x_2 \in \ker(I - L)$, proving uniqueness. Otherwise part (a) gives $0 < n < \infty$, and a basis of the kernel consists of n linearly independent homogeneous solutions. $\square$

Proof of Theorem 22.1, parts (a)–(c). (a) Put $Y = \ker(I - L)$. On Y , $Ly = y$. Thus the closed unit ball of Y is contained in the compact closure of the image under L of the unit ball of X . It is closed, hence compact. A normed space with compact closed unit ball is finite-dimensional (see Rudin, pp. 17–18, or supply a Banach-space proof).

(b) Suppose $y_n = x_n - Lx_n \rightarrow y_0$. Let $d_n = \text{dist}(x_n, \ker(I - L))$. Since this kernel is finite-dimensional, choose $\tilde{x}_n \in \ker(I - L)$ attaining the distance: $d_n = \|x_n - \tilde{x}_n\|$. Notice that

$$(22.1) \quad y_n = (x_n - \tilde{x}_n) - L(x_n - \tilde{x}_n).$$

We claim that $\{x_n - \tilde{x}_n\}$ is bounded. Otherwise pass to indices n_k with $d_{n_k} \rightarrow \infty$ and set $z_k = (x_{n_k} - \tilde{x}_{n_k})/d_{n_k}$. Then $\|z_k\| = 1$ and $z_k - Lz_k = y_{n_k}/d_{n_k} \rightarrow 0$. Compactness gives a subsequence with $Lz_k \rightarrow w$, hence $z_k \rightarrow w$. Taking limits gives $w = Lw$, so $w \in \ker(I - L)$. On the other hand,

$$\text{dist}(z_k, \ker(I - L)) = \frac{\text{dist}(x_{n_k}, \ker(I - L))}{d_{n_k}} = 1.$$

Passing to the limit contradicts $w \in \ker(I - L)$. Thus the reduced sequence is bounded. Pass to a subsequence for which $L(x_{n_k} - \tilde{x}_{n_k}) \rightarrow x_0$. Equation (22.1) gives $x_{n_k} - \tilde{x}_{n_k} \rightarrow z_0 = y_0 + x_0$, and hence $z_0 - Lz_0 = y_0$. Therefore $y_0 \in \text{Ran}(I - L)$.

(c) The general annihilator identity ${}^\perp \ker(I - L^*) = \overline{\text{Ran}(I - L)}$, together with (b), proves the assertion. $\square$

Proof of part (d). First suppose $\ker(I - L) = \{0\}$. If $X_1 = \text{Ran}(I - L) \neq X$, then X_1 is a Banach space by (b). Also $LX_1 \subset X_1$, since $L(I - L)x = (I - L)Lx$. The restriction $L_1 = L|_{X_1}$ is compact. Its range $X_2 = (I - L)X_1$ is closed by (b), and $X_2 \neq X_1$: otherwise $(I - L)X_1 = (I - L)X$ and injectivity would give $X_1 = X$. Continuing gives strictly decreasing closed subspaces

$$X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots, \quad X_n = (I - L)^n X.$$

We use Riesz's lemma: for a proper closed subspace M of a normed space Y and $0 < \varepsilon < 1$, there is $y \in Y$ with $\|y\| = 1$ and $\text{dist}(y, M) \geq 1 - \varepsilon$. Indeed, choose $z \notin M$, put $d = \text{dist}(z, M) > 0$, and choose $m \in M$ with $d \leq \|z - m\| \leq d/(1 - \varepsilon)$. Then $y = (z - m)/\|z - m\|$ has distance $d/\|z - m\| \geq 1 - \varepsilon$ from M . Choose $x_n \in X_n$ with $\|x_n\| = 1$ and $\text{dist}(x_n, X_{n+1}) \geq 1/2$. For $n > m$,

$$Lx_n - Lx_m = -(x_n - Lx_n) + (x_m - Lx_m) + x_n - x_m \in X_{m+1} - x_m.$$

Consequently $\|Lx_n - Lx_m\| \geq 1/2$, contradicting compactness. Thus $\text{Ran}(I - L) = X$.

Conversely, if $\text{Ran}(I - L) = X$, then $\ker(I - L^*) = \text{Ran}(I - L)^\perp = \{0\}$. By Schauder's theorem, L^* is compact. The implication just proved, applied to L^* , gives $\text{Ran}(I - L^*) = X^*$. Hence $\ker(I - L) = {}^\perp \text{Ran}(I - L^*) = \{0\}$. $\square$

22.3. Complemented subspaces.

Definition 22.1. A closed subspace M of a TVS X is *complemented* if there is a closed subspace N with $X = M + N$ and $M \cap N = \{0\}$.

Example 22.1. If X is locally convex and $\dim M = n < \infty$, then M is complemented. Let $e_1, \dots, e_n$ be a basis, and write $x = \sum_i \alpha_i(x)e_i$ on M . The coordinate isomorphism $M \rightarrow \mathbb{F}^n$ is a homeomorphism, so its coordinates α_i are continuous linear functionals. Hahn-Banach extends them to continuous linear functionals L_i on X . Put $N = \bigcap_i \ker L_i$. It is closed, $M \cap N = \{0\}$, and for every $x \in X$ the vector $x_0 = \sum_i L_i(x)e_i$ satisfies $x - x_0 \in N$. Thus $X = M + N$.

Example 22.2. If M is closed and $\dim(X/M) = n < \infty$, then M is complemented. Let $\pi : X \rightarrow X/M$ be the quotient map. Choose representatives x_i of a basis e_i of X/M , and put $N = \text{span}\{x_i\}$. Then N is closed because it is finite-dimensional. If $\sum_i a_i x_i \in M$, applying π gives $\sum_i a_i e_i = 0$, hence all $a_i = 0$. Thus $M \cap N = \{0\}$. Writing any $\pi(x)$ in the basis also shows $x - \sum_i a_i x_i \in M$, proving $X = M + N$.

Proof of part (e). Write $d = \dim \ker(I - L)$ and $d^* = \dim \ker(I - L^*)$. Suppose $d < d^*$. The finite-dimensional kernel is complemented; let P be a continuous projection onto it, constructed as in the first example. By (c), the closed range $\text{Ran}(I - L)$ has codimension d^* (exercise). The second example provides a closed complement N of dimension d^* . Choose an injective but nonsurjective linear map $\ell : \ker(I - L) \rightarrow N$ and set $\tilde{L} = L + \ell P$. This is compact because ℓP has finite rank. If $(I - \tilde{L})x = 0$, then $(I - L)x = \ell Px$. The left side lies in $\text{Ran}(I - L)$, the right in N , so both vanish. Thus $x \in \ker(I - L)$ and $Px = 0$, which implies $x = 0$.

Part (d) now implies $\text{Ran}(I - \tilde{L}) = X$. But choose $y \in N \setminus \text{Ran } \ell$. The equation $(I - \tilde{L})x = y$ would imply $(I - L)x \in N$, and hence $(I - L)x = 0$. Then $y = -\ell Px \in$

$\text{Ran } \ell$, a contradiction. Therefore $d^* \leq d$. Apply this inequality to the compact operator L^* :

$$\dim \ker(I - L^{**}) \leq \dim \ker(I - L^*) \leq \dim \ker(I - L).$$

The canonical embedding $X \hookrightarrow X^{**}$ intertwines L and L^{**} , so it embeds $\ker(I - L)$ into $\ker(I - L^{**})$. All the displayed dimensions must therefore be equal, proving (e). $\square$

Editorial note. In the final comparison the source writes $L^{**} = L$ on X . This means equality after the canonical embedding; no reflexivity of X is assumed. Riesz's lemma is stated with $0 < \varepsilon < 1$, the range needed for its displayed proof.

23. FREDHOLM OPERATORS

23.1. Review (December 4, 2015). For a compact operator L on a Banach space, $\ker(I - L)$ is finite-dimensional, $\text{Ran}(I - L)$ is closed, and

$$\text{Ran}(I - L) = {}^\perp \ker(I - L^*).$$

The range of $I - L^*$ is also closed. Moreover, $I - L$ is surjective if and only if it is injective, and $\dim \ker(I - L) = \dim \ker(I - L^*)$.

23.2. Definitions and examples. Let X, Y be Banach spaces.

Definition 23.1. A continuous linear operator $L \in \mathcal{L}(X, Y)$ is *Fredholm* if its kernel is finite-dimensional, its range is closed, and

$$\text{codim Ran } L := \dim(Y / \text{Ran } L) < \infty.$$

Write $\mathcal{Fr}(X, Y)$ for the collection of Fredholm operators. Its *index* is the integer

$$\text{ind } L = \dim \ker L - \text{codim Ran } L.$$

This is an important class of operators in modern mathematics.

Remark 23.1. For operators between Banach spaces the closed-range condition follows from finite algebraic codimension and need not be imposed separately. More generally, if the range of a bounded operator admits a closed algebraic complement, its range is closed.

Example 23.1. If $K \in \mathcal{K}(X)$, then $I - K$ is Fredholm of index zero, since

$$\text{codim Ran}(I - K) = \dim \ker(I - K^*) = \dim \ker(I - K)$$

(compare Problem Set 12, Part 2, Problem 1(1)).

Example 23.2. On ℓ^2 , consider the operator

$$L(a_1, a_2, a_3, \dots) = (a_3, a_4, a_5, \dots).$$

Its kernel is $\{(a_1, a_2, 0, 0, \dots)\}$ and its range is all of ℓ^2 , so $\text{ind } L = 2$. Its adjoint is

$$L^*(a_1, a_2, \dots) = (0, 0, a_1, a_2, \dots),$$

with zero kernel and range of codimension two. Hence $\text{ind } L^* = -2$.

Example 23.3. The differentiation operator $L : C^1([0, 1]) \rightarrow C([0, 1])$, $Lf = f'$, is surjective: $L(\int_0^x f(t) dt) = f(x)$. Its kernel consists of constant functions. It is Fredholm of index one (using the usual C^1 and uniform norms).

For a Fredholm operator, injectivity means $\dim \ker L = 0$, and surjectivity means $\text{ind } L = \dim \ker L$.

Proposition 23.1. If $L \in \mathcal{Fr}(X, Y)$, then $L^* \in \mathcal{Fr}(Y^*, X^*)$ and $\text{ind } L^* = -\text{ind } L$.

Proof. The closed range theorem gives closedness of $\text{Ran } L^*$. The annihilator and quotient identities give

$$\dim \ker L^* = \text{codim Ran } L, \quad \text{codim Ran } L^* = \dim \ker L.$$

Subtract the dimensions. □

Lemma 23.1. For bounded operators $L_1 : X \rightarrow Y$ and $L_2 : Y \rightarrow Z$, if two of $L_1, L_2, L_2 L_1$ are Fredholm, then so is the third, and

$$\text{ind}(L_2 L_1) = \text{ind } L_2 + \text{ind } L_1.$$

Proof. Exercise, as in the source. $\square$

Proposition 23.2. $\mathcal{Fr}(X, Y)$ is open in $\mathcal{L}(X, Y)$, and the index is locally constant.

Proof. For L Fredholm, choose closed complements

$$X = \ker L \oplus X_1, \quad Y = \operatorname{Ran} L \oplus Y_1.$$

Let $i_1 : X_1 \hookrightarrow X$ and $P_2 : Y \rightarrow \operatorname{Ran} L$ be inclusion and projection. The operator $L_0 = P_2 L i_1 = L|_{X_1} : X_1 \rightarrow \operatorname{Ran} L$ is a continuous linear isomorphism, with continuous inverse. For $\tilde{L}$ sufficiently close to L ,

$$\|P_2 \tilde{L} i_1 - L_0\| \leq \|P_2\| \|\tilde{L} - L\| \|i_1\|$$

is small, so $\tilde{L}_0 = P_2 \tilde{L} i_1$ is invertible (compare Problem Set 5, Part 1, Problem 4). The operators i_1, P_2 are Fredholm of indices $-\dim \ker L$ and $\dim Y_1$, respectively. Applying Lemma 23.1 yields Fredholmness of $\tilde{L}$ and

$$0 = \operatorname{ind} \tilde{L}_0 = \operatorname{codim} \operatorname{Ran} L + \operatorname{ind} \tilde{L} - \dim \ker L.$$

Thus $\operatorname{ind} \tilde{L} = \operatorname{ind} L$. $\square$

23.3. Parametrices and compact perturbations.

Theorem 23.3. For $L \in \mathcal{L}(X, Y)$ the following are equivalent:

- (1) L is Fredholm.
- (2) There is a parametrix (pseudo-inverse) $\tilde{L} \in \mathcal{L}(Y, X)$ such that $\tilde{L}L = I_X + K_1$ and $L\tilde{L} = I_Y + K_2$, with K_1, K_2 finite rank.
- (3) The same identities hold for some $\tilde{L}$ with K_1, K_2 compact.
- (4) There are $\tilde{L}_1, \tilde{L}_2 \in \mathcal{L}(Y, X)$ and compact K_1, K_2 such that $\tilde{L}_1 L = I_X + K_1$ and $L\tilde{L}_2 = I_Y + K_2$.

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate. For (4) $\Rightarrow$ (1), the space

$$\ker L \subset \ker(\tilde{L}_1 L) = \ker(I_X + K_1)$$

is finite-dimensional. Also

$$\operatorname{Ran} L \supset \operatorname{Ran}(L\tilde{L}_2) = \operatorname{Ran}(I_Y + K_2)$$

has finite codimension. The preceding remark supplies closedness of the range. For (1) $\Rightarrow$ (2), use the decompositions from the preceding proof and define $\tilde{L} = i_1 L_0^{-1} P_2$, or

$$\tilde{L}(y + y_1) = L_0^{-1} y \quad (y \in \operatorname{Ran} L, y_1 \in Y_1).$$

This is bounded. Let $-K_1$ be projection onto $\ker L$, and $-K_2$ projection onto Y_1 . They have finite rank, and direct calculation gives $\tilde{L}L - I_X = K_1$, $L\tilde{L} - I_Y = K_2$. $\square$

Remark 23.2. The construction gives $LK_1 = 0$ and $\tilde{L}K_2 = 0$, so we may choose the parametrix with $L\tilde{L}L = L$ and $\tilde{L}L\tilde{L} = \tilde{L}$.

Theorem 23.4. If L is Fredholm and $K \in \mathcal{K}(X, Y)$, then $L + K$ is Fredholm and $\operatorname{ind}(L + K) = \operatorname{ind} L$.

Proof. Choose a parametrix $\tilde{L}$ as above. It is Fredholm with $\text{ind } \tilde{L} = -\text{ind } L$. The identities

$$(L + K)\tilde{L} = I_Y + K_2 + K\tilde{L}, \quad \tilde{L}(L + K) = I_X + K_1 + \tilde{L}K$$

show that $L + K$ is Fredholm. The composition on the first side is an identity plus a compact operator, hence has index zero. Index additivity therefore gives $\text{ind}(L + K) = -\text{ind } \tilde{L} = \text{ind } L$. $\square$

Theorem 23.5. *A Fredholm operator has index zero if and only if it is a compact perturbation of an invertible bounded operator: $L = L_0 + K$, with L_0 invertible and K compact.*

Proof. Exercise, as in the source. $\square$

24. THE SPECTRUM

24.1. Review (December 8, 2015). An operator between Banach spaces is Fredholm when its kernel and cokernel are finite-dimensional; its range is then closed. The index is $\dim \ker L - \operatorname{codim} \operatorname{Ran} L$. Fredholm operators form an open set, and the index is locally constant. They are precisely the operators admitting a two-sided parametrix modulo compact operators. The index changes sign under duality, is additive under composition, and is unchanged by compact perturbations.

24.2. Resolvent and spectrum. In finite dimensions, λ is an eigenvalue of $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ if and only if $\ker(L - \lambda I) \neq \{0\}$, if and only if $\operatorname{Ran}(L - \lambda I) \neq \mathbb{C}^n$, if and only if $L - \lambda I$ is not invertible. Now let X be a complex Banach space and $L \in \mathcal{L}(X)$. (The definitions extend to closed operators with domains contained in X .)

Definition 24.1. The *resolvent set* is

$$\rho(L) = \{\lambda \in \mathbb{C} : \lambda I - L \text{ is bijective and } (\lambda I - L)^{-1} \in \mathcal{L}(X)\}.$$

Its members are called regular points. The *resolvent* at λ is $R_\lambda(L) = (\lambda I - L)^{-1}$. The *spectrum* is $\sigma(L) = \mathbb{C} \setminus \rho(L)$.

For bounded L , the inverse mapping theorem shows that bijectivity of $\lambda I - L$ already implies boundedness of its inverse.

Definition 24.2 (Types of spectrum). (1) The *point spectrum* $\sigma_p(L)$ consists of eigenvalues: $\lambda \in \sigma_p(L)$ exactly when $\ker(\lambda I - L) \neq \{0\}$. The dimension of this kernel is the (geometric) multiplicity.
 (2) The *continuous spectrum* $\sigma_c(L)$ consists of $\lambda \in \sigma(L)$ for which $\lambda I - L$ is injective and has dense range.
 (3) The *residual spectrum* $\sigma_r(L)$ consists of $\lambda \in \sigma(L)$ for which $\lambda I - L$ is injective and its range is not dense.

Example 24.1 (The unilateral shift). On ℓ^2 , define

$$L(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

It has no eigenvalues: $La = \lambda a$ forces every coordinate to vanish. Since L is injective with nondense range, $0 \in \sigma_r(L)$. Also $1 \in \sigma_c(L)$. Indeed,

$$(I - L)(a_1, a_2, \dots) = (a_1, a_2 - a_1, a_3 - a_2, \dots)$$

is injective. Its range is dense because

$$\overline{\operatorname{Ran}(I - L)} = {}^\perp \ker(I - L^*), \quad L^*(a_1, a_2, \dots) = (a_2, a_3, \dots),$$

and L^* has no nonzero fixed vector in ℓ^2 . The full residual and continuous spectra are problem-set exercises.

Editorial note. To finish the assertion $1 \in \sigma_c(L)$, note that $I - L$ is not surjective: the preimage of $(1, 0, 0, \dots)$ would be $(1, 1, 1, \dots)$, which is not in ℓ^2 .

Proposition 24.1. If $|\lambda| > \|L\|$, then $\lambda \in \rho(L)$.

Proof. Since $\|\lambda^{-1}L\| < 1$, the operator $I - \lambda^{-1}L$ is invertible by the Neumann series, so $\lambda I - L$ is invertible. $\square$

The Neumann-series assertion (Problem Set 5, Part 1, Problem 4) is

$$A^{-1} = \sum_{k=0}^{\infty} (I - A)^k \quad \text{if } \|I - A\| < 1, \quad \|A^{-1}\| \leq \frac{1}{1 - \|I - A\|}.$$

The series converges to a bounded operator (also compare the pointwise limit corollary of Banach–Steinhaus in Lecture 9).

Proposition 24.2. $\rho(L)$ is open.

Proof. For $\lambda_0 \in \rho(L)$, factor

$$\lambda I - L = (\lambda_0 I - L)[I + (\lambda - \lambda_0)(\lambda_0 I - L)^{-1}].$$

The bracket is invertible whenever $|\lambda - \lambda_0| < \|(\lambda_0 I - L)^{-1}\|^{-1}$. $\square$

Corollary 24.1. $\sigma(L)$ is a compact subset of $\mathbb{C}$.

Definition 24.3. The *spectral radius* is $r(L) = \sup\{|\lambda| : \lambda \in \sigma(L)\}$.

