

LAPLACE EIGENVALUES AND PÓLYA'S CONJECTURE

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ABSTRACT. These notes were taken during Professor Zuoqin Wang's short course, *Laplace Eigenvalues and Pólya's Conjecture*. The course covers basic spectral theory and Weyl's law, followed by Pólya's conjecture for tiling domains, product domains, thin products, and domains with small holes. The digital version was prepared with the assistance of AI (ChatGPT-6.1 Sol). Please email me at yijiemiao0420@gmail.com or yijiemiao@seu.edu.cn if you spot any typos or errors. Thank you!

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1. BASIC SPECTRAL THEORY FOR THE LAPLACIAN

1.1. Laplacian eigenvalues. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Green's identity reads

$$\int_{\Omega} f_1 \Delta f_2 \, dx - \int_{\Omega} f_2 \Delta f_1 \, dx = \int_{\partial\Omega} \left(f_1 \frac{\partial f_2}{\partial \nu} - f_2 \frac{\partial f_1}{\partial \nu} \right) dS.$$

The boundary term need not vanish for general smooth functions. It vanishes when both functions satisfy the same one of the following boundary conditions:

- Dirichlet: $f = 0$ on $\partial\Omega$.
- Neumann: $\partial_{\nu} f = 0$ on $\partial\Omega$.
- Mixed (Zaremba): write $\partial\Omega = (\partial\Omega)_1 \cup (\partial\Omega)_2$ and impose $f = 0$ on $(\partial\Omega)_1$ and $\partial_{\nu} f = 0$ on $(\partial\Omega)_2$.
- Robin: $\alpha f + \partial_{\nu} f = 0$ on $\partial\Omega$, with the same real coefficient α .

For the last case, use

$$f_1 \partial_{\nu} f_2 - f_2 \partial_{\nu} f_1 = f_1 (\alpha f_2 + \partial_{\nu} f_2) - f_2 (\alpha f_1 + \partial_{\nu} f_1).$$

From the PDE point of view, the eigenvalue problems are

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \partial_{\nu} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Their eigenvalues are called Dirichlet and Neumann eigenvalues, respectively.

Physical background. Laplacian eigenvalues occur in wave-like problems. For example, a vibrating drumhead satisfies

$$\begin{cases} f_{tt} = \Delta f & \text{on } [0, \infty) \times \Omega, \\ f = 0 & \text{on } [0, \infty) \times \partial\Omega. \end{cases}$$

Separation of variables $f(t, x) = T(t)X(x)$ gives the Dirichlet eigenvalue problem and $T'' + \lambda T = 0$. One solution is $T(t) = \cos(\sqrt{\lambda}t)$, so λ is the square of the angular frequency.

1.2. Basic examples.

Example 1.1 (Rectangles and boxes). Let $\Omega = (0, a_1) \times \cdots \times (0, a_n)$. Separation of variables gives

$$\lambda = \pi^2 \sum_{i=1}^n \frac{m_i^2}{a_i