Proposition 24.3 (Resolvent identity). For $\lambda, \mu \in \rho(L)$,

$$R_\lambda(L) - R_\mu(L) = (\mu - \lambda)R_\lambda(L)R_\mu(L).$$

Proof. Insert $I = (\mu I - L)R_\mu$ in R_λ , and write $\mu I - L = (\mu - \lambda)I + (\lambda I - L)$. $\square$

24.3. Spectrum of a compact operator. Let K be compact on a complex Banach space X .

Proposition 24.4. If $\dim X = \infty$, then $0 \in \sigma(K)$.

Proof. Otherwise K is invertible, so $I = KK^{-1}$ is compact. Its closed unit ball would be compact, implying $\dim X < \infty$, a contradiction. $\square$

Proposition 24.5. Every nonzero point of $\sigma(K)$ is an eigenvalue.

Proof. If $\lambda \neq 0$ is not an eigenvalue, then $\ker(I - \lambda^{-1}K) = \{0\}$. The Fredholm alternative makes $I - \lambda^{-1}K$ surjective, hence invertible. Thus $\lambda \in \rho(K)$. $\square$

Theorem 24.6. Zero is the only possible accumulation point of $\sigma(K)$.

Proof. Suppose distinct nonzero eigenvalues λ_n converge to $\lambda \neq 0$, and choose eigenvectors x_n with $Kx_n = \lambda_n x_n$. The vectors $x_1, \dots, x_n$ are linearly independent. To see this, if $\sum_{i=1}^n \alpha_i x_i = 0$ with $\alpha_1 \neq 0$, apply the polynomial

$$p(K), \quad p(t) = \prod_{i=2}^n \frac{t - \lambda_i}{\lambda_1 - \lambda_i}.$$

It gives $\alpha_1 x_1 = 0$, a contradiction. Let $E_n = \text{span}\{x_1, \dots, x_n\}$, a strictly increasing sequence. By Riesz's lemma choose $y_n \in E_n$ with $\|y_n\| = 1$ and $\text{dist}(y_n, E_{n-1}) \geq 1/2$. For $n > m \geq 2$,

$$\frac{Ky_n}{\lambda_n} - \frac{Ky_m}{\lambda_m} = \frac{Ky_n - \lambda_n y_n}{\lambda_n} - \frac{Ky_m - \lambda_m y_m}{\lambda_m} + y_n - y_m.$$

All terms except y_n belong to E_{n-1} : writing $y_n = \sum_{i=1}^n a_i x_i$ gives $(K - \lambda_n)y_n = \sum_{i=1}^{n-1} a_i(\lambda_i - \lambda_n)x_i$. Hence the displayed norm is at least $1/2$.

Compactness gives a convergent subsequence of Ky_n . Since $\lambda_n \rightarrow \lambda \neq 0$, it also gives a convergent subsequence of Ky_n/λ_n , a contradiction. $\square$

Corollary 24.2. *For each $\varepsilon > 0$, the set $\{\lambda \in \sigma(K) : |\lambda| > \varepsilon\}$ is finite. If X is infinite-dimensional, the spectrum has one of the forms $\{0\}$, $\{0, \lambda_1, \dots, \lambda_n\}$, or $\{0, \lambda_1, \lambda_2, \dots\}$ with $\lambda_n \rightarrow 0$.*

Each nonzero eigenvalue has finite multiplicity. This follows directly from the Fredholm theorem, or by using independent eigenvectors for one fixed eigenvalue in the preceding construction.

Proposition 24.7. *For $\lambda \neq 0$, $\lambda \in \sigma_p(K)$ if and only if $\lambda \in \sigma_p(K^*)$, where K^* is the Banach dual operator.*

Proof. The Fredholm theorem gives $\dim \ker(\lambda I - K) = \dim \ker(\lambda I - K^*)$ for $\lambda \neq 0$. □

Editorial note. The spectral forms including zero require $\dim X = \infty$; in finite dimensions a compact operator can be invertible. The restriction $\lambda \neq 0$ is also necessary in the last proposition; the source's proof divides by λ .

For a general bounded operator the preceding eigenvalue assertion need not hold. One can prove the following instead.

Proposition 24.8. *For $L \in \mathcal{L}(X)$:*

- (a) *If $\lambda \in \sigma_p(L)$, then $\lambda \in \sigma_p(L^*) \cup \sigma_r(L^*)$.*
- (b) *If $\lambda \in \sigma_r(L)$, then $\lambda \in \sigma_p(L^*)$.*

Proof. Exercise, as in the source. □

25. ORTHONORMAL BASES IN HILBERT SPACES

25.1. Review (December 11, 2015). The resolvent set is open and its complement, the spectrum, is compact. The spectrum is the disjoint union of point, continuous, and residual spectra, according to failure of injectivity, or injectivity with dense proper range, or injectivity with nondense range. For a compact operator on an infinite-dimensional space, zero belongs to the spectrum, every nonzero spectral point is an eigenvalue, and zero is the only possible accumulation point.

25.2. Hilbert-space facts. A Hilbert space H is a complete inner product space. Our inner product is linear in the first argument and conjugate-linear in the second:

$$(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z), \quad (x, y) = \overline{(y, x)}, \quad (x, x) > 0 \ (x \neq 0).$$

Its norm is $\|x\| = \sqrt{(x, x)}$. We have already proved:

- (1) Cauchy–Schwarz: $|(x, y)| \leq \|x\| \|y\|$ (Problem Set 4, Part 1).
- (2) The parallelogram law: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.
- (3) Every nonempty closed convex set E has a unique element of smallest norm (Lecture 7).
- (4) For a closed vector subspace M , $H = M \oplus M^\perp$ (Lecture 17). The orthogonal projection is continuous with norm at most one, and norm one if $M \neq \{0\}$ (Lecture 21).
- (5) Every $L \in H^*$ has a unique representing vector $y \in H$ such that $L(x) = (x, y)$ and $\|L\| = \|y\|$ (Lecture 17).
- (6) A compact map from a normed space into H is an operator-norm limit of finite-rank maps (Lecture 21).
- (7) $x_n \rightarrow x$ in norm if and only if $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ (midterm).

Examples include $\mathbb{R}^n$, $\mathbb{C}^n$, ℓ^2 , L^2 , and $H^1(D)$ (compare Problem Set 4, Part 1). An important distinction from general Banach spaces is orthogonality: $x \perp y$ means $(x, y) = 0$. For nonzero vectors the source also uses

$$\theta = \arccos \frac{|(x, y)|}{\|x\| \|y\|}$$

as an angle. This is an acute angle between their one-dimensional subspaces; in real Euclidean geometry the usual oriented-vector angle uses (x, y) without the absolute value.

Definition 25.1. A family $\{e_\alpha\}_{\alpha \in A}$ is *orthogonal* if $(e_\alpha, e_\beta) = 0$ for distinct indices. It is *orthonormal* if additionally $\|e_\alpha\| = 1$, or equivalently $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$.

Every orthonormal set is linearly independent.

25.3. Bases and expansions. For any finite or countably infinite linearly independent sequence $x_1, x_2, \dots$, Gram–Schmidt produces an orthonormal sequence with $\text{span}\{e_1, \dots, e_n\} = \text{span}\{x_1, \dots, x_n\}$ for every n :

$$y_1 = x_1, \quad e_1 = y_1 / \|y_1\|, \quad y_n = x_n - \sum_{i=1}^{n-1} (x_n, e_i) e_i, \quad e_n = y_n / \|y_n\|.$$

Definition 25.2. An orthonormal set is *complete*, or an *orthonormal basis*, if it is maximal under inclusion among orthonormal sets.

The collection of orthonormal sets is partially ordered by inclusion. The union of a chain is an upper bound, so Zorn's lemma gives:

Proposition 25.1. *Every nonzero Hilbert space has an orthonormal basis.*

A basis may be uncountable. Recall that a space is separable if it has a countable dense subset.

Proposition 25.2. *A Hilbert space has a finite or countably infinite orthonormal basis if and only if it is separable.*

Proof. Exercise, as in the source. $\square$

Theorem 25.3. *For an orthonormal set $S = \{e_\alpha\}$ in H and $x \in H$:*

- (1) *Bessel's inequality holds: $\sum_\alpha |(x, e_\alpha)|^2 \leq \|x\|^2$. Only countably many summands can be nonzero.*
- (2) *If S is a basis, then*

$$x = \sum_\alpha (x, e_\alpha) e_\alpha, \quad \|x\|^2 = \sum_\alpha |(x, e_\alpha)|^2$$

(the latter is Parseval's identity).

- (3) *Conversely, if either displayed identity holds for every x , then S is an orthonormal basis.*

Proof. For any finite orthonormal subset $\{e_1, \dots, e_n\}$,

$$(25.1) \quad 0 \leq \left\| x - \sum_{i=1}^n (x, e_i) e_i \right\|^2 = \|x\|^2 - \sum_{i=1}^n |(x, e_i)|^2.$$

Thus, for each positive integer m , only finitely many coefficients have absolute value at least $1/m$. Their union is countable, so list all nonzero coefficients as (x, e_i) , with either finitely many or a sequence of indices. Taking suprema of the finite sums proves (1).

The finite-dimensional case of (2) is familiar. In the infinite case, Bessel's inequality makes the series of squared coefficients convergent. The partial sums $x_n = \sum_{i=1}^n (x, e_i) e_i$ satisfy

$$\|x_n - x_m\|^2 = \sum_{i=m+1}^n |(x, e_i)|^2 \longrightarrow 0.$$

Completeness gives a limit $\hat{x}$. For each listed e_j , $(x - \hat{x}, e_j) = 0$ by continuity of the inner product; for every unlisted e_α , both (x, e_α) and $(\hat{x}, e_\alpha)$ vanish. Hence $x - \hat{x} \perp S$. Maximality of S implies $x = \hat{x}$, since otherwise its normalization could be added to S . Taking limits in (25.1) proves Parseval's identity. Conversely, if S were not maximal, there would be a nonzero x_0 orthogonal to all its members. The expansion would give $x_0 = 0$; Parseval's identity would give $\|x_0\|^2 = 0$. Either is a contradiction. $\square$

Editorial note. The reduction to a countable list in the source is a list of the nonzero coefficients of a fixed vector, not an assumption that the whole orthonormal basis is countable.

The proof also shows that $\sum_\alpha c_\alpha e_\alpha$ converges whenever $\sum_\alpha |c_\alpha|^2 < \infty$, and that an orthonormal set is a basis exactly when $S^\perp = \{0\}$.

Corollary 25.1. *Every infinite orthonormal sequence e_n converges weakly to zero.*

Proof. Bessel's inequality gives $(x, e_n) \rightarrow 0$ for every x . By Riesz representation, these inner products test all continuous linear functionals, so $e_n \rightharpoonup 0$. $\square$

Nevertheless $\|e_n - e_m\| = \sqrt{2}$ for $n \neq m$.

25.4. Bounded operators on a Hilbert space.

Proposition 25.4. *Let H be a complex Hilbert space and $L \in \mathcal{L}(H)$. If $(Lx, x) = 0$ for every $x \in H$, then $L = 0$.*

Proof. Expanding the hypothesis at $x + y$ gives $0 = (Ly, x) + (Lx, y)$. Expanding at $x + iy$ gives $0 = i(Ly, x) - i(Lx, y)$. Therefore $(Ly, x) = 0$ for every x, y , so $Ly = 0$ for every y . $\square$

Remark 25.1. The assertion fails for real Hilbert spaces. Rotation of $\mathbb{R}^2$ by $\pi/2$ is a nonzero bounded linear operator with $(Lx, x) = 0$ for all x .

26. ADJOINTS, NORMAL, UNITARY, AND SELF-ADJOINT OPERATORS

26.1. Review (December 15, 2015). Every Hilbert space has an orthonormal basis, finite or countable precisely when it is separable. An orthonormal set $\{e_\alpha\}$ is a basis exactly when every vector has its orthonormal expansion, or equivalently when Parseval's identity holds for every vector.

26.2. The Hilbert-space adjoint. Take Hilbert spaces H_1, H_2 and a bounded linear map $L : H_1 \rightarrow H_2$. Riesz representation gives conjugate-linear isometric bijections

$$\Phi_i : H_i \rightarrow H_i^*, \quad y \mapsto L_y = (\cdot, y), \quad L_{\alpha y} = \bar{\alpha} L_y.$$

Thus the Banach dual operator $L_{\text{dual}}^* : H_2^* \rightarrow H_1^*$ corresponds to the operator

$$L^* = \Phi_1^{-1} L_{\text{dual}}^* \Phi_2 : H_2 \rightarrow H_1.$$

It is linear: the two conjugate-linear identifications cancel.

Proposition 26.1. *The adjoint $L^* \in \mathcal{L}(H_2, H_1)$ is uniquely characterized by*

$$(Lx, y)_{H_2} = (x, L^*y)_{H_1} \quad (x \in H_1, y \in H_2).$$

For densely defined operators an adjoint can also be defined; see Problem Set 6, Part 2, Problem 2.

Example 26.1. On $H = L^2([0, 1])$, let $Lf(x) = \int_0^1 k(x, y)f(y) dy$ for continuous k on $[0, 1]^2$. Then Fubini's theorem gives

$$\begin{aligned} (Lf, g) &= \int_0^1 \int_0^1 k(x, y)f(y)\overline{g(x)} dy dx \\ &= \int_0^1 f(x) \overline{\int_0^1 k(y, x)g(y) dy} dx. \end{aligned}$$

Consequently $L^*g(x) = \int_0^1 \overline{k(y, x)}g(y) dy$.

In finite dimensions, the adjoint is represented by the conjugate transpose matrix (in orthonormal coordinates). For Hilbert spaces, “adjoint” will henceforth mean this Hilbert-space adjoint.

Many properties agree with the Banach dual adjoint:

$$(L_2 L_1)^* = L_1^* L_2^*, \quad (L^*)^* = L, \quad \|L^*\| = \|L\|.$$

Scalar multiplication differs:

$$(\alpha L_1 + \beta L_2)^* = \bar{\alpha} L_1^* + \bar{\beta} L_2^*.$$

The Banach dual version has α, β instead of their conjugates.

Proposition 26.2. *For $L \in \mathcal{L}(H_1, H_2)$, $\|L^* L\| = \|L\|^2 = \|L L^*\|$.*

Proof. $\|L^* L\| \leq \|L^*\| \|L\| = \|L\|^2$. Conversely,

$$\|Lx\|^2 = (L^* Lx, x) \leq \|L^* L\| \|x\|^2,$$

so $\|L\|^2 \leq \|L^* L\|$. Apply this to L^* as well. $\square$

In Hilbert spaces annihilators can be replaced by orthogonal complements inside the original space: $M^\perp = \{x : (x, y) = 0 \text{ for all } y \in M\}$.

Proposition 26.3. $\ker L = (\text{Ran } L^*)^\perp$ and $\ker L^* = (\text{Ran } L)^\perp$.

Proof. $Lx = 0$ if and only if $(Lx, y) = (x, L^*y) = 0$ for every y . This proves the first equality; apply it to L^* for the second. $\square$

Corollary 26.1. $L \in \mathcal{L}(H)$ is invertible if and only if $\ker L^* = \{0\}$ and some $\alpha > 0$ satisfies $\|Lx\| \geq \alpha \|x\|$ for all x .

Proof. The lower bound is equivalent to injectivity with closed range (Problem Set 5, Part 2, Problem 4). The kernel condition makes the range dense, so it is all of H . $\square$

Corollary 26.2. If L is invertible, then L^* is invertible and $(L^*)^{-1} = (L^{-1})^*$.

Proof. Take adjoints of $LL^{-1} = L^{-1}L = I$. $\square$

26.3. Normal and unitary operators. From now on H is a complex Hilbert space.

Definition 26.1. A bounded operator L on H is *normal* if $LL^* = L^*L$, *self-adjoint* if $L = L^*$, and *unitary* if $LL^* = L^*L = I$. Every self-adjoint or unitary operator is normal.

Proposition 26.4. L is normal if and only if $\|Lx\| = \|L^*x\|$ for every x .

Proof. The equality of the two squared norms is equivalent to $((LL^* - L^*L)x, x) = 0$ for every x . On a complex Hilbert space, the proposition at the end of Lecture 25 makes this equivalent to $LL^* = L^*L$. $\square$

Corollary 26.3. For normal L , $\ker L = \ker L^*$. Such an L is invertible exactly when $\|Lx\| \geq \alpha \|x\|$ for some $\alpha > 0$ and every x .

The operator $L - \lambda I$ is normal whenever L is normal. Thus $\ker(L - \lambda I) = \ker(L^* - \bar{\lambda}I)$.

Corollary 26.4. For a normal operator, $Lx = \lambda x$ if and only if $L^*x = \bar{\lambda}x$. Eigenvectors for distinct eigenvalues are orthogonal.

Proof. Only the last assertion remains. If $Lx_i = \lambda_i x_i$, then

$$\lambda_1(x_1, x_2) = (Lx_1, x_2) = (x_1, L^*x_2) = (x_1, \bar{\lambda}_2 x_2) = \lambda_2(x_1, x_2).$$

For $\lambda_1 \neq \lambda_2$, this forces $(x_1, x_2) = 0$. $\square$

Proposition 26.5. The following are equivalent:

- (1) L is unitary;
- (2) L is surjective and $(Lx, Ly) = (x, y)$ for all x, y ;
- (3) L is surjective and $\|Lx\| = \|x\|$ for all x .

Proof. (1) $\Rightarrow$ (2) follows from $L^{-1} = L^*$ and $(Lx, Ly) = (L^*Lx, y) = (x, y)$. The implication (2) $\Rightarrow$ (3) is immediate. For (3) $\Rightarrow$ (1), the norm equality gives $((L^*L - I)x, x) = 0$, hence $L^*L = I$. The norm equality and surjectivity make L invertible, so $L^* = L^{-1}$ and $LL^* = I$. $\square$

A unitary operator on a nonzero Hilbert space has norm one.

Proposition 26.6. The spectrum of a unitary operator lies on the unit circle.

Proof. Since $\|L\| = 1$, its spectrum lies in $\{|\lambda| \leq 1\}$. For $|\lambda| < 1$, $\|\lambda L^*\| < 1$, so $\lambda I - L = -L(I - \lambda L^*)$ is invertible. Thus no such λ belongs to the spectrum. $\square$

26.4. Self-adjoint operators. Self-adjoint operators play a major role in spectral theory and mathematical physics. The integral operator above is self-adjoint exactly when $k(x, y) = \overline{k(y, x)}$. For every L , the operators LL^* , L^*L , $(L + L^*)/2$, and $(L - L^*)/(2i)$ are self-adjoint, and

$$L = \frac{L + L^*}{2} + i \frac{L - L^*}{2i}.$$

Proposition 26.7. *The spectrum of a self-adjoint operator is real.*

Proof. Write $\lambda = \alpha + i\beta$. Self-adjointness makes the cross terms cancel in

$$\|(L - \lambda I)x\|^2 = \|(L - \alpha I)x\|^2 + \beta^2 \|x\|^2.$$

If $\beta \neq 0$, this is bounded below by $\beta^2 \|x\|^2$. Since $L - \lambda I$ is normal, the preceding invertibility criterion shows it is invertible. Thus spectral points must have $\beta = 0$. $\square$

Corollary 26.5. *A self-adjoint operator has no residual spectrum.*

Proof. If $\lambda \in \sigma_r(L)$, the range of $\lambda I - L$ is nondense, so $\bar{\lambda} \in \sigma_p(L^*)$ by the orthogonal-complement identity. For self-adjoint L , the spectrum is real and $L^* = L$, so $\lambda \in \sigma_p(L)$, contradicting the disjointness of the point and residual spectra (compare Problem Set 13, Part 1). $\square$

Proposition 26.8. *L is self-adjoint if and only if $(Lx, x) \in \mathbb{R}$ for every $x \in H$.*

Proof. If $L = L^*$, then $(Lx, x) = (x, Lx) = \overline{(Lx, x)}$. Conversely, expanding the reality condition at $x + y$ and $x + iy$ yields

$$(Lx, y) + (Ly, x) \in \mathbb{R}, \quad -i(Lx, y) + i(Ly, x) \in \mathbb{R}.$$

These two identities imply $(Ly, x) = (y, Lx)$ for every x, y , hence $L = L^*$. $\square$

27. COMPACT SELF-ADJOINT OPERATORS AND OPERATOR IDEALS

27.1. Review (December 18, 2015). Hilbert adjoints satisfy $(L_2 L_1)^* = L_1^* L_2^*$, $(\alpha L)^* = \bar{\alpha} L^*$, $(L^*)^* = L$, and $\|L^*\| = \|L\|$. Normality is equivalent to $\|Lx\| = \|L^*x\|$; eigenvectors for distinct eigenvalues are orthogonal. A unitary is a surjective isometry and has spectrum on the unit circle. A self-adjoint operator is characterized by real quadratic forms, has real spectrum, and has no residual spectrum. Problem Set 14, Part 1 also gives, for self-adjoint L ,

$$\|L\| = \sup_{x \neq 0} \frac{|(Lx, x)|}{(x, x)}, \quad \|L\| \in \sigma(L) \text{ or } -\|L\| \in \sigma(L).$$

Consequently a self-adjoint operator with spectrum $\{0\}$ is zero.

27.2. Spectral theorem for compact self-adjoint operators. Suppose $L \neq 0$ is compact and self-adjoint on H . Its spectrum is real, and every nonzero spectral point is an eigenvalue. One of $\pm \|L\|$ is therefore an eigenvalue; for a corresponding unit eigenvector x , $|(Lx, x)| = \|L\|$. Thus the supremum of $|(Lx, x)|$ over the unit sphere is attained at an eigenvector. The nonzero eigenvalues can be listed, with multiplicities, as a finite sequence or as $\lambda_n \rightarrow 0$, ordered by $|\lambda_1| \geq |\lambda_2| \geq \dots$. Each eigenspace is finite-dimensional. Choose an orthonormal basis in each eigenspace. For example, the list $3, 3, -2.5, 2, -2, -2, -2, 1, \dots$ denotes multiplicity two for 3 and multiplicity three for -2 .

Let the corresponding orthonormal eigenvectors be $e_1, e_2, \dots$, so $Le_i = \lambda_i e_i$, and put $M = \overline{\text{span}\{e_i\}}$.

Lemma 27.1. $L(M) \subset M$ and $L(M^\perp) \subset M^\perp$.

Proof. If $x \in M$, write $x = \sum_i \alpha_i e_i$. Continuity gives $Lx = \sum_i \alpha_i \lambda_i e_i \in M$; convergence also follows from $|\lambda_i| \leq \|L\|$. If $y \in M^\perp$, then for $x \in M$, $(x, Ly) = (Lx, y) = 0$, so $Ly \in M^\perp$. $\square$

The restriction $L_1 = L|_{M^\perp}$ is compact and self-adjoint. It has no nonzero spectral point: such a point would be an eigenvalue, and its eigenvector would belong both to M and $M^\perp$. Thus $L_1 = 0$ by the self-adjoint norm formula, and $M^\perp \subset \ker L$. Conversely, for $y \in \ker L$ and $x = \sum_i \alpha_i e_i \in M$,

$$\left(\sum_{i=1}^n \alpha_i e_i, y \right) = \left(L \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} e_i, y \right) = \left(\sum_{i=1}^n \frac{\alpha_i}{\lambda_i} e_i, Ly \right) = 0.$$

Taking limits shows $y \in M^\perp$. Hence $M^\perp = \ker L$. Choose an orthonormal basis $\{f_\alpha\}$ of this kernel, whose vectors are zero-eigenvalue eigenvectors. Its union with $\{e_i\}$ is an orthonormal basis of H .

Theorem 27.1 (Spectral theorem). *Every compact self-adjoint operator on a Hilbert space admits an orthonormal basis of eigenvectors. If the space is separable, the basis is finite or countable.*

Every $x \in H$ can be written $x = \sum_i \alpha_i e_i + x_0$ with $x_0 \in \ker L$, so

$$Lx = \sum_i \lambda_i (x, e_i) e_i.$$

The finite-rank truncations $L_k x = \sum_{i=1}^k \lambda_i (x, e_i) e_i$ converge to L in operator norm. Zero may belong to the continuous spectrum rather than the point spectrum. For

example, if $b_i \neq 0$ and $b_i \rightarrow 0$, the diagonal map $(a_i) \mapsto (a_i b_i)$ on ℓ^2 is compact and injective; its point spectrum is the set of nonzero diagonal values (Problem Set 12, Part 1).

Corollary 27.1 (Singular-value decomposition). *For a compact $L \in \mathcal{L}(H)$ there are orthonormal sets $\{e_i\}, \{f_i\}$ and positive numbers λ_i , a finite list or sequence tending to zero, such that*

$$Lx = \sum_i \lambda_i (x, e_i) f_i.$$

Proof. L^*L is compact, self-adjoint, and positive. Let its nonzero eigenvalues and orthonormal eigenvectors be μ_i, e_i . Then $\mu_i = (L^*Le_i, e_i) = \|Le_i\|^2 > 0$. Set $\lambda_i = \sqrt{\mu_i}$ and $f_i = Le_i/\lambda_i$. Since

$$(f_i, f_j) = \frac{(L^*Le_i, e_j)}{\lambda_i \lambda_j} = \delta_{ij},$$

this is an orthonormal set. If $x = \sum_i (x, e_i) e_i + x_0$ with $L^*Lx_0 = 0$, then $\|Lx_0\|^2 = (L^*Lx_0, x_0) = 0$, so the expansion for Lx follows. $\square$

The λ_i are the *singular values*, the positive square roots of the nonzero eigenvalues of L^*L . The operator $|L|x = \sum_i \lambda_i (x, e_i) e_i$ is bounded (indeed compact) and satisfies $|L|^2 = L^*L$; write $|L| = \sqrt{L^*L}$. More generally, every positive bounded operator L has a unique positive square root. Here positivity means $(Lx, x) \geq 0$ for every x and implies self-adjointness.

27.3. Hilbert–Schmidt operators. Let H be separable with orthonormal basis $\{e_n\}$ and $L \in \mathcal{L}(H)$.

Lemma 27.2. *For another orthonormal basis $\{f_m\}$,*

$$\sum_n \|Le_n\|^2 = \sum_{n,m} |(Le_n, f_m)|^2 = \sum_m \|L^*f_m\|^2.$$

The sums may be infinite. In particular, $\sum_n \|Le_n\|^2$ is independent of the orthonormal basis.

Proof. Parseval and $(Le_n, f_m) = (e_n, L^*f_m)$ give the displayed identities, with interchange valid for nonnegative sums. Repeating with a different basis in the domain proves independence. $\square$

Definition 27.1. L is *Hilbert–Schmidt* if

$$\|L\|_{\text{HS}}^2 := \sum_n \|Le_n\|^2 < \infty.$$

For $b = (b_i) \in \ell^\infty$, the diagonal operator $L(a_i) = (a_i b_i)$ on ℓ^2 has $\|L\|_{\text{HS}}^2 = \sum_i |b_i|^2$, so it is Hilbert–Schmidt exactly when $b \in \ell^2$.

Proposition 27.2. *The Hilbert–Schmidt operators have the following properties:*

- (1) L is Hilbert–Schmidt if and only if L^* is.
- (2) $\|L\| \leq \|L\|_{\text{HS}}$.
- (3) If L_1 is Hilbert–Schmidt and L_2 is bounded, both $L_1 L_2$ and $L_2 L_1$ are Hilbert–Schmidt.
- (4) They form a vector space $\mathcal{L}^2$; the Hilbert–Schmidt norm makes it a Hilbert space.

Proof. Exercise, as in the source. $\square$

Theorem 27.3. *Every Hilbert–Schmidt operator is compact.*

Proof. Let P_N be projection onto $\text{span}\{e_1, \dots, e_N\}$. Then $L_N = LP_N$ is finite rank, and

$$\|L - L_N\| \leq \|L - L_N\|_{\text{HS}} = \left(\sum_{n>N} \|Le_n\|^2 \right)^{1/2} \longrightarrow 0.$$

Thus L is compact. This also proves that finite-rank operators are dense in $\mathcal{L}^2$ for the Hilbert–Schmidt norm. $\square$

If L is additionally self-adjoint, $\|L\|_{\text{HS}}^2 = \sum_j \lambda_j^2$, with eigenvalues counted with multiplicity.

Editorial note. The source defines the Hilbert–Schmidt “norm” as the sum of squares, omitting its square root; that sum is the squared norm. The norm comparisons, convergence estimate, and eigenvalue formula above use the standard square-root normalization consistently. The basis-change identity uses L^*f_m , as its proof requires.

27.4. Trace-class operators. For general bounded L , the positive operator $|L| = \sqrt{L^*L}$ is defined by the positive-square-root theorem. The sum $\sum_n (|L|e_n, e_n)$ is independent of the orthonormal basis, by the preceding basis-independence result applied to $\sqrt{|L|}$.

Definition 27.2. L is of *trace class* if

$$\|L\|_1 := \text{tr } |L| := \sum_n (|L|e_n, e_n) < \infty.$$

Write $\mathcal{L}^1$ for the trace-class operators.

The analogues of the Hilbert–Schmidt properties hold:

Proposition 27.4. (1) $\mathcal{L}^1$ is a vector space.

(2) If $L_1 \in \mathcal{L}^1$ and L_2 is bounded, then $L_1L_2, L_2L_1 \in \mathcal{L}^1$.

(3) $(\mathcal{L}^1, \|\cdot\|_1)$ is a Banach space and $\|L\| \leq \|L\|_1$.

(4) $L \in \mathcal{L}^1$ implies $L^* \in \mathcal{L}^1$.

(5) Every trace-class operator is compact.

Proof. Exercise, as in the source. $\square$

For a positive trace-class operator, $\text{tr } L = \sum_i \lambda_i$, where the eigenvalues are counted with multiplicity. More generally:

Theorem 27.5. *A compact operator is of trace class if and only if its singular values satisfy $\sum_i \lambda_i < \infty$.*

Editorial note. The source’s last page writes $|L|^2 = LL^*$. We retain the convention $|L|^2 = L^*L$ introduced on page 3, which is the standard absolute value and matches the singular-value definition.

28. FUNCTIONS OF SELF-ADJOINT OPERATORS

28.1. Review (December 22, 2015). A compact self-adjoint operator has real nonzero eigenvalues, listed with multiplicities in decreasing absolute value and tending to zero if the list is infinite. Together with an orthonormal basis of its kernel, its nonzero eigenvectors form an orthonormal basis of H . Thus $L = \sum_i \lambda_i(\cdot, e_i)e_i$. For a general compact operator, the singular-value expansion is $L = \sum_i \lambda_i(\cdot, e_i)f_i$, where λ_i are the positive square roots of the nonzero eigenvalues of L^*L . On a separable Hilbert space, Hilbert–Schmidt and trace-class operators are characterized by

$$\|L\|_{\text{HS}}^2 = \sum_n \|Le_n\|^2 < \infty, \quad \|L\|_1 = \text{tr } |L| = \sum_n (|L|e_n, e_n) < \infty.$$

The latter equals the sum of the singular values.

28.2. Ordering self-adjoint operators.

Definition 28.1. A bounded operator A on a complex Hilbert space is *positive* if $(Ax, x) \geq 0$ for every x , written $A \geq 0$. For self-adjoint A, B , write $A \geq B$ when $A - B \geq 0$.

Positivity means nonnegativity here. Both LL^* and L^*L are positive. Since real quadratic forms characterize self-adjointness, every positive operator is self-adjoint; its eigenvalues are nonnegative.

Proposition 28.1. For self-adjoint L , $-\|L\|I \leq L \leq \|L\|I$.

Proof. By the quadratic-form norm formula (Problem Set 14, Part 1, Problem 1), $|(Lx, x)| \leq \|L\| \|x\|^2$. Thus both $((\|L\|I - L)x, x)$ and $((\|L\|I + L)x, x)$ are nonnegative. $\square$

If $L \geq 0$ and $L \leq 0$, then $L = 0$.

Lemma 28.1 (Cauchy–Schwarz for a positive operator). If $L \geq 0$, then $|(Lx, y)| \leq \sqrt{(Lx, x)}\sqrt{(Ly, y)}$.

Proof. Exercise, as in the source. $\square$

Lemma 28.2. $0 \leq A \leq B$ implies $\|A\| \leq \|B\|$.

Proof. Apply the quadratic-form norm formula. $\square$

Theorem 28.2 (Monotone strong convergence). An increasing sequence of self-adjoint operators $L_0 \leq L_1 \leq \dots \leq B$, bounded above by a self-adjoint B , has a strong limit $L_\infty \in \mathcal{L}(H)$: for every $x \in H$, $L_n x \rightarrow L_\infty x$ in norm.

Proof. For each x , the scalar sequence $(L_n x, x)$ increases and is bounded above by (Bx, x) , hence converges. For $n \geq m$ put $A = L_n - L_m$. Then $0 \leq A \leq B - L_0$ and $\|A\| \leq C := \|B - L_0\|$. The positive-operator Cauchy–Schwarz inequality, with $y = Ax$, gives

$$\begin{aligned} \|Ax\|^2 &= (Ax, Ax) \leq \sqrt{(Ax, x)}\sqrt{(A^2x, Ax)} \\ &\leq \sqrt{(L_n x, x) - (L_m x, x)} C^{3/2} \|x\|. \end{aligned}$$

The right side tends to zero. Thus $\{L_n x\}$ is Cauchy for every x , and its limit defines $L_\infty x$. The pointwise-limit corollary of Banach–Steinhaus (Lecture 9) makes L_∞ bounded and linear. Its quadratic form is the limit of those of L_n , so it is self-adjoint and $L_\infty \leq B$. $\square$

Editorial note. The source's estimate uses $0 \leq L_n - L_m \leq L_n \leq B$, which presumes $L_0 \geq 0$. The bound by $B - L_0$ above removes this unstated restriction.

28.3. Polynomial functions of an operator.

Theorem 28.3. *Suppose $mI \leq L \leq MI$ and a real polynomial p satisfies $p(t) \geq 0$ on $[m, M]$. Then $p(L) \geq 0$.*

Proof. The case $m = M$ is immediate. For $m < M$, factor the nonzero polynomial into real linear factors with roots outside the interval and real quadratic factors:

$$p(t) = c \prod_j (t - \alpha_j) \prod_k (\beta_k - t) \prod_l ((t - \gamma_l)^2 + \delta_l^2),$$

where $c > 0$, $\alpha_j \leq m$, $\beta_k \geq M$. Interior real roots have even multiplicity and are included by allowing $\delta_l = 0$. Every corresponding operator factor is positive, and all commute. A product of commuting positive operators is positive: if $A, B \geq 0$ and $AB = BA$, the positive square root $A^{1/2}$ commutes with B (compare Problem Set 14, Part 2), and $(ABx, x) = (BA^{1/2}x, A^{1/2}x) \geq 0$. This proves the theorem; $p = 0$ is trivial. $\square$

Corollary 28.1. *If $p_1(t) \leq p_2(t)$ on $[m, M]$, then $p_1(L) \leq p_2(L)$.*

Proposition 28.4. *If $mI \leq L \leq MI$, then $\sigma(L) \subset [m, M]$.*

Proof. First, $-MI \leq L \leq MI$ gives $\|L\| \leq M$ by the quadratic-form formula, so its real spectrum lies in $[-M, M]$. In general, center at the midpoint: $L' = L - (M + m)I/2$ satisfies

$$-\frac{M - m}{2}I \leq L' \leq \frac{M - m}{2}I.$$

Apply the first case to L' and translate the spectrum. $\square$

Choose $a < m \leq M < b$. Let $K[a, b]$ denote the bounded real-valued piecewise functions on $[a, b]$ which are pointwise limits of decreasing sequences of real continuous functions. We use the source's class in constructing functions of L .

Lemma 28.3. *Every $\varphi \in K[a, b]$ is a pointwise limit of a decreasing sequence of real polynomials p_n on $[a, b]$.*

Proof. Choose continuous $\varphi_n \downarrow \varphi$ by definition. Weierstrass approximation gives a real polynomial p_n with

$$\left| p_n(t) - \left(\varphi_n(t) + \frac{3}{2^{n+2}} \right) \right| \leq \frac{1}{2^{n+2}} \quad (t \in [a, b]).$$

Then

$$p_{n+1}(t) \leq \varphi_{n+1}(t) + \frac{1}{2^{n+1}} \leq \varphi_n(t) + \frac{1}{2^{n+1}} \leq p_n(t),$$

and $p_n(t) \rightarrow \varphi(t)$. $\square$

For any such sequence, polynomial order gives $p_1(L) \geq p_2(L) \geq \cdots \geq (\inf \varphi)I$. The monotone convergence theorem, applied after reversing signs, therefore gives a self-adjoint strong limit.

Definition 28.2. For $\varphi \in K[a, b]$, define $\varphi(L) = \lim_{n \rightarrow \infty} p_n(L)$ in the strong operator sense.

Independence of the approximation follows from the next lemma.

Lemma 28.4. *Let $p_n \downarrow \varphi$ and $q_n \downarrow \psi$ be polynomial approximations, with $\varphi, \psi \in K[a, b]$ and $\psi(t) \leq \varphi(t)$ on $[m, M]$. Then $\lim_n q_n(L) \leq \lim_n p_n(L)$.*

Proof. Fix n . For each $t_0 \in [m, M]$, choose $N_0(t_0)$ such that $q_N(t_0) < \psi(t_0) + 1/n \leq \varphi(t_0) + 1/n \leq p_n(t_0) + 1/n$ for all $N \geq N_0(t_0)$. By continuity, the strict inequality $q_{N_0(t_0)}(t) < p_n(t) + 1/n$ persists in a neighborhood of t_0 . Choose finitely many of these neighborhoods covering $[m, M]$, and let N_0 be the largest of their indices. Monotonicity gives $q_N(t) < p_n(t) + 1/n$ on the whole interval whenever $N \geq N_0$. Polynomial order yields $q_N(L) \leq p_n(L) + 1/n$. Take the strong limit first in N and then in n . $\square$

Applying the lemma in both directions when $\psi = \varphi$ proves well-definedness of $\varphi(L)$. The following properties are easy to prove:

- (1) $(\varphi_1 + \varphi_2)(L) = \varphi_1(L) + \varphi_2(L)$.
- (2) For $c \geq 0$, $(c\varphi)(L) = c\varphi(L)$.
- (3) For nonnegative $\varphi_1, \varphi_2 \in K[a, b]$, $(\varphi_1\varphi_2)(L) = \varphi_1(L)\varphi_2(L)$.
- (4) $\varphi_1 \geq \varphi_2$ on $[m, M]$ implies $\varphi_1(L) \geq \varphi_2(L)$.

Extend the definition to differences: if $f = \varphi - \psi$ with $\varphi, \psi \in K[a, b]$, put $f(L) = \varphi(L) - \psi(L)$. This is independent of the representation, since $\varphi - \psi = \tilde{\varphi} - \tilde{\psi}$ implies $\varphi + \tilde{\psi} = \tilde{\varphi} + \psi$, and additivity gives identical operator differences. On the extended class $K[a, b] - K[a, b]$, scalar multiplication by every real c and products are allowed:

$$(cf)(L) = cf(L), \quad (fg)(L) = f(L)g(L).$$

Editorial note. The first construction gives closure under multiplication by nonnegative scalars and products of nonnegative functions in the original class of monotone limits. The difference extension in the final paragraph of the source is essential before using arbitrary real scalars and products. Strong limits, rather than operator-norm limits, are meant throughout.

29. SPECTRAL PROJECTIONS

29.1. Review (December 25, 2015). For self-adjoint operators, $L_1 \leq L_2$ means $L_2 - L_1 \geq 0$. A monotone sequence $L_0 \leq L_1 \leq \dots \leq L$ converges strongly to a bounded self-adjoint operator $L_\infty \leq L$. Let $K[a, b]$ consist of the bounded piecewise-continuous real functions that are decreasing pointwise limits of continuous functions (equivalently, of suitable polynomials). The preceding lecture constructs $\varphi(L)$ by strong limits for self-adjoint L with $\sigma(L) \subset [m, M] \subset (a, b)$, and extends the definition to the difference class $K[a, b] - K[a, b]$. On this extended class, the calculus respects addition, multiplication by real scalars, and products:

$$\begin{aligned} (\varphi_1 + \varphi_2)(L) &= \varphi_1(L) + \varphi_2(L), & (c\varphi)(L) &= c\varphi(L), \\ (\varphi_1\varphi_2)(L) &= \varphi_1(L)\varphi_2(L). \end{aligned}$$

It also preserves order: $\varphi_1 \geq \varphi_2$ implies $\varphi_1(L) \geq \varphi_2(L)$.

29.2. Projections.

Definition 29.1. $P \in \mathcal{L}(H)$ is a *projection* if $P^2 = P$. It is an *orthogonal projection* if in addition $P = P^*$.

Proposition 29.1. For a projection P :

- (1) $Px = x$ for $x \in R(P)$;
- (2) $I - P$ is a projection, $R(I - P) = N(P)$ and $N(I - P) = R(P)$;
- (3) $H = R(P) + N(P)$ and $R(P) \cap N(P) = \{0\}$.

In particular, $R(P)$ is closed.

Proof. If $x = Py$, then $Px = P^2y = Py = x$. Also $(I - P)^2 = I - P$, and $Px = 0$ if and only if $(I - P)x = x$, which is if and only if $x \in R(I - P)$. Applying the same identity to $I - P$ gives the other equality. Finally, $x = Px + (I - P)x$ gives the sum decomposition, and a vector belonging to both summands is fixed and annihilated by P , hence is zero. $\square$

For a closed subspace $E \subset H$, the decomposition $H = E \oplus E^\perp$ defines $P_E(x + y) = x$ for $x \in E$, $y \in E^\perp$. It is continuous with norm one if $E \neq \{0\}$, and $R(P_E) = E$, $N(P_E) = E^\perp$.

Proposition 29.2. A projection P is orthogonal if and only if $R(P) \perp N(P)$. In this case $P = P_{R(P)}$.

Proof. If $P = P^*$, for $x \in R(P)$ and $y \in N(P) = R(I - P)$,

$$(x, y) = (Px, (I - P)y) = (x, P(I - P)y) = 0.$$

Conversely, the direct-sum decomposition and orthogonality give $N(P) = R(P)^\perp$, so $P = P_{R(P)}$. For $x = x_1 + x_2$ and $y = y_1 + y_2$ in the orthogonal decomposition,

$$(Px, y) = (x_1, y_1) = (x, Py),$$

showing $P = P^*$. $\square$

Corollary 29.1. If P is an orthogonal projection, then $0 \leq P \leq I$.

Proof. $(Px, x) = (Px, Px) = \|Px\|^2 \geq 0$, and $\|Px\|^2 \leq \|x\|^2$ gives the upper bound. $\square$

Proposition 29.3. *For a projection P and $T \in \mathcal{L}(H)$, $PT = TP$ if and only if both $R(P)$ and $N(P)$ are invariant under T .*

Proof. If they commute, $Px = 0$ implies $PTx = TPx = 0$, and $x = Py$ implies $Tx = TP y = PT y \in R(P)$. Conversely, decompose $x = x_1 + x_2$ with $x_1 \in R(P)$, $x_2 \in N(P)$. Invariance gives $PTx = PTx_1 + PTx_2 = Tx_1 = TPx$. $\square$

Proposition 29.4. *For an orthogonal projection P , $\sigma(P) = \{0, 1\}$ if both its range and kernel are nonzero. For $P = I$ it is $\{1\}$, and for $P = 0$ it is $\{0\}$, assuming $H \neq \{0\}$.*

Proof. We already know $\sigma(P) \subset [0, 1]$. For $\lambda \notin \{0, 1\}$, write $y = y_1 + y_2$ in $R(P) \oplus N(P)$ and set

$$x = \frac{y_1}{\lambda - 1} + \frac{y_2}{\lambda}.$$

Then $(\lambda I - P)x = y$. The same decomposition shows injectivity. Nonzero vectors in the range and kernel give eigenvalues one and zero, respectively, whenever those spaces are nonzero. $\square$

29.3. Spectral projections. Let $L \in \mathcal{L}(H)$ be self-adjoint with $mI \leq L \leq MI$. For real λ define

$$\chi_\lambda(t) = \begin{cases} 1, & t \leq \lambda, \\ 0, & t > \lambda. \end{cases}$$

The continuous functions

$$\chi_{n,\lambda}(t) = \begin{cases} 1, & t \leq \lambda, \\ 1 - n(t - \lambda), & \lambda < t < \lambda + 1/n, \\ 0, & t \geq \lambda + 1/n \end{cases}$$

decrease pointwise to χ_λ . Thus the functional calculus defines the bounded self-adjoint operator $E_\lambda = \chi_\lambda(L)$.

Lemma 29.1. *E_λ is an orthogonal projection; $E_\lambda = 0$ for $\lambda < m$, $E_\lambda = I$ for $\lambda > M$; and $E_\lambda E_\mu = E_\lambda$ for $\lambda < \mu$.*

Proof. The identities $\chi_\lambda^2 = \chi_\lambda$ and $\chi_\lambda \chi_\mu = \chi_\lambda$ give idempotence and the product formula. The endpoint assertions follow because the functions are identically zero or one on $[m, M]$ in the indicated cases. $\square$

Proposition 29.5. *E_λ is right-continuous in the strong operator topology:*

$$E_{\lambda+\varepsilon}x \rightarrow E_\lambda x \quad (\varepsilon \downarrow 0).$$

Proof. Choose decreasing continuous φ_n with $\varphi_n \geq \chi_{\lambda+1/n}$ and $\varphi_n \downarrow \chi_\lambda$. Then $\varphi_n(L) \rightarrow E_\lambda$ strongly. For $0 < \varepsilon < 1/n$,

$$\varphi_n(L) \geq E_{\lambda+1/n} \geq E_{\lambda+\varepsilon} \geq E_\lambda.$$

The positive-operator squeeze gives the asserted strong limit. $\square$

Proposition 29.6. *$R(E_\lambda)$ is invariant under L .*

Proof. $LE_\lambda = E_\lambda L$, since E_λ is a strong limit of polynomials in L . $\square$

If $\lambda_1 < \lambda_2$, the pointwise inequalities

$$\lambda_1(\chi_{\lambda_2} - \chi_{\lambda_1}) \leq t(\chi_{\lambda_2} - \chi_{\lambda_1}) \leq \lambda_2(\chi_{\lambda_2} - \chi_{\lambda_1})$$

give

$$\lambda_1(E_{\lambda_2} - E_{\lambda_1}) \leq L(E_{\lambda_2} - E_{\lambda_1}) \leq \lambda_2(E_{\lambda_2} - E_{\lambda_1}).$$

In particular, $LE_\lambda \leq \lambda E_\lambda$ and $\lambda(I - E_\lambda) \leq L(I - E_\lambda)$.

29.4. Construction of the spectral integral. Take a partition

$$a < \lambda_0 < m \leq \lambda_1 < \cdots < \lambda_{n-1} \leq M < \lambda_n < b$$

with mesh $\Delta = \max_k(\lambda_{k+1} - \lambda_k) < \varepsilon$. Choose $\mu_k \in [\lambda_k, \lambda_{k+1}]$ and put $Q_k = E_{\lambda_{k+1}} - E_{\lambda_k}$. The Q_k are mutually orthogonal projections with $\sum_k Q_k = I$. Summing the preceding inequalities yields

$$\sum_{k=0}^{n-1} \lambda_k Q_k \leq L \leq \sum_{k=0}^{n-1} \lambda_{k+1} Q_k.$$

Because $-\varepsilon \leq \lambda_k - \mu_k \leq \lambda_{k+1} - \mu_k \leq \varepsilon$,

$$-\varepsilon I \leq \sum_k (\lambda_k - \mu_k) Q_k \leq L - \sum_k \mu_k Q_k \leq \sum_k (\lambda_{k+1} - \mu_k) Q_k \leq \varepsilon I.$$

For a self-adjoint operator, these order bounds imply

$$\left\| L - \sum_{k=0}^{n-1} \mu_k (E_{\lambda_{k+1}} - E_{\lambda_k}) \right\| \leq \varepsilon.$$

Thus for any choices of sample points, the sums converge in operator norm as the mesh tends to zero. This defines the spectral integral:

$$L = \int_a^b \lambda \, dE_\lambda = \int_{m-\eta}^M \lambda \, dE_\lambda = \int_{-\infty}^{\infty} \lambda \, dE_\lambda \quad (\eta > 0).$$

30. CONSEQUENCES OF THE SPECTRAL THEOREM

30.1. Review (December 29, 2015). For bounded self-adjoint L , set $E_\lambda = \chi_\lambda(L)$, where $\chi_\lambda(t) = 1$ for $t \leq \lambda$ and zero otherwise. These orthogonal projections form the *spectral family*:

- (1) $E_\lambda = 0$ for $\lambda < m$ and $E_\lambda = I$ for $\lambda > M$ when $mI \leq L \leq MI$;
- (2) $E_{\lambda+\varepsilon} \rightarrow E_\lambda$ strongly as $\varepsilon \downarrow 0$;
- (3) for $\lambda_1 < \lambda_2$, $E_{\lambda_1}E_{\lambda_2} = E_{\lambda_1} = E_{\lambda_2}E_{\lambda_1}$.

The spectral theorem gives

$$L = \int_a^b \lambda \, dE_\lambda = \int_{m-\eta}^M \lambda \, dE_\lambda = \int_{-\infty}^{\infty} \lambda \, dE_\lambda,$$

where the integral is the operator-norm limit of the sums $\sum_k \mu_k (E_{\lambda_{k+1}} - E_{\lambda_k})$ as the mesh tends to zero.

30.2. Continuity of the spectral family. The spectral family is right-continuous but may have jumps from the left.

Example 30.1. Let $L = \text{diag}(1, 2)$ on $\mathbb{C}^2$, with standard eigenvectors e_1, e_2 . Then

$$L = 1(\cdot, e_1)e_1 + 2(\cdot, e_2)e_2, \quad p(L) = p(1)(\cdot, e_1)e_1 + p(2)(\cdot, e_2)e_2$$

for every polynomial p . Passing to the characteristic functions gives

$$E_\lambda = \begin{cases} 0, & \lambda < 1, \\ P_{\mathbb{C}e_1}, & 1 \leq \lambda < 2, \\ I, & \lambda \geq 2. \end{cases}$$

Thus the left limit differs from E_λ at $\lambda = 1, 2$. For small $\varepsilon > 0$,

$$\begin{aligned} E_{1-\varepsilon}e_1 &= 0 \neq E_1e_1 = e_1, & E_{1-\varepsilon}e_2 &= 0 = E_1e_2, \\ E_{2-\varepsilon}e_1 &= e_1 = E_2e_1, & E_{2-\varepsilon}e_2 &= 0 \neq E_2e_2 = e_2. \end{aligned}$$

Editorial note. The final branch includes $\lambda = 2$ by the convention $\chi_\lambda = 1$ on $(-\infty, \lambda]$; the handwritten case display uses only $\lambda > 2$ and leaves that endpoint unstated.

Proposition 30.1. *If $\lambda_0 \in \sigma_p(L)$, then $\lim_{\varepsilon \downarrow 0} E_{\lambda_0-\varepsilon} \neq E_{\lambda_0}$ in the strong operator topology.*

Proof. Let $Lx_0 = \lambda_0 x_0$ with $x_0 \neq 0$. Recall

$$\begin{aligned} \lambda_1(E_{\lambda_2} - E_{\lambda_1}) &\leq L(E_{\lambda_2} - E_{\lambda_1}) \leq \lambda_2(E_{\lambda_2} - E_{\lambda_1}), & \lambda_1 < \lambda_2, \\ \lambda(I - E_\lambda) &\leq L(I - E_\lambda), & LE_\lambda &\leq \lambda E_\lambda. \end{aligned}$$

For $Q = I - E_{\lambda_0+\varepsilon}$, apply the first of the latter two inequalities to x_0 :

$$(\lambda_0 + \varepsilon)(Qx_0, x_0) \leq (LQx_0, x_0) = (Qx_0, Lx_0) = \lambda_0(Qx_0, x_0).$$

Since Q is orthogonal, $(Qx_0, x_0) = \|Qx_0\|^2 \geq 0$, so $Qx_0 = 0$. Hence $E_{\lambda_0+\varepsilon}x_0 = x_0$, and right-continuity gives $E_{\lambda_0}x_0 = x_0$. Similarly,

$$\lambda_0(E_{\lambda_0-\varepsilon}x_0, x_0) \leq (\lambda_0 - \varepsilon)(E_{\lambda_0-\varepsilon}x_0, x_0)$$

forces $E_{\lambda_0-\varepsilon}x_0 = 0$. Thus its left limit on x_0 is zero, while $E_{\lambda_0}x_0 = x_0$. $\square$

Theorem 30.2. *Conversely, if the strong left limit of E_λ at λ_0 differs from E_{λ_0} , then $\lambda_0 \in \sigma_p(L)$.*

Proof. For $\lambda < \lambda_0$, set $Q_\lambda = E_{\lambda_0} - E_\lambda$ and $H_\lambda = R(Q_\lambda)$. The increasing bounded family E_λ has a strong left limit $P_0 \leq E_{\lambda_0}$. It is still an orthogonal projection: the uniform bound $\|E_\lambda\| \leq 1$ permits passage to the strong limit in $E_\lambda^2 = E_\lambda$.

By hypothesis $E_{\lambda_0} - P_0 \neq 0$. Choose a nonzero x_0 in its range, so $(E_{\lambda_0} - P_0)x_0 = x_0$. For $\lambda < \lambda_1 < \lambda_0$,

$$(E_{\lambda_0} - E_\lambda)(E_{\lambda_0} - E_{\lambda_1}) = E_{\lambda_0} - E_{\lambda_1}.$$

Taking $\lambda_1 \uparrow \lambda_0$ shows $Q_\lambda x_0 = x_0$, so $x_0 \in H_\lambda$ for every $\lambda < \lambda_0$. The spectral bounds on this invariant subspace give

$$\lambda I|_{H_\lambda} \leq L|_{H_\lambda} \leq \lambda_0 I|_{H_\lambda}.$$

Therefore $\|(L - \lambda_0 I)x_0\| \leq (\lambda_0 - \lambda)\|x_0\|$. Letting $\lambda \uparrow \lambda_0$ gives $Lx_0 = \lambda_0 x_0$. $\square$

For self-adjoint operators the residual spectrum is empty, so $\sigma(L) = \sigma_p(L) \cup \sigma_c(L)$. The preceding results imply strong continuity of the spectral family at every real point of the continuous spectrum or the resolvent set.

Theorem 30.3. *Suppose the spectral family is continuous at λ_0 , and there are $\lambda_n \uparrow \lambda_0$, $\mu_n \downarrow \lambda_0$ with $P_n = E_{\mu_n} - E_{\lambda_n} \neq 0$ for every n . Then $\lambda_0 \in \sigma_c(L)$.*

Proof. On $H_n = R(P_n)$, $\lambda_n I \leq L \leq \mu_n I$. Choose $x_n \in H_n$ with $\|x_n\| = 1$. Then

$$\|Lx_n - \lambda_0 x_n\| \leq \mu_n - \lambda_n \longrightarrow 0.$$

Thus $\lambda_0 \in \sigma(L)$, since a bounded inverse would contradict this estimate. Continuity excludes an eigenvalue, so it lies in the continuous spectrum. $\square$

Theorem 30.4 (Resolvent intervals). (1) *If $E_\lambda = E_\mu$ for $\lambda < \mu$, then $(\lambda, \mu) \subset \rho(L)$.*

(2) *If $[\lambda, \mu] \subset \rho(L)$, then $E_\lambda = E_\mu$.*

Proof. For (1), put $H_1 = R(E_\lambda)$ and $H_2 = R(I - E_\lambda)$. These are orthogonal invariant subspaces and $H = H_1 \oplus H_2$. We have $L|_{H_1} \leq \lambda I$ and $L|_{H_2} \geq \mu I$. For $\lambda_0 \in (\lambda, \mu)$ it suffices to invert $L - \lambda_0 I$ on both summands.

On H_2 , $(L - \lambda_0 I)|_{H_2} \geq (\mu - \lambda_0)I$; on H_1 , $(\lambda_0 I - L)|_{H_1} \geq (\lambda_0 - \lambda)I$. Both are bounded below by a positive constant and are self-adjoint, hence invertible. Their direct sum gives the required inverse. For (2), at every $\hat{\lambda} \in [\lambda, \mu]$ the family is continuous, and the preceding theorem implies it is constant in some interval about $\hat{\lambda}$: otherwise arbitrarily small enclosing intervals would have nonzero projection differences and yield continuous spectrum. A finite subcover, or local constancy on the connected interval, then shows $E_\lambda = E_\mu$. $\square$

30.3. Functions of self-adjoint operators.

Theorem 30.5. *For every continuous real function φ on $[a, b]$,*

$$\varphi(L) = \int_a^b \varphi(\lambda) dE_\lambda = \lim_{\Delta \rightarrow 0} \sum_k \varphi(\mu_k)(E_{\lambda_{k+1}} - E_{\lambda_k})$$

in operator norm.

Proof. First consider $\varphi(t) = t^m$. Let $Q_k = E_{\lambda_{k+1}} - E_{\lambda_k}$. Orthogonality gives $Q_k Q_l = 0$ for $k \neq l$ and $Q_k^2 = Q_k$, so

$$\left(\sum_k \mu_k Q_k \right)^m = \sum_k \mu_k^m Q_k.$$

As the mesh tends to zero, the left side converges to L^m . Linearity proves the formula for every polynomial p . For continuous φ , choose a polynomial p such that $|\varphi(t) - p(t)| < \varepsilon$ on $[a, b]$.

Order preservation gives $-\varepsilon I \leq \varphi(L) - p(L) \leq \varepsilon I$, hence $\|\varphi(L) - p(L)\| \leq \varepsilon$. For a sufficiently fine partition, the polynomial case gives

$$\left\| p(L) - \sum_k p(\mu_k) Q_k \right\| < \varepsilon.$$

Finally, orthogonality and the uniform approximation imply

$$\left\| \sum_k (\varphi(\mu_k) - p(\mu_k)) Q_k \right\| \leq \varepsilon.$$

Combining the three estimates bounds the error for φ by 3ε , proving the theorem. $\square$

Corollary 30.1. *If a continuous φ vanishes on $\sigma(L)$, then $\varphi(L) = 0$.*

Proof. In the approximating spectral sums, choose a sample point in $[\lambda_k, \lambda_{k+1}] \cap \sigma(L)$ whenever this intersection is nonempty; then its coefficient is zero. If the closed interval does not meet the spectrum, the resolvent-interval theorem gives $E_{\lambda_k} = E_{\lambda_{k+1}}$. Thus every term is zero, and passage to the limit proves the claim. $\square$

Corollary 30.2. *If $\varphi(t) \neq 0$ for all $t \in \sigma(L)$, then $\varphi(L)$ is invertible.*

Proof. Define a continuous $\tilde{\varphi}$ on $[a, b]$ equal to $1/\varphi$ on the compact set $\sigma(L)$. On each complementary interval interpolate linearly between endpoint values; on outer intervals extend constantly. This gives a continuous extension. Since $\varphi\tilde{\varphi} = 1$ on $\sigma(L)$, the preceding corollary and multiplicativity give

$$\varphi(L)\tilde{\varphi}(L) - I = 0 = \tilde{\varphi}(L)\varphi(L) - I.$$

Thus $\tilde{\varphi}(L)$ is the inverse. $\square$

THE END

APPENDIX A. PROBLEM SETS, SUPPLEMENTARY MATERIALS, AND QUIZ
SOLUTIONS

A.1. Problem Set 1, Part 1. Assigned September 1, 2015; due September 8, before class.

- (1) Suppose $(X, \|\cdot\|)$ is a normed vector space. Let $d(x, y) = \|x - y\|$. Prove that d is a metric compatible with the vector-space structure.
- (2) Let $X = \{x = (a_1, a_2, \dots) : a_i \in \mathbb{R}\}$. Prove:
 - (1) X is an infinite-dimensional vector space.
 - (2) For $x = (a_i)$, $y = (b_i)$, the function

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{|a_i - b_i|}{1 + |a_i - b_i|}$$

is a metric on X .

- (3) There is no norm on X with $d(x, y) = \|x - y\|$.
- (3) A subset A of a vector space is convex if $\alpha x + (1 - \alpha)y \in A$ for $x, y \in A$ and $0 \leq \alpha \leq 1$. Prove:
 - (1) Every open ball in a normed space is convex.
 - (2) If d is translation-invariant and $d(\alpha x, \alpha y) \leq |\alpha|d(x, y)$, then every open ball is convex.
 - (3) The function

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{|x_1 - x_2|} + \sqrt{|y_1 - y_2|}$$

is a translation-invariant metric on $\mathbb{R}^2$, but $B(0, 1)$ is not convex.

- (4) Let X be a normed space.
 - (1) Prove that its unit sphere contains a nontrivial line segment if and only if there are linearly independent vectors x, y with $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$.
 - (2) Find a normed vector space satisfying (1).

A.2. Problem Set 1, Part 2. Assigned September 6, 2015; due September 8, before class.

- (1) Prove the following convergence properties:
 - (1) $x_n \rightarrow x_0$, $y_n \rightarrow y_0$ imply $x_n + y_n \rightarrow x_0 + y_0$.
 - (2) $x_n \rightarrow x_0$, $\alpha_n \rightarrow \alpha_0$ imply $\alpha_n x_n \rightarrow \alpha_0 x_0$.
 - (3) In a metric space, $x_n \rightarrow x_0$ implies $d(x_n, y) \rightarrow d(x_0, y)$ for every y .

Editorial note. The source assumes only a metric space for this entire problem. Parts (1)–(2) require a topological vector space, for example a normed space; vector operations need not exist in a general metric space.

- (2) Let $\mathcal{P}$ be all polynomials on $[0, 1]$, and put

$$d_0(f, g) = \max_{[0, 1]} |f - g|, \quad d_1(f, g) = \int_0^1 |f - g| \, dx.$$

Is $(\mathcal{P}, d_0)$ complete? Is $(\mathcal{P}, d_1)$ complete?

- (3) Define

$$\ell^\infty = \{(a_i) : a_i \in \mathbb{R}, \sup_i |a_i| < \infty\}, \quad \|(a_i)\| = \sup_i |a_i|.$$

Prove this is a Banach space.

- (4) A series $\sum_i x_i$ in a normed space is absolutely convergent if $\sum_i \|x_i\| < \infty$. Prove that the space is Banach if and only if every absolutely convergent series converges.

A.3. Problem Set 2, Part 1. Assigned September 8, 2015; due September 15, before class.

- (1) Prove that a closed subset of a complete metric space is complete, and that a complete subspace of a metric space is closed.
- (2) Prove the equivalence of:
 - (1) (X, d) is complete;
 - (2) every nested sequence of nonempty closed balls $B_1 \supset B_2 \supset \cdots$ with radii $r_i \rightarrow 0$ has a unique point in $\bigcap_i B_i$.
- (3) Prove that nowhere differentiable continuous functions form a set of second category in $(C([0, 1]), d_0)$. Hint: let A_n consist of those f for which some $s \in [0, 1]$ satisfies

$$\left| \frac{f(s+h) - f(s)}{h} \right| \leq n \quad \text{whenever } 0 < |h| < 1/n, \quad 0 \leq s+h \leq 1.$$

Prove that each A_n is nowhere dense. Cf. Zhang Gongqing and Lin Yuanqu, pp. 92–93.

- (4) Let (X, d) be a metric space.
 - (1) Call two Cauchy sequences $(x_n), (y_n)$ equivalent if $\lim_n d(x_n, y_n) = 0$. Prove this is an equivalence relation.
 - (2) Let $\tilde{X}$ be the equivalence classes. Define

$$D(\bar{x}, \bar{y}) = \lim_{n \rightarrow \infty} d(x_n, y_n), \quad (x_n) \in \bar{x}, \quad (y_n) \in \bar{y}.$$

Prove that the limit exists, is independent of representatives, and makes D a metric on $\tilde{X}$.

- (3) Prove $(\tilde{X}, D)$ is complete.
- (4) Let X_0 consist of classes of constant sequences. Prove that $D(\bar{x}_0, \bar{y}_0) = d(x_0, y_0)$, so X can be regarded as a metric subspace of $\tilde{X}$.
- (5) Prove X_0 is dense in $\tilde{X}$.
- (6) Prove that if X is complete, then $X_0 = \tilde{X}$.

The space $\tilde{X}$ is called the completion of X .

A.4. Problem Set 2, Part 2. Assigned September 11, 2015; due September 15, before class.

- (1) On $\mathbb{R}$, consider

$$d_1(x, y) = |x - y|, \quad d_2(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|.$$

Prove that d_2 is incomplete (whereas d_1 is complete), and that they induce the same topology.

- (2) Let X be a TVS. Prove that the sum of two open sets is open. Construct closed sets A, B for which $A + B$ is not closed.
- (3) The closure of $A \subset X$ is the smallest closed set containing A . For a TVS X , prove:
 - (1) $\bar{A} = \{x \in X : U \cap A \neq \emptyset \text{ for every open neighborhood } U \text{ of } x\}$.
 - (2) If A is a vector subspace, so is $\bar{A}$.

(3) If A is convex, so is $\overline{A}$.

Editorial note. In (1), $x \in X$ corrects the source's $x \in A$.

(4) Let $0 < p < 1$ and let $L^p([0, 1])$ consist of Lebesgue measurable functions satisfying $\int_0^1 |f|^p < \infty$. Prove:

(1) $d(f, g) = \int_0^1 |f - g|^p dx$ is a translation-invariant metric.

(2) Its only convex open sets are $\emptyset$ and $L^p([0, 1])$.

Cf. Rudin, p. 37.

Editorial note. As usual, functions equal almost everywhere are identified; otherwise the displayed distance is only a pseudometric.

A.5. Problem Set 3, Part 1. Assigned September 15, 2015; due September 22, before class.

(1) Let $\mathcal{B}_0$ be a nonempty family of subsets of a vector space X satisfying:

(i) $U, V \in \mathcal{B}_0$ imply some $W \in \mathcal{B}_0$ has $W \subset U \cap V$;

(ii) for each $U \in \mathcal{B}_0$, some $V \in \mathcal{B}_0$ satisfies $V + V \subset U$;

(iii) for each $U \in \mathcal{B}_0$, some $V \in \mathcal{B}_0$ satisfies $\alpha V \subset U$ for all $|\alpha| \leq 1$;

(iv) for every $U \in \mathcal{B}_0$ and $x \in X$, some $\delta > 0$ has $\alpha x \in U$ for $|\alpha| \leq \delta$.

Define

$$\mathcal{T} = \{A \subset X : \text{for every } x \in A \text{ there is } U \in \mathcal{B}_0 \text{ with } x + U \subset A\}.$$

Prove that $\mathcal{T}$ is a translation-invariant topology. If it is Hausdorff and zero is interior to every $U \in \mathcal{B}_0$, prove that $(X, \mathcal{T})$ is a TVS and $\{\text{Int } U : U \in \mathcal{B}_0\}$ is a local base at zero.

(2) Prove that every convex neighborhood of zero in a TVS contains a balanced convex neighborhood of zero.

(3) Prove:

(1) Closures and sums of bounded sets are bounded.

(2) E is bounded if and only if $\alpha_n x_n \rightarrow 0$ for every sequence $x_n \in E$ and every scalar sequence $\alpha_n \rightarrow 0$.

(3) If V is a bounded open neighborhood of zero and $0 \neq \alpha_n \rightarrow 0$, then $\{\alpha_n V\}$ is a local base at zero.

Editorial note. The nonzero assumption on α_n in (3) is necessary for these to be open neighborhoods and is implicit in the source.

(4) Prove:

(1) The closure of a balanced set is balanced.

(2) A union of balanced sets is balanced.

(3) If E is convex and contains zero, then E is balanced if and only if $\alpha x \in E$ whenever $x \in E$ and $|\alpha| = 1$.

(4) The convex hull of A is the intersection of all convex sets containing A . Prove the convex hull of a balanced set is balanced.

A.6. Problem Set 3, Part 2. Assigned September 18, 2015; due September 22, before class.

(1) Read pages 2–3 of Lecture 6 for the proof of its theorem.

(2) If the topology of X is defined by a separating sequence (p_n) of seminorms, prove that for every positive sequence $c_n \rightarrow 0$,

$$d(x, y) = \max_n \frac{c_n p_n(x - y)}{1 + p_n(x - y)}$$

is a compatible translation-invariant metric.

- (3) If d_1, d_2 are compatible translation-invariant metrics on a TVS and d_1 is complete, prove that d_2 is complete.
- (4) Let X be the space of rapidly decreasing functions on $\mathbb{R}$:

$$X = \left\{ f \in C^\infty(\mathbb{R}) : \|f\|_{n,m} = \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty \text{ for all } m, n \geq 0 \right\}.$$

Prove that X is a vector space, that $p_{n,m}(f) = \|f\|_{n,m}$ form a separating family of seminorms, and that X is Fréchet.

A.7. Problem Set 4, Part 1. Assigned September 22, 2015; due September 29, before class.

- (1) Prove:
 - (1) If p is a seminorm on X , then $A = \{x : p(x) < 1\}$ is convex, balanced, absorbing, and $p = \mu_A$.
 - (2) For convex balanced absorbing A , its Minkowski functional μ_A is a norm if and only if $\bigcap_{t>0} tA = \{0\}$.
- (2) Let X be an inner product space.
 - (1) Prove $|(x, y)| \leq \|x\| \|y\|$ (Schwarz's inequality).
 - (2) Prove the parallelogram law $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.
 - (3) Conversely, if a norm satisfies the parallelogram law, prove there is an inner product inducing it.
- (3) (1) Read the proof of the last theorem in Lecture 7.
- (2) In $X = C([-1, 1])$ with the uniform norm, let

$$K = \left\{ g(x) = 2x + f(x) : f \in C([-1, 1]), \int_{-1}^0 f(x) dx = \int_0^1 f(x) dx = 0 \right\}.$$

Prove K is closed and convex, $\inf_{g \in K} \|g\| = 1$, but no $g \in K$ has norm one. Thus the last theorem of Lecture 7 fails for general Banach spaces.

- (3) A Banach space is uniformly convex if

$$\|x_n\| \leq 1, \quad \|y_n\| \leq 1, \quad \|x_n + y_n\| \rightarrow 2 \implies \|x_n - y_n\| \rightarrow 0.$$

Prove the last theorem in Lecture 7 holds for uniformly convex Banach spaces. See p. 345, Exercise 32.

- (4) Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and

$$H^2(D) = \{f : f \text{ holomorphic in } D, \int_D |f(z)|^2 dA < \infty\}.$$

Prove that $(f, g) = \int_D f(z) \overline{g(z)} dA$ is an inner product and that $H^2(D)$ is a Hilbert space.

A.8. Problem Set 4, Part 2. Assigned September 25, 2015; due September 29, before class.

- (1) Read pages 4–5 of the lecture notes and reprove Corollary 4 on page 5 by a simpler method assuming X is metrizable.

Editorial note. The source names Lecture 7 here, but the finite-dimensional closed-subspace corollary appears on page 5 of Lecture 8.

- (2) (1) Prove that the image and inverse image of a convex (respectively, balanced) set under a linear map are convex (respectively, balanced).
- (2) Construct $L_1, L_2 : X \rightarrow X$ with $L_1 L_2 = I$ but neither operator invertible.
- (3) On the rapidly decreasing functions on $\mathbb{R}$ (Problem Set 3, Part 2, Problem 4), prove that $L_{m,n} f = x^m f^{(n)}$ is linear and continuous for every $m, n \geq 0$.
- (4) (1) If X is a TVS and $L : X \rightarrow \mathbb{C}^n$ is surjective and linear, prove L is open.
- (2) For a closed subspace N of a TVS X , give X/N the quotient topology: $V \subset X/N$ is open if and only if $\pi^{-1}(V)$ is open, where $\pi : X \rightarrow X/N$ is the canonical projection. Prove X/N is a TVS.
- (3) If a surjective linear map $L : X \rightarrow \mathbb{C}^n$ has closed kernel, prove it is continuous.

A.9. Problem Set 5, Part 1. Assigned September 29, 2015; due October 13, before class.

- (1) Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges for every $b \in \ell^1$ if and only if $a \in \ell^\infty$.
- (2) (1) For normed spaces X, Y , prove a linear map $L : X \rightarrow Y$ is continuous if and only if $\|Lx\|_Y \leq C$ whenever $\|x\|_X < 1$, for some $C > 0$.
- (2) Let H be Hilbert and $L : H \rightarrow H$ linear, with $(Lx, y) = (x, Ly)$ for every $x, y \in H$. Prove L is continuous.
- (3) Let $X \subset \ell^1$ be the finitely supported sequences, and $L_n : X \rightarrow \mathbb{R}$, $L_n(a) = na_n$. Prove L_n is linear, $\sup_n |L_n(a)| < \infty$ for each $a \in X$, but $\sup_n \|L_n\| = \infty$.
- (4) If X is Banach and $L : X \rightarrow X$ is continuous and linear with $\|I - L\| < 1$, prove L is invertible and L^{-1} is continuous.

A.10. Problem Set 5, Part 2. Assigned October 9, 2015; due October 13, before class.

- (1) Give $X = C^1([a, b])$ the norm $\|f\|_X = \|f\|_\infty + \|f'\|_\infty$, and give $Y = C^1([a, b])$ the norm $\|f\|_Y = \|f\|_\infty$. Are X, Y complete? For the identity $L : X \rightarrow Y$, are L and L^{-1} continuous?
- (2) A nonzero continuous linear functional on an F-space is open. Prove this, then prove the same under the weaker assumption that X is a TVS.
- (3) Suppose $X \subset C([0, 1]) \subset L^2([0, 1])$ is a closed subspace of $L^2([0, 1])$. Prove it is closed in $C([0, 1])$ and that $\|f\|_\infty \leq C \|f\|_2$ for $f \in X$, for some $C > 0$.
- (4) For Banach spaces X, Y and continuous linear $L : X \rightarrow Y$, prove

$$\exists c > 0 \forall x \in X : \|Lx\| \geq c \|x\| \iff \ker L = \{0\} \text{ and } \operatorname{Im} L \text{ is closed.}$$

A.11. Problem Set 6, Part 1. Assigned October 10, 2015; due October 20, before class.

- (1) Prove that for $1 < p < \infty$, ℓ^p with norm $\|a\|_p = (\sum_{i=1}^{\infty} |a_i|^p)^{1/p}$ is Banach.
- (2) Let X consist of sequences $a = (a_k)$ whose partial sums $A_n = \sum_{k=1}^n a_k$ converge. Set $\|a\| = \sup_n |\sum_{k=1}^n a_k|$. Prove X is Banach and $\ell^1 \subset X$ is dense.
- (3) Read Rudin, Definition 1.40 and Theorem 1.41, pp. 30–32. If X is an F-space, Y a TVS, and $L : X \rightarrow Y$ an open continuous surjective linear map, prove $Y \cong X/\ker L$, and conclude Y is an F-space.

Editorial note. The source calls L an “open mapping” without explicitly restating continuity, linearity, or surjectivity. These are required for the quotient conclusion.

- (4) (1) For F-spaces X, Y and a TVS Z , prove that a separately continuous bilinear map $B : X \times Y \rightarrow Z$ is jointly continuous.
 (2) Let $X = \mathcal{P}([0, 1])$ have norm $\|p\| = \int_0^1 |p(x)| dx$. Prove $B(p, q) = \int_0^1 p(x)q(x) dx$ is bilinear and separately continuous, but not jointly continuous.

A.12. Problem Set 6, Part 2. Assigned October 16, 2015; due October 20, before class.

- (1) Prove that the product of two F-spaces is an F-space.
 (2) Let L be a densely defined linear operator in a Hilbert space X . Define its adjoint by

$$\text{Dom}(L^*) = \{y \in X : \exists y^* \in X \forall x \in \text{Dom}(L), (Lx, y) = (x, y^*)\}, \quad L^*y = y^*.$$

Prove L^* is closed. Deduce another proof of Problem Set 5, Part 1, Problem 2(2).

- (3) Suppose a Banach space has closed subspaces Y, Z with $X = Y \oplus Z$. For the unique decomposition $x = y + z$, define $P_Y x = y$ and $P_Z x = z$. Prove $P_Y^2 = P_Y$, $P_Y P_Z = 0$, and P_Y is continuous. Find an example with $\|P_Y\| > 1$.
 (4) (1) For a linear operator between Banach spaces, prove it admits a closure if and only if $x_n \in \text{Dom}(L)$, $x_n \rightarrow 0$, $Lx_n \rightarrow y$ always imply $y = 0$.
 (2) On $L^2(\Omega)$ consider the differential operator

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha.$$

Prove P admits a closure. Cf. Example 1.8 in the typed Lecture 8.

Editorial note. The source refers to a typed lecture for the coefficient regularity and initial domain; these are not specified on this problem sheet. The usual initial domain is $C_c^\infty(\Omega)$ with smooth coefficients.

A.13. Problem Set 7, Part 1. Assigned October 20, 2015; due October 27, before class.

- (1) Let X be complex and $f : X \rightarrow \mathbb{C}$, $u = \text{Re } f$. Prove that f is complex-linear if and only if u is real-linear and $f(x) = u(x) - iu(ix)$. Prove the complex Hahn–Banach theorem.
 (2) Let Y be a subspace of a normed space X , $x_0 \notin Y$, and $Z = Y + \mathbb{F}x_0$. Define $\ell(y + \alpha x_0) = \alpha$. Prove that ℓ is linear, and continuous if and only if $x_0 \notin \bar{Y}$. Deduce

$$\bar{Y} = \bigcap \{\ker L : L \in X^*, Y \subset \ker L\}.$$

- (3) Suppose a linear functional ℓ and seminorms $p_1, \dots, p_n$ on X satisfy $|\ell(x)| \leq \sum_{k=1}^n p_k(x)$. Prove $\ell = \sum_{k=1}^n \ell_k$ with linear ℓ_k satisfying $|\ell_k(x)| \leq p_k(x)$. Hint: consider $\tilde{X} = X^n$, $p(x_1, \dots, x_n) = \sum_k p_k(x_k)$, and the diagonal subspace $\tilde{Y} = \{(x, \dots, x) : x \in X\}$.

- (4) Use Zorn's lemma to prove every vector space has a Hamel basis. Prove no infinite-dimensional F -space has a countable Hamel basis.

A.14. Problem Set 7, Part 2. Assigned October 23, 2015; due October 27, before class.

- (1) Prove Corollaries 1, 2, and 3 on page 5 of the course notes.
 (2) In $X = L^2([-1, 1])$, let

$$E_a = \{f \in C([-1, 1]) : f(0) = a\}, \quad a \in \mathbb{R}.$$

Prove E_a is convex and dense in X . Prove that for $a \neq b$, E_a and E_b cannot be separated by a nonzero continuous linear functional on X .

- (3) Let A, B be nonempty disjoint convex subsets of a real TVS, and suppose A contains an interior point. Prove there exist $0 \neq L \in X^*$ and $c \in \mathbb{R}$ with $L(x) \leq c \leq L(y)$ for $x \in A, y \in B$. Construct an example showing that in general one cannot replace this by $L(x) < c \leq L(y)$.
 (4) Let X be the convergent scalar sequences and $Y = c_0$, with the supremum norm. Show both are Banach. If a continuous linear map $L : X \rightarrow Y$ extends the identity on Y , prove $\|L\| \geq 2$. Thus the norm-preserving Hahn-Banach extension theorem does not extend to general vector-valued operators.

A.15. Problem Set 8, Part 1. Assigned October 27, 2015; due November 3, before class.

- (1) If the dual of a TVS X separates points, prove that $E \subset X$ is weakly bounded if and only if each $L \in X^*$ is bounded as a scalar function on E .
 (2) In $C([0, 1])$ with the supremum norm, put

$$f_n(x) = \begin{cases} nx, & 0 \leq x \leq 1/n, \\ 2 - nx, & 1/n \leq x \leq 2/n, \\ 0, & 2/n \leq x \leq 1. \end{cases} \quad (n \geq 2).$$

Prove $f_n \not\rightarrow 0$ in norm, but $f_n \rightarrow 0$.

- (3) For Fréchet spaces X, Y , denote their weak topologies by X^w, Y^w . Prove a linear map $L : X \rightarrow Y$ is continuous if and only if $L : X^w \rightarrow Y^w$ is continuous.
 (4) For $1 < p < \infty$, prove $(\ell^p)^* = \ell^q$ where $1/p + 1/q = 1$. Show ℓ^p has sequences converging weakly but not strongly.

A.16. Problem Set 9, Part 1. Assigned November 3, 2015; due November 10, before class.

- (1) Read the proofs of Theorem 2.2(b), p. 43; Theorem 2.9, p. 47; and Theorem 3.18, pp. 70–71.
 (2) (1) Define weak-* convergence.
 (2) Prove a weak-* convergent sequence has a unique limit.
 (3) Prove $(\ell^1)^* = \ell^\infty$.
 (4) For $x = (a_i) \in \ell^\infty$, put $x_n = (0, \dots, 0, a_n, a_{n+1}, \dots)$. Prove $x_n \rightarrow 0$ weak-*.
 (3) (1) Prove $E \subset X^*$ is weak-* bounded if and only if $\sup_{L \in E} |Lx| < \infty$ for every $x \in X$.
 (2) Prove every weak-* convergent sequence is weak-* bounded.

- (3) If X is separable and $(L_n) \subset X^*$ is weak-* bounded, prove it has a weak-* convergent subsequence.

Editorial note. Part (3), as stated on the sheet, needs an additional hypothesis such as X Banach (so pointwise boundedness gives a common norm bound). For a general separable normed space the conclusion need not hold.

- (4) (1) Let $L_1, \dots, L_n, L$ be linear functionals and suppose, for some $\varepsilon, \delta > 0$,
- $$\{x : |L_j x| < \varepsilon \ (1 \leq j \leq n)\} \subset \{x : |Lx| < \delta\}.$$

Prove L is a linear combination of $L_1, \dots, L_n$.

- (2) If X^* separates points and the weak topology of X is metrizable, prove X^* has a countable Hamel basis.
- (3) Prove the weak topology of an infinite-dimensional normed space is not metrizable.
- (4) Prove the weak-* topology of the dual of an infinite-dimensional Banach space is not metrizable.

A.17. Problem Set 9, Part 2. Assigned November 6, 2015; due November 10, before class.

- (1) Prove Lemmas 1 and 2 on page 4 of the written notes.
- (2) Find the extreme points of each following unit ball K :
- (a) $X_1 = \ell^2$, $K = \{a : \sum_i |a_i|^2 \leq 1\}$;
 - (b) $X_2 = c_0$, $K = \{a : \sup_i |a_i| \leq 1\}$;
 - (c) $X_3 = L^1([0, 1])$, $K = \{f : \|f\|_1 \leq 1\}$;
 - (d) $X_4 = C([0, 1])$, $K = \{f : \sup_{[0, 1]} |f| \leq 1\}$;
 - (e) $X_5 = \ell^\infty$, $K = \{a : \sup_i |a_i| \leq 1\}$.

Which of $X_1, \dots, X_5$ are duals of Banach spaces? Explain.

- (3) For pairwise orthogonal unit vectors u_n in a Hilbert space, let

$$K = \{0, u_1, \tfrac{1}{2}u_2, \tfrac{1}{3}u_3, \dots\}.$$

Prove K is compact and $\text{co}(K)$ bounded but not closed. Find all extreme points of $\overline{\text{co}(K)}$.

- (4) Prove that the closure of a totally bounded set in a complete metric space is compact. Read Ascoli's theorem in Rudin, p. 394.

A.18. Problem Set 10, Part 1. Assigned November 10, 2015; due November 17, before class.

- (1) For a subspace M of a Hilbert space, prove $(M^\perp)^\perp = \overline{M}$.
- (2) Prove $(\ell^\infty)^*$ strictly contains the canonical copy of ℓ^1 . Hint: on the convergent sequences $c \subset \ell^\infty$, $\ell((a_n)) = \lim_n a_n$ is continuous and linear. Extend it to $L \in (\ell^\infty)^*$ and show $L \notin \ell^1$.
- (3) Let B be the closed unit ball of a Banach space X , and B^{**} that of X^{**} . Prove the canonical image of B is weak-* dense in B^{**} .
- (4) Suppose a bilinear (real case) or sesquilinear (complex case) form $B : H \times H \rightarrow \mathbb{F}$ satisfies

$$|B(x, y)| \leq C \|x\| \|y\|, \quad |B(y, y)| \geq b \|y\|^2 \quad (b > 0).$$

Prove that every continuous linear functional L on H has a unique representative $y \in H$ with $Lx = B(x, y)$ for all $x \in H$.

Editorial note. The sheet writes $Lx = (x, y)$ in the last line. The version using $B(x, y)$ is the intended Lax–Milgram conclusion suggested by the hypotheses; with (x, y) it is simply the Riesz representation theorem.

A.19. Problem Set 10, Part 2. Assigned November 13, 2015; due November 17, before class.

- (1) If a normed space X has separable dual, prove X is separable. Show the converse fails: prove ℓ^1 is separable, whereas ℓ^∞ is not (the latter was done in class).
- (2) Let B be the closed unit ball.
 - (1) If X is reflexive and $x^* \in X^*$, show there is $x_0 \in B$ with $\langle x_0, x^* \rangle = \|x^*\| = \sup_{x \in B} |\langle x, x^* \rangle|$.
 - (2) For $x^* : c_0 \rightarrow \mathbb{R}$, $x^*((a_n)) = \sum_{n=1}^\infty a_n/2^n$, prove $x^* \in c_0^*$ but no $x_0 \in B$ attains $\|x^*\|$.
- (3) Prove every uniformly convex Banach space is reflexive. See Problem Set 4, Part 1, Problem 3(3).
- (4) Let $\varphi : X \rightarrow (-\infty, +\infty]$ on a topological space. It is lower semicontinuous if $\varphi_c = \{x : \varphi(x) \leq c\}$ is closed for every $c \in \mathbb{R}$. On a vector space, it is convex if $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$ for $0 < t < 1$. Prove:
 - (1) If E is nonempty compact and φ lower semicontinuous, then φ attains its infimum on E .
 - (2) The sublevel sets of a convex function are convex.
 - (3) On a locally convex TVS, a convex function lower semicontinuous for the original topology is also weakly lower semicontinuous.
 - (4) If X is reflexive Banach, E nonempty closed and convex, and $\varphi : E \rightarrow (-\infty, +\infty]$ is convex and lower semicontinuous with $\varphi(x) \rightarrow +\infty$ as $x \in E$, $\|x\| \rightarrow \infty$, then its infimum on E is attained. The coercivity condition may be omitted when E is bounded.

Editorial note. For (3) the source assumes only that the dual separates points; the usual closed-convex separation argument requires local convexity. This hypothesis is made explicit here.

A.20. Problem Set 11, Part 1. Assigned November 17, 2015; due November 24, before class. Here $M^\perp$ denotes an annihilator and ${}^\perp N$ a preannihilator.

- (1) (1) For closed subspaces M_1, M_2 of a Banach space, prove

$$M_1 \cap M_2 = {}^\perp(M_1^\perp + M_2^\perp), \quad M_1^\perp \cap M_2^\perp = (M_1 + M_2)^\perp.$$
 (2) For densely defined $L : \text{Dom}(L) \subset X \rightarrow Y$, with graph Γ_L , prove

$$\Gamma_L \cap (X \times \{0\}) = \ker L \times \{0\},$$

$${}^\perp \text{Im}(L^*) \times \{0\} = {}^\perp(\Gamma_L^\perp + (X \times \{0\})^\perp).$$
 (3) If L is also closed, prove $\ker L = {}^\perp \text{Im}(L^*)$.
- (2) For a closed subspace M of a Banach space X , prove: if X is reflexive, so is X/M ; if M and X/M are reflexive, then X is reflexive.
- (3) Find the adjoints in these examples:
 - (1) the inclusion $L : X \hookrightarrow Y$ of a closed Banach subspace;
 - (2) $L : \ell^1 \rightarrow \ell^1$, $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$;
 - (3) $L : \ell^2 \rightarrow \ell^2$, $(x_1, x_2, \dots) \mapsto (x_1, x_2/2, x_3/3, \dots)$;

- (4) $L : C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $Lf = f''$.
- (4) (1) If X is reflexive and $N \subset X^*$ is a norm-closed subspace, prove $({}^\perp N)^\perp = N$.
- (2) Identify $c_0^* = \ell^1$ and set

$$N = \{(a_n) \in \ell^1 : \sum_{n=1}^{\infty} (-1)^n a_n = 0\}.$$

Prove N is closed but strictly contained in $({}^\perp N)^\perp$. Hint: prove ${}^\perp N = \{0\}$.

A.21. Problem Set 11, Part 2. Assigned November 20, 2015; due November 24, before class.

- (1) A densely defined operator L in a Hilbert space is symmetric if $\text{Dom}(L) \subset \text{Dom}(L^*)$ and $L^*|_{\text{Dom}(L)} = L$. Prove that if L is symmetric, then L^{**} is symmetric.
- (2) Prove the new version of the open mapping theorem on page 4 of the written notes.
- (3) For a densely defined closed operator between Banach spaces, prove L is surjective if and only if there exists $C > 0$ such that

$$\|y^*\| \leq C \|L^*y^*\| \quad (y^* \in \text{Dom}(L^*)).$$

- (4) For $L \in \mathcal{L}(X, Y)$ between Banach spaces, prove: L is invertible if and only if L^* is invertible; when invertible, $(L^*)^{-1} = (L^{-1})^*$.

A.22. Problem Set 12, Part 1. Assigned November 27, 2015; due December 8, before class. All spaces in these problems are Banach. Write $\mathcal{K}(X, Y)$ for compact operators.

- (1) Prove:
- (1) If $L \in \mathcal{K}(X, Y)$ and $X_1 \subset X$ is closed, then $L|_{X_1} \in \mathcal{K}(X_1, Y)$.
 - (2) If $L_1 \in \mathcal{L}(X, Y)$ and $L_2 \in \mathcal{L}(Y, Z)$ and one is compact, then $L_2 L_1$ is compact.
 - (3) If $L \in \mathcal{L}(X, Y)$ has closed range, it is compact if and only if it has finite rank.
- (2) For $L \in \mathcal{L}(X, Y)$, prove:
- (1) If L is compact, $x_n \rightharpoonup x$ implies $Lx_n \rightarrow Lx$.
 - (2) If X is reflexive and $x_n \rightharpoonup x$ always implies $Lx_n \rightarrow Lx$, then L is compact.
- (3) Examples of compact operators:
- (1) For $a = (a_i) \in \ell^\infty$ and $1 \leq p \leq \infty$, let $L_a : \ell^p \rightarrow \ell^p$, $L_a((b_i)) = (a_i b_i)$. Prove L_a is continuous and compact if and only if $a \in c_0$.
 - (2) For $k \in C([0, 1]^2)$, prove $L_k f(x) = \int_0^1 k(x, y) f(y) dy$ is compact on $C([0, 1])$.
 - (3) If k is continuous on $\{(x, y) : 0 \leq y \leq x \leq 1\}$, prove the Volterra operator $L_k f(x) = \int_0^x k(x, y) f(y) dy$ is compact on $C([0, 1])$.
- (4) A sequence (x_n) is a Schauder basis of X if every $x \in X$ has a unique scalar expansion $x = \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n x_n$. Prove that a space with a Schauder basis is separable, and that every compact $L : X \rightarrow X$ on such a space can be approximated in operator norm by finite-rank operators.

A.23. Problem Set 12, Part 2. Assigned December 1, 2015; due December 8, before class.

- (1) If $L \in \mathcal{L}(X, Y)$ has closed range, prove the dimension relations

$$\dim \ker L = \dim(X^*/\operatorname{Im} L^*), \quad \dim \ker L^* = \dim(Y/\operatorname{Im} L).$$

Editorial note. These equalities are interpreted in the finite-dimensional case: either side is finite exactly when the other is, and then the dimensions agree. In general the associated spaces are dual to one another, which does not imply equality of infinite Hamel dimensions.

- (2) Read Rudin, Theorem 1.22, pp. 17–18. Use the lemma on page 3 of the notes to prove that if the closed unit ball of a normed space is compact, the space is finite-dimensional.
- (3) Prove:
- (1) If $L \in \mathcal{L}(X, Y)$ is surjective, $\ker L$ is complemented in X if and only if some $\tilde{L} \in \mathcal{L}(Y, X)$ satisfies $L\tilde{L} = I_Y$.
 - (2) If L is injective, its range is closed and complemented in Y if and only if some $\tilde{L} \in \mathcal{L}(Y, X)$ satisfies $\tilde{L}L = I_X$.
- (4) If $L \in \mathcal{L}(X, X)$ and L^n is compact for some positive integer n , prove $\dim \ker(I - L) < \infty$ and $\operatorname{Im}(I - L)$ is closed.

A.24. Problem Set 12, Part 3. Assigned December 4, 2015; due December 8.

- (1) If $L \in \mathcal{L}(X, Y)$ between Banach spaces has a range admitting a closed complementary subspace, prove its range is closed.
- (2) (1) For $L \in \mathcal{L}(X, Y)$, prove the equivalence of: $\ker L$ finite-dimensional and $\operatorname{Im} L$ closed; every bounded sequence (x_n) with (Lx_n) convergent has a convergent subsequence.
- (2) Suppose X is a Banach subspace of a Banach space Y , the inclusion is compact, and $L \in \mathcal{L}(X, Y)$ satisfies

$$\|x\|_X \leq C(\|x\|_Y + \|Lx\|_Y).$$

Prove $\dim \ker L < \infty$ and $\operatorname{Im} L$ is closed.

Editorial note. The source calls X a “closed subspace” in part (2). A compact inclusion of a closed subspace with its induced Banach norm would force finite dimension. The estimate is usually applied when X has its own stronger Banach norm.

- (3) Prove a Fredholm operator has index zero if and only if it is $L = L_0 + K$ with L_0 invertible and bounded and K compact.
- (4) A sequence $X_0 \xrightarrow{L_0} X_1 \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} X_n$ is exact if $\ker L_k = \operatorname{Im} L_{k-1}$ at each intermediate term.
- (1) For a sequence $X_0 \rightarrow X_1 \rightarrow X_2$ exact at X_1 , prove that finite dimensions of X_0 and X_2 imply finite dimension of X_1 . Prove that for a finite-dimensional exact sequence $0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow 0$, $\sum_k (-1)^k \dim X_k = 0$.
 - (2) For $L_1 : X \rightarrow Y$, $L_2 : Y \rightarrow Z$, prove exactness of

$$\begin{aligned} 0 \longrightarrow \ker L_1 \longrightarrow \ker(L_2 L_1) \xrightarrow{L_1} \ker L_2 \xrightarrow{\pi} Y/\operatorname{Im} L_1 \\ \xrightarrow{L_2} Z/\operatorname{Im}(L_2 L_1) \longrightarrow Z/\operatorname{Im} L_2 \longrightarrow 0. \end{aligned}$$

- (3) If two of L_1, L_2, L_2L_1 are Fredholm, prove the third is, and

$$\text{ind}(L_2L_1) = \text{ind } L_2 + \text{ind } L_1.$$

A.25. Problem Set 13, Part 1. Assigned December 8, 2015; due December 15, before class.

- (1) Let $X = \ell^2$ and $L : X \rightarrow X$ be the shift

$$L(a_1, a_2, a_3, \dots) = (0, a_1, a_2, \dots).$$

Find $\sigma_p(L)$, $\sigma_r(L)$, $\sigma_c(L)$, and $\sigma_p(L^*)$, $\sigma_r(L^*)$, $\sigma_c(L^*)$.

- (2) Let X be a Banach space and $L \in \mathcal{L}(X)$. Prove:
- (1) If $\lambda \in \sigma_p(L)$, then either $\lambda \in \sigma_p(L^*)$ or $\lambda \in \sigma_r(L^*)$.
 - (2) If $\lambda \in \sigma_r(L)$, then $\lambda \in \sigma_p(L^*)$.
- (3) Find compact operators K with each of the following spectra:
- (1) $\sigma(K) = \{0\}$;
 - (2) $\sigma(K) = \{0, 1, 2, 3, 4\}$;
 - (3) $\sigma(K) = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.
- (4) (1) If $L \in \mathcal{L}(X)$ is invertible, prove

$$\sigma(L^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(L)\}.$$

- (2) Let X be the bounded continuous functions on $\mathbb{R}$, with norm

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|.$$

For $f \in X$ define $(Lf)(x) = f(x+1)$. Find $\sigma(L)$.

A.26. Problem Set 13, Part 2. Assigned December 11, 2015; due December 15, before class.

- (1) (1) Prove that a Hilbert space has a finite or countable orthonormal basis if and only if it is separable.
- (2) Let H be an infinite-dimensional separable Hilbert space. Prove that H is isomorphic to ℓ^2 as a Hilbert space.
- (2) Find an orthonormal basis for each space below:
- (1) $H = \{p(t) = a_3t^3 + a_2t^2 + a_1t + a_0 : a_i \in \mathbb{R}, 0 \leq t \leq 1\}$, with $(p, q) = \int_0^1 p(t)q(t) dt$;
 - (2) $H = L^2(S^1)$;
 - (3) $H = H^2(D)$, as in Problem Set 4, Part 1.
- (3) (1) Let $\{e_1, e_2, \dots\}$ be an orthonormal basis of a Hilbert space H . For $L \in H^*$, set

$$y = \sum_{n=1}^{\infty} \overline{L(e_n)} e_n.$$

Prove that $L = L_y$, where $L_y(x) = (x, y)$.

- (2) For $\xi \in D$, define $L : H^2(D) \rightarrow \mathbb{C}$ by $L(f) = f(\xi)$. Prove L is continuous and find $g \in H^2(D)$ such that $L(f) = (f, g)$.
- (4) Let

$$X = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} : f(t) = \sum_{k=1}^n c_k e^{i\alpha_k t}, \quad n \in \mathbb{N}, \quad c_k \in \mathbb{C}, \quad \alpha_k \in \mathbb{R} \right\}.$$

- (1) Show that

$$(f, g) = \lim_{A \rightarrow \infty} \frac{1}{2A} \int_{-A}^A f(t) \overline{g(t)} dt$$

defines an inner product on X .

- (2) If the α_k in $f(t) = \sum_{k=1}^n c_k e^{i\alpha_k t}$ are pairwise distinct, prove $\|f\|^2 = |c_1|^2 + \cdots + |c_n|^2$.
- (3) Prove that X is not complete.
- (4) Let H be the completion of X for this norm. Prove that H is not separable.

A.27. Problem Set 14, Part 1. Assigned December 15, 2015; due December 22, before class.

- (1) Let $L \in \mathcal{L}(H)$ be self-adjoint. Prove:
- (1) $\|L\| = \sup_{x \neq 0} \frac{|(Lx, x)|}{\|x\|^2}$;
- (2) either $\|L\|$ or $-\|L\|$ belongs to $\sigma(L)$.
- (2) (1) Read Theorem 12.16 in Rudin, pp. 315–316.
- (2) If $L_1, L_2 \in \mathcal{L}(H)$ are normal and $L_1 L_2 = L_2 L_1$, prove that $L_1 + L_2$ and $L_1 L_2$ are normal.
- (3) If $L \in \mathcal{L}(H)$ is normal, prove there is a unitary $U \in \mathcal{L}(H)$ such that $L^* = UL$.
- (3) (1) Let X be a Banach space and $L \in \mathcal{L}(X)$. Prove the spectral-radius formula

$$r(L) = \lim_{n \rightarrow \infty} \|L^n\|^{1/n}.$$

The spectral radius was defined at the bottom of Lecture 24, p. 2.

- (2) If $L \in \mathcal{L}(H)$ is normal, prove $r(L) = \|L\|$.
- (4) Consider $L : L^2([0, 1]) \rightarrow L^2([0, 1])$ given by

$$(Lf)(x) = \int_0^1 k(x, y) f(y) dy, \quad k(x, y) = \min\{x, y\}.$$

- (1) Prove that L is compact and self-adjoint.
- (2) Find its spectrum.
- (3) Find $\|L\|$.

A.28. Problem Set 14, Part 2. Assigned December 18, 2015; due December 22, before class.

- (1) An operator $L \in \mathcal{L}(H)$ is positive if $(Lx, x) \geq 0$ for every $x \in H$. Let $L \geq 0$.
- (1) Prove there is a unique positive $\tilde{L} \in \mathcal{L}(H)$ with $L = \tilde{L}^2$.
- (2) If $L_1 L = L L_1$, prove $L_1 \tilde{L} = \tilde{L} L_1$.
- Can the exponent 2 be replaced by n ?
- (2) Prove the properties of Hilbert–Schmidt operators stated in the middle of p. 4 of the lecture notes.
- (3) Prove every trace-class operator is compact.

- (4) Let H be a Hilbert space and $L \in \mathcal{K}(H)$ positive. List its eigenvalues, counting multiplicity, as $\lambda_1 \geq \lambda_2 \geq \cdots \rightarrow 0$. Prove the min-max formulas

$$\lambda_n = \max_{\dim V=n} \min_{v \in V \setminus \{0\}} \frac{(Lv, v)}{\|v\|^2},$$

$$\lambda_n = \min_{\dim V=n-1} \max_{v \in V^\perp \setminus \{0\}} \frac{(Lv, v)}{\|v\|^2}.$$

The source assumes $V \subset \ker(L)^\perp$.

A.29. Problem Set 15, Part 1. Assigned December 22, 2015; due December 29, before class.

- (1) Suppose $L \geq 0$. Prove

$$|(Lx, y)| \leq \sqrt{(Lx, x)} \sqrt{(Ly, y)}.$$

- (2) Suppose $L_1, L_2 \geq 0$ and $L_1 L_2 = L_2 L_1$. Prove:

(1) $L_1 \geq L_2$ if and only if $\sqrt{L_1} \geq \sqrt{L_2}$;

(2) $\frac{L_1 + L_2}{2} \geq \sqrt{L_1 L_2}$.

Editorial note. The handwritten commutation relation reads $L_1 L_2 = L_2 L_2$; it is corrected to the intended $L_1 L_2 = L_2 L_1$.

- (3) (1) Let $L \in \mathcal{L}(H)$ be positive. Prove that L is bijective if and only if $L \geq \varepsilon I$ for some $\varepsilon > 0$.
 (2) Let $L \in \mathcal{L}(H)$. Prove that L is bijective if and only if for some $\varepsilon > 0$, both $LL^* \geq \varepsilon I$ and $L^*L \geq \varepsilon I$.
 (4) (1) Suppose $L \geq 0$. Prove the equivalences: L is compact, $\sqrt{L}$ is compact, and $|L|$ is compact.
 (2) If $0 \leq L_1 \leq L_2$ and L_2 is compact, prove L_1 is compact.

A.30. Problem Set 15, Part 2. Assigned December 25, 2015; due December 29, before class. *This is the last problem set.*

- (1) Prove that for a projection $P \in \mathcal{L}(H)$, the following are equivalent:
 (1) P is an orthogonal projection;
 (2) $\|P\| \leq 1$;
 (3) $PP^* = P^*P$;
 (4) $P \geq 0$.
 (2) Let $\Lambda \subset \mathcal{L}(H)$. A subspace $E \subset H$ is *invariant under* Λ if $L(E) \subset E$ for all $L \in \Lambda$. Suppose the only closed invariant subspaces are $\{0\}$ and H . Prove that if a self-adjoint $L_0 \in \mathcal{L}(H)$ commutes with every $L \in \Lambda$, then $L_0 = cI$ for some real c .
 (3) Let $E_1 \supset E_2 \supset \cdots$ be closed subspaces of H and $E = \bigcap_{n=1}^\infty E_n$. Prove

$$P_E = \lim_{n \rightarrow \infty} P_{E_n}.$$

Editorial note. The limit is in the strong operator topology: for each $x \in H$, $P_{E_n}x \rightarrow P_E x$. Operator-norm convergence is not asserted.

- (4) (1) Let L be compact and self-adjoint. Prove that its spectral projections satisfy

$$E_\lambda x = \begin{cases} \sum_{\lambda_k \leq \lambda} (x, e_k) e_k, & \lambda < 0, \\ x - \sum_{\lambda_k > \lambda} (x, e_k) e_k, & \lambda > 0. \end{cases}$$

- (2) On $H = L^2([0, 1])$, let $(Lx)(t) = tx(t)$. Prove

$$(E_\lambda x)(t) = \chi_\lambda(t)x(t),$$

where χ_λ denotes the indicator of $(-\infty, \lambda]$.

A.31. Supplementary notes (Ex2).

A.31.1. Outline of the lecture notes.

Problem Set 6, Part 2, Problem 2. Density of $\text{Dom}(L)$ ensures that the adjoint is well-defined. If two possible values of L^*y differ by a , then $(x, a) = 0$ for every $x \in \text{Dom}(L)$. Approximating a by such x gives $(a, a) = 0$, hence $a = 0$.

Problem Set 6, Part 2, Problem 4(2). Suppose $f_n \rightarrow 0$ and $Pf_n \rightarrow g$ in L^2 . To prove closability we must show $g = 0$. Integration by parts gives, for $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} Pf_n(x)\varphi(x) \, dx = \int_{\Omega} f_n(x)P^t\varphi(x) \, dx,$$

where the formal transpose is

$$P^t\varphi = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \partial^\alpha (a_\alpha \varphi).$$

Taking limits yields $\int_{\Omega} g\varphi = 0$ for every test function, so $g = 0$.

Editorial note. The source denotes the formal transpose by P^* and moves the limit inside the integral. Here the limit is justified by Cauchy–Schwarz and L^2 convergence; pointwise convergence need not hold. Complex inner-product notation instead uses the conjugated formal adjoint.

Notes on Lecture 14. At the bottom of page 3 the notes say: if $\dim X = \infty$, every weak neighborhood of zero contains an infinite-dimensional subspace. Consequently, the weak topology is not locally bounded and hence not normable. Indeed, if it were locally bounded, dilation of a bounded neighborhood containing an infinite-dimensional subspace would force that same subspace into every zero neighborhood, contradicting the Hausdorff property. “Not normable” cannot here be replaced by “not metrizable”, as the next example illustrates.

Take $X = \mathbb{R}^{\mathbb{N}}$ with coordinate seminorms $p_k(x) = |x_k|$. Then

$$d(x, y) = \max_{k \geq 1} 2^{-k} \frac{p_k(x - y)}{1 + p_k(x - y)}$$

is a translation-invariant metric defining a standard Fréchet space. This space is metrizable but not locally bounded: every neighborhood of zero contains an infinite-dimensional coordinate subspace. For every $\varepsilon > 0$, $\varepsilon B_d(0, 1)$ need not be contained in $B_d(0, 1/2)$. Thus a metric ball need not be bounded in the TVS sense, although norm balls are bounded. In particular, metrizability does not imply local boundedness. The author notes that they had previously made a mistake about this point.

Editorial note. The source describes this as “a space of countably many dimensions” and writes x_k as a seminorm. This example is the full countable product, not a countable-dimensional Hamel space, and its seminorms are $|x_k|$.

Problem Set 8, Part 1, Problem 2. One can prove $f_n \rightarrow 0$ without knowing the dual of $C([0, 1])$. Otherwise, for some continuous real functional L , a subsequence satisfies $L(f_{n_k}) > \varepsilon$ for some $\varepsilon > 0$. Choose a further subsequence n'_k so that

$$\left\| \sum_{k=1}^m f_{n'_k} \right\|_{\infty} \leq 2 \quad (m \geq 1).$$

But $L(\sum_{k=1}^m f_{n'_k}) > m\varepsilon$, contradicting continuity of L .

Editorial note. For complex scalars one first takes a suitable rotated real part of a functional, so the displayed strict inequalities make sense.

Problem Set 8, Part 1, Problem 4(1). The identity $(\ell^p)^* = \ell^q$ is an identification of normed vector spaces, not merely sets. We must prove that every $L \in (\ell^p)^*$ has

$$L(a) = \sum_i a_i b_i \quad (a \in \ell^p)$$

for suitable coefficients b_i . We must also prove $b \in \ell^q$ and $\|L\| = \|b\|_q$; the norm equality is often omitted. The representation uses linearity, and the membership and norm calculation use Hölder's inequality and its equality conditions. This also reproves completeness of ℓ^p for $1 < p < \infty$, since $(\ell^q)^* = \ell^p$ and a dual space is complete; likewise $\ell^1 = (c_0)^*$.

A.31.2. Additional topics.

- An absorbing set need not have zero as an interior point. For a discontinuous real linear functional f , take $E = \{x : f(x) \leq 1\}$.
- Find a closed set E and point x for which $d(x, E) = d$ but $d(x, y) > d$ for every $y \in E$: the distance need not be attained.
- Differentiation, with domain $\{f \in C^1([0, 1]) : f(0) = 0\}$ in $C([0, 1])$, is a closed operator into $C([0, 1])$. Its inverse is the continuous integration operator $g \mapsto \int_0^x g(t) dt$.

A.31.3. Problem Set 10, Part 2, Problem 3: Milman–Pettis.

Theorem. Every uniformly convex Banach space is reflexive.

Proof. Identify X with its canonical image in X^{**} . Let B, B^{**} be the respective closed unit balls, and fix x^{**} with $\|x^{**}\| = 1$. Since X is complete, B is norm closed in X^{**} . It suffices to approximate x^{**} in norm by points of B . For $\varepsilon > 0$, choose $\delta > 0$ by uniform convexity so that

$$\|x\|, \|y\| \leq 1, \quad \|x - y\| > \varepsilon \implies \left\| \frac{x + y}{2} \right\| < 1 - \delta.$$

Choose $f \in X^*$ of norm one with $\operatorname{Re} x^{**}(f) > 1 - \delta/2$, and let

$$V = \{y^{**} : |(y^{**} - x^{**})(f)| < \delta/2\}.$$

This is a weak-* neighborhood of x^{**} . By weak-* density of B in B^{**} (Problem Set 10, Part 1, Problem 3), choose $x \in B \cap V$. We claim $x^{**} \in x + \varepsilon B^{**}$. If not, the

set $W = X^{**} \setminus (x + \varepsilon B^{**})$ is a weak-* neighborhood of x^{**} , because B^{**} is weak-* closed. Density again gives $y \in B \cap V \cap W$. Thus $\|x - y\| > \varepsilon$, while

$$|f(x) - x^{**}(f)| < \delta/2, \quad |f(y) - x^{**}(f)| < \delta/2.$$

It follows that

$$2 - \delta < 2 \operatorname{Re} x^{**}(f) < \operatorname{Re} f(x + y) + \delta \leq \|x + y\| + \delta,$$

so $\|(x + y)/2\| > 1 - \delta$, contradicting uniform convexity. Consequently $\|x^{**} - x\| \leq \varepsilon$ for some $x \in B$ for every $\varepsilon > 0$. Closedness of B implies $x^{**} \in B$. Scaling gives surjectivity of the canonical embedding, hence reflexivity. $\square$

Editorial note. Real parts are included to cover complex spaces. They reduce to the ordered scalar pairings used in the handwritten real-space proof.

A.32. Supplementary solutions (Ex3).

A.32.1. *Problem Set 15, Part 1, Problem 2(1): forward implication.* Let $L_1, L_2 \geq 0$ commute and suppose $L_1 \geq L_2$. Set

$$A = \sqrt{L_1} - \sqrt{L_2}, \quad L_3 = \sqrt{\sqrt{L_1} + \sqrt{L_2}}.$$

Then L_3 is positive and self-adjoint, and

$$H = \overline{\operatorname{Im} L_3} \oplus (\operatorname{Im} L_3)^\perp = \overline{\operatorname{Im} L_3} \oplus \ker L_3.$$

Write $y = y_1 + y_2$ accordingly. Expanding gives

$$(Ay, y) = (Ay_1, y_1) + (Ay_1, y_2) + (Ay_2, y_1) + (Ay_2, y_2).$$

Since A commutes with L_3 , it preserves $\overline{\operatorname{Im} L_3}$; thus $(Ay_1, y_2) = 0$. Also $Ay_2 = 0$, since $y_2 \in \ker L_3$ implies $\sqrt{L_1}y_2 = \sqrt{L_2}y_2 = 0$. Therefore $(Ay, y) = (Ay_1, y_1)$. For every x ,

$$(AL_3x, L_3x) = (L_3^2Ax, x) = ((L_1 - L_2)x, x) \geq 0.$$

Thus $(Az, z) \geq 0$ on $\operatorname{Im} L_3$ and, by continuity, on its closure. It follows that $A \geq 0$.

Editorial note. The definition of L_3 has two layers of square roots, as required by the identity $L_3^2A = L_1 - L_2$. The source's range statement is interpreted with closure when passing to y_1 .

A.32.2. *Problem Set 15, Part 2, Problem 2.* Suppose L_0 is self-adjoint and commutes with each $L \in \Lambda$, where the only common closed invariant subspaces are $\{0\}$ and H . Commutation gives

$$L(cI - L_0) = (cI - L_0)L.$$

Hence both $\operatorname{Im}(cI - L_0)$ and $\ker(cI - L_0)$ are invariant; the latter is closed. If c is an eigenvalue of L_0 , the nonzero kernel must be all of H , so $L_0 = cI$. Otherwise $\sigma_p(L_0) = \emptyset$, and its spectrum is continuous. The spectral family E_λ then has no jumps:

$$\lim_{\varepsilon \downarrow 0} E_{\lambda - \varepsilon} = E_\lambda = \lim_{\varepsilon \downarrow 0} E_{\lambda + \varepsilon} \quad \text{strongly.}$$

If $mI \leq L_0 \leq MI$, then $E_\lambda = 0$ for $\lambda < m$ and $E_\lambda = I$ for $\lambda > M$. Every $L \in \Lambda$ commutes with the spectral projections, so $\operatorname{Im} E_\lambda$ is a closed invariant subspace. It must therefore be either $\{0\}$ or H , meaning $E_\lambda = 0$ or I for every λ . Such a family cannot pass from zero to I while having no jumps, a contradiction. Thus an eigenvalue exists, reducing to the first case.

A.32.3. *Problem Set 14, Part 1, Problem 1.* For every bounded operator,

$$\|L\| = \sup_{x, y \neq 0} \frac{|(Lx, y)|}{\|x\| \|y\|}.$$

One inequality is immediate from Cauchy–Schwarz, and the other follows by taking $y = Lx$ when $Lx \neq 0$. For self-adjoint L , let $K = \sup_{x \neq 0} |(Lx, x)| / \|x\|^2$. Clearly $K \leq \|L\|$. To prove the reverse inequality it suffices to show $|(Lx, y)| \leq K$ for unit x, y . Rotating y by a scalar of modulus one reduces this to a bound on the real part. By polarization and the parallelogram law,

$$\begin{aligned} \operatorname{Re}(Lx, y) &= \frac{1}{4} ((L(x+y), x+y) - (L(x-y), x-y)) \\ &\leq \frac{K}{4} (\|x+y\|^2 + \|x-y\|^2) = K. \end{aligned}$$

Thus $\|L\| = K$.

If an operator A has unit vectors x_n with $Ax_n \rightarrow 0$, then it cannot have a bounded inverse. Since (Lx, x) is real, choose unit x_n with $(Lx_n, x_n) \rightarrow \|L\|$ or $-\|L\|$. In the first case put $A = \|L\|I - L \geq 0$. Then $(Ax_n, x_n) \rightarrow 0$. Positivity gives

$$|(Ax, y)| \leq \sqrt{(Ax, x)} \sqrt{(Ay, y)}.$$

Consequently

$$\|Ax\| = \sup_{\|y\|=1} |(Ax, y)| \leq \sqrt{\|A\|} \sqrt{(Ax, x)}.$$

It follows that $Ax_n \rightarrow 0$, so $\|L\| \in \sigma(L)$. The other case gives $-\|L\| \in \sigma(L)$.

Editorial note. The last line of the source places $\|L\|$ in $\sigma(L_0)$; the operator here is L .

A.32.4. *Problem Set 14, Part 1, Problem 2.* Part (2) is immediate from the commutation conclusions of the cited theorem. For part (3), let L be normal and define U on its range by

$$U(Lx) = L^*x.$$

Since $\|L^*x\| = \|Lx\|$, this is well-defined and isometric from $\operatorname{Im} L$ onto $\operatorname{Im} L^*$. It extends to a unitary map $\overline{\operatorname{Im} L} \rightarrow \overline{\operatorname{Im} L^*}$. Also $\ker L = \ker L^*$, so

$$\overline{\operatorname{Im} L^*} = (\ker L)^\perp = (\ker L^*)^\perp = \overline{\operatorname{Im} L}.$$

Thus it is unitary on $\overline{\operatorname{Im} L}$. Direct sums of unitary maps are unitary. Using $H = \overline{\operatorname{Im} L} \oplus \ker L^*$, extend U by the identity on $\ker L^*$. The resulting unitary on all of H satisfies $L^* = UL$.

A.32.5. *Problem Set 14, Part 1, Problem 3: spectral radius.* Write $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. First,

$$r(A) \leq \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}.$$

Indeed, the root test shows that if $|\lambda|$ exceeds the right-hand side, then the series

$$\sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}$$

converges and equals $(\lambda I - A)^{-1}$. Thus λ is in the resolvent set.

For the opposite inequality, take $f \in \mathcal{L}(X)^*$ and set $u_f(\lambda) = f((\lambda I - A)^{-1})$. This is holomorphic for $|\lambda| > r(A)$, with Laurent expansion

$$u_f(\lambda) = \sum_{n=0}^{\infty} \frac{f(A^n)}{\lambda^{n+1}}.$$

For every $\varepsilon > 0$, this series converges at $\lambda = r(A) + \varepsilon$. Hence $f(A^n/(r(A) + \varepsilon)^{n+1})$ is bounded in n for each f . Banach–Steinhaus, applied to the canonical images in the bidual of $\mathcal{L}(X)$, gives

$$\frac{\|A^n\|}{(r(A) + \varepsilon)^{n+1}} \leq M_\varepsilon.$$

Therefore $\limsup_n \|A^n\|^{1/n} \leq r(A) + \varepsilon$, and letting $\varepsilon \downarrow 0$ proves the reverse inequality.

Finally, the limit exists. Set $s = \inf_{n \geq 1} \|A^n\|^{1/n}$, so $\liminf_n \|A^n\|^{1/n} \geq s$. Given $\varepsilon > 0$, choose m with $\|A^m\|^{1/m} < s + \varepsilon$. Write $n = pm + q$, $0 \leq q < m$. Then

$$\|A^n\|^{1/n} \leq \|A^m\|^{p/n} \|A^q\|^{1/n} \leq (s + \varepsilon)^{pm/n} \|A\|^{q/n}.$$

For $A \neq 0$, the right-hand side tends to $s + \varepsilon$. The case $A = 0$ is immediate. Thus $\limsup_n \|A^n\|^{1/n} \leq s$, proving existence and the spectral-radius formula.

Editorial note. Here $0 \leq q < m$ includes zero, correcting the source's strict lower bound on the remainder.

If L is self-adjoint, $\|L^2\| = \|L\|^2$, and induction gives $\|L^{2^k}\| = \|L\|^{2^k}$. Hence

$$r(L) = \lim_{k \rightarrow \infty} \|L^{2^k}\|^{1/2^k} = \|L\|.$$

For normal L , the operator LL^* is self-adjoint and

$$\|(LL^*)^{2^k}\| = \|LL^*\|^{2^k} = \|L\|^{2^{k+1}}.$$

Normality gives $(LL^*)^{2^k} = L^{2^k} (L^{2^k})^*$, whose norm is $\|L^{2^k}\|^2$. Thus again $\|L^{2^k}\| = \|L\|^{2^k}$ and $r(L) = \|L\|$.

A.32.6. *Problem Set 14, Part 1, Problem 4: the minimum kernel.* Compactness has already been proved. Symmetry and reality of the kernel give self-adjointness by Fubini:

$$\begin{aligned} (Lf, g) &= \int_0^1 \int_0^1 k(x, y) f(y) \overline{g(x)} \, dy \, dx \\ &= \int_0^1 f(y) \overline{\int_0^1 k(y, x) g(x) \, dx} \, dy = (f, Lg). \end{aligned}$$

Since L is compact, it remains to find its nonzero eigenvalues. The equation $Lf = \lambda f$ becomes

$$\int_0^x y f(y) \, dy + x \int_x^1 f(y) \, dy = \lambda f(x).$$

Differentiating gives $\lambda f'(x) = \int_x^1 f(y) \, dy$, hence

$$\lambda f'' + f = 0, \quad f(0) = 0, \quad f'(1) = 0.$$

The nonzero solutions have

$$f(x) = C \sin \frac{x}{\sqrt{\lambda}}, \quad \lambda = \frac{4}{(2k+1)^2\pi^2}, \quad k = 0, 1, 2, \dots$$

Therefore

$$\sigma(L) = \left\{ \frac{4}{(2k+1)^2\pi^2} : k \geq 0 \right\} \cup \{0\}, \quad \|L\| = r(L) = \frac{4}{\pi^2}.$$

A.33. Quiz 1: Questions and solutions. September 18, 2015.

1. (5 points). Your name: *Solution* (as written on the solution sheet).
2. (5 points). Let the following letters denote classes of spaces:

- A* Vector spaces
- B* Topological vector spaces
- C* Normable topological vector spaces
- D* Metrizable topological vector spaces
- E* Complete metric spaces
- F* Banach spaces
- G* Spaces of second Baire category
- H* Locally convex topological vector spaces
- I* Locally bounded topological vector spaces
- J* Topological vector spaces with a countable local base

Write down all possible inclusion and equality relations between them. For example, $A \supset B \supset C, D, H, I, J$.

Solution. The red annotations also introduce K = Fréchet spaces and L = Hilbert spaces. The displayed inclusion diagram gives

$$\begin{aligned} A &\supset B \supset D \supset C \supset F \supset L, \\ B &\supset H \supset K \supset F, \quad E \supset K, \quad E \supset F, \quad E \subset G, \\ D &= J \supset I \supset C. \end{aligned}$$

Editorial note. As in the diagram in Lecture 7, the categories are understood within the compatible vector-space setting: E denotes TVSs admitting a complete compatible translation-invariant metric. If E literally means all complete metric spaces, it is not a subclass of vector spaces. The implication to second category assumes a nonempty space. The source diagram appears also to place C inside K ; incomplete normed spaces show that this inclusion requires completeness, so the valid relation used above is $F \subset K$.

3. (1) (2 points). Describe all balanced neighborhoods of zero in $\mathbb{C}$ as a complex vector space.

Solution. The open disks $B(0, r)$, $r > 0$; the whole plane is also allowed.

3. (2) (3 points). Describe all balanced neighborhoods of zero in $\mathbb{C} = \mathbb{R}^2$ as a real vector space.

Solution. The open neighborhoods of zero that are symmetric and star-shaped about zero.

Editorial note. The word neighborhood uses the course's open-neighborhood convention. Without openness in that convention, closed disks and other balanced neighborhoods would also be allowed in (1).

A.34. Quiz 2: Questions and solutions. September 29, 2015.

1. (5 points). Your name: *Solution*.

2. (5 points). Let X, Y be normed vector spaces and

$$\mathcal{L}(X, Y) = \{L : X \rightarrow Y : L \text{ is continuous and linear}\}.$$

For $L \in \mathcal{L}(X, Y)$ define

$$\|L\| = \sup_{\|x\|_X=1} \|Lx\|_Y = \sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X}.$$

Prove that this is a norm on $\mathcal{L}(X, Y)$, by verifying:

- (1) $L \neq 0$ implies $\|L\| \neq 0$;
- (2) $\|\alpha L\| = |\alpha| \|L\|$;
- (3) $\|L_1 + L_2\| \leq \|L_1\| + \|L_2\|$.

Solution. If $L \neq 0$, some $x \neq 0$ has $Lx \neq 0$, so $\|L\| \geq \|Lx\|_Y / \|x\|_X > 0$. Also,

$$\|\alpha L\| = \sup_{\|x\|=1} \|\alpha Lx\| = |\alpha| \sup_{\|x\|=1} \|Lx\| = |\alpha| \|L\|.$$

Finally,

$$\begin{aligned} \|L_1 + L_2\| &= \sup_{\|x\|=1} \|L_1x + L_2x\| \\ &\leq \sup_{\|x\|=1} \|L_1x\| + \sup_{\|x\|=1} \|L_2x\| = \|L_1\| + \|L_2\|. \end{aligned}$$

□

3. (5 points). (1) What is a Banach space? (2) Prove that if X, Y are Banach spaces, so is $\mathcal{L}(X, Y)$ for the norm above. The sheet leaves the hint blank.

Solution to (1). A complete normed vector space.

Solution to (2). Let $\{L_n\}$ be Cauchy in the operator norm. For each $x \in X$,

$$\|(L_n - L_m)x\| \leq \|L_n - L_m\| \|x\| \rightarrow 0.$$

Since Y is Banach, define $Lx = \lim_n L_nx$. The pointwise-limit corollary discussed in class gives continuity of L ; linearity follows by taking limits. Given $\varepsilon > 0$, choose N with $\|L_n - L_m\| < \varepsilon$ for $n, m > N$. For fixed $n > N$ and every unit vector x , passing to $m \rightarrow \infty$ gives $\|(L_n - L)x\| \leq \varepsilon$. Taking the supremum gives $\|L_n - L\| \leq \varepsilon$, as required. □

Editorial note. The last step writes out the uniform Cauchy estimate rather than interchanging a supremum and a limit as in the handwritten formula. For a zero domain the operator norm is defined to be zero.

A.35. Quiz 3: Questions and solutions. October 23, 2015.

1. (5 points). Your name: *Solution*.

2. (5 points). Let X be a normed vector space.

- (1) If $x \neq y$ are in X , prove that some $\ell \in X^*$ has $\ell(x) \neq \ell(y)$.
- (2) If $Y \subset X$ is a subspace and $x_0 \notin \overline{Y}$, prove there is $L \in X^*$ with $L(x_0) = 1$ and $L|_Y = 0$.

Solution to (1). On $\mathbb{F}(x - y)$ define $\ell_0(\alpha(x - y)) = \alpha$. This is bounded. Extend it by Hahn–Banach to $\ell \in X^*$. Then $\ell(x - y) = 1 \neq 0$, so $\ell(x) \neq \ell(y)$. □

Solution to (2), method 1. Set $Z = Y + \mathbb{F}x_0$ and define $\ell_0(y + \alpha x_0) = \alpha$. This is well-defined and linear. Since $x_0 \notin \overline{Y}$, choose $\varepsilon > 0$ with $B(x_0, \varepsilon) \cap \overline{Y} = \emptyset$. For $\alpha \neq 0$,

$$\frac{|\ell_0(y + \alpha x_0)|}{\|y + \alpha x_0\|} = \frac{1}{\|x_0 + y/\alpha\|} \leq \frac{1}{\varepsilon}.$$

The case $\alpha = 0$ is immediate, so ℓ_0 is bounded. Extend it by Hahn–Banach. $\square$

Solution to (2), method 2. The quotient $X/\overline{Y}$ is normed and $[x_0] \neq [0]$. Apply (1) and rescale to find $\tilde{L} \in (X/\overline{Y})^*$ with $\tilde{L}([x_0]) = 1$ and $\tilde{L}([0]) = 0$. Then $L = \tilde{L} \circ \pi$, where π is the quotient map, works. $\square$

3. (5 points). Let X be a TVS and $A \subset X$ convex and absorbing. (1) Define its Minkowski functional μ_A . (2) If A is open, prove that $x \in A$ implies $\mu_A(x) < 1$.
Solution to (1). $\mu_A(x) = \inf\{t > 0 : x \in tA\}$.

Solution to (2), method 1. The map $m_x : \mathbb{F} \rightarrow X$, $\alpha \mapsto \alpha x$, is continuous and $m_x(1) = x \in A$. Thus $m_x^{-1}(A)$ contains an interval of real numbers around 1. Choose $\varepsilon > 0$ with $(1 + \varepsilon)x \in A$. Then $\mu_A(x) \leq (1 + \varepsilon)^{-1} < 1$. $\square$

Solution to (2), method 2. Choose a neighborhood U of zero with $x + U \subset A$. For sufficiently small real $\varepsilon > 0$, $\varepsilon x \in U$. Then $(1 + \varepsilon)x \in A$, so again $\mu_A(x) \leq (1 + \varepsilon)^{-1} < 1$. $\square$

A.36. Quiz 4: Questions and solutions. November 10, 2015.

- (5 points). Your name: *Solutions*.
- (5 points). Are the following spaces duals of Banach spaces? If yes, give a Banach space whose dual it is; otherwise write “nothing”.

Space	Answer	Source annotation
ℓ^2	ℓ^2	$\frac{1}{2} + \frac{1}{2} = 1$
$C([0, 1])$	nothing	Problem Set 9, Part 2
c_0	nothing	Lecture 16; Problem Set 9, Part 2
ℓ^1	c_0	Problem 3 below
ℓ^3	$\ell^{3/2}$	$\frac{1}{3} + \frac{1}{3/2} = 1$
ℓ^∞	ℓ^1	Problem Set 9, Part 1
$C(\mathbb{R})$	nothing	Fréchet, not Banach

Editorial note. The dual identifications and exclusions in the table refer to the stated natural norms and isometric dual-space realizations, as in the cited lecture arguments.

- (5 points). Recall

$$c_0 = \{(a_1, a_2, \dots) : a_n \rightarrow 0\} \subset \ell^\infty.$$

Prove that $(c_0)^* = \ell^1$. Hint: prove a one-to-one correspondence between $y \in \ell^1$ and $L_y \in (c_0)^*$, with $\|y\|_1 = \|L_y\|$.

Solution. For simplicity, take $\mathbb{F} = \mathbb{R}$, as in the source. For $y = (b_n) \in \ell^1$, define

$$L_y(x) = \sum_{n=1}^{\infty} a_n b_n, \quad x = (a_n) \in c_0.$$

This is linear and bounded, since

$$|L_y(x)| \leq \sum_n |b_n| \sup_n |a_n| = \|y\|_1 \|x\|_\infty.$$

Conversely, for $L \in (c_0)^*$ let $b_n = L(e_n)$, where e_n is the n th standard unit vector. Put

$$z_N = (\operatorname{sgn} b_1, \dots, \operatorname{sgn} b_N, 0, 0, \dots),$$

where the source takes $\operatorname{sgn} t = 1$ for $t \geq 0$ and -1 for $t < 0$. Then $z_N \in c_0$ and $\|z_N\|_\infty = 1$, so

$$\sum_{n=1}^N |b_n| = Lz_N \leq \|L\|.$$

Thus $y = (b_n) \in \ell^1$. For $x = (a_n) \in c_0$, its truncations $x_N = (a_1, \dots, a_N, 0, \dots)$ converge uniformly to x . Therefore

$$Lx = \lim_N Lx_N = \lim_N \sum_{n=1}^N a_n b_n = \sum_{n=1}^\infty a_n b_n = L_y x.$$

The first estimate gives $\|L_y\| \leq \|y\|_1$; the estimates with z_N give the opposite inequality after $N \rightarrow \infty$. The coefficients $b_n = L(e_n)$ are unique, proving the correspondence. $\square$

A.37. Quiz 5: Questions and solutions. December 1, 2015.

- (5 points). Your name: *Solution*.
- (5 points). (a) Give the definition of a Hilbert space. (b) List as many properties as possible studied this semester that hold for Hilbert spaces but not for general Banach spaces.

Solution to (a). A Hilbert space is a complete inner product space. Its inner product is linear in the first variable, conjugate symmetric, and positive definite:

$$(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z), \quad (x, y) = \overline{(y, x)}, \quad (x, x) > 0 \ (x \neq 0).$$

It is complete for $\|x\| = \sqrt{(x, x)}$.

Solution to (b). The source lists:

- (1) Cauchy–Schwarz: $|(x, y)| \leq \|x\| \|y\|$.
- (2) The parallelogram law: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.
- (3) Every nonempty closed convex $E \subset H$ has a unique point c with $\|c\| = \inf_{x \in E} \|x\|$.
- (4) For every closed subspace M , $H = M \oplus M^\perp$. Its orthogonal projection $P : H \rightarrow M$ has norm one when $M \neq \{0\}$.
- (5) Every $L \in H^*$ is uniquely of the form $L = (\cdot, y)$ for $y \in H$.
- (6) Every compact operator on H can be approximated in operator norm by a sequence of finite-rank operators.
- (7) $x_n \rightarrow x$ in norm if and only if $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$.

Editorial note. The projection's domain and target are written explicitly as $H \rightarrow M$; the handwritten arrow is reversed.

- (5 points). Let X be a complex Banach space and $L \in \mathcal{L}(X)$. (a) Define the resolvent set $\rho(L)$. (b) For $\lambda, \mu \in \rho(L)$, prove that the resolvents commute and

$$R_\lambda(L) - R_\mu(L) = (\mu - \lambda)R_\lambda(L)R_\mu(L).$$

Operators A, B commute when $AB = BA$.

Solution to (a).

$$\rho(L) = \{\lambda \in \mathbb{C} : \lambda I - L \text{ is invertible}\}, \quad R_\lambda(L) = (\lambda I - L)^{-1} \in \mathcal{L}(X).$$

Solution to (b). Insert $\mu I - L = (\mu - \lambda)I + (\lambda I - L)$:

$$\begin{aligned} R_\lambda &= R_\lambda(\mu I - L)R_\mu \\ &= (\mu - \lambda)R_\lambda R_\mu + R_\mu. \end{aligned}$$

This gives the resolvent identity. If $\lambda \neq \mu$, exchanging them gives

$$R_\lambda R_\mu = \frac{R_\lambda - R_\mu}{\mu - \lambda} = \frac{R_\mu - R_\lambda}{\lambda - \mu} = R_\mu R_\lambda.$$

For $\lambda = \mu$, commutation is immediate. □

A.38. **Quiz 6: True or false, with solutions.** December 29, 2015.

1. (5 points). Your name: *Solution*.

2. Which statements are true? Write T before the true ones and F before the false ones. All Hilbert spaces in this quiz are complex. The T/F answers and short annotations below are those on the solution sheet.

- (1) **T.** If $L \in \mathcal{L}(H)$ and $(Lx, x) = 0$ for all $x \in H$, then $L = 0$.
- (2) **T.** Every complete metric space is of second category.
- (3) **F.** If d_1, d_2 induce the same topology on X and d_1 is complete, then d_2 is complete.

Annotation: Need translation invariance; cf. Problem Set 2, Part 2.

- (4) **T.** Every vector space, complete or not, has a Hamel basis.
- (5) **T.** If X is an infinite-dimensional Banach space, the weak-star topology on X^* is not metrizable.
- (6) **F.** Every Fredholm operator on X can be written as $I + K$ with $K \in \mathcal{K}(X)$.

Annotation: “Invertible + compact.”

- (7) **F.** For every compact operator $K \in \mathcal{K}(X)$, $0 \in \sigma(K)$.

Annotation: Need infinite dimension.

- (8) **T.** If $L \geq 0$ and $L \leq 0$, then $L = 0$.
- (9) **T.** A projection $P \in \mathcal{L}(H)$ is an orthogonal projection if and only if $P \geq 0$.
- (10) **F.** $L \in \mathcal{L}(H)$ is unitary if and only if $(Lx, Ly) = (x, y)$ for all $x, y \in H$.

Annotation: Need surjectivity.

- (11) **F.** For every compact set E , its convex hull $\text{co}(E)$ is compact.

Annotation: “Fréchet.”

- (12) **F.** If X is reflexive and $E \subset X$ is nonempty, closed, and convex, there is a unique $x_0 \in E$ with $\|x_0\| = \inf_{x \in E} \|x\|$.

Annotation: Not unique.

Editorial note. In (2), nonemptiness is implicit. The note in (3) concerns compatible translation-invariant metrics on a TVS. In (6), the stronger assertion that every Fredholm operator is an invertible operator plus a compact operator also needs qualification: such a decomposition forces Fredholm index zero. The written annotation is retained rather than used as a general theorem. In (11), a standard valid statement uses the *closed* convex hull of a compact set in a Fréchet space; local convexity and completeness alone do not make the unclosed convex hull compact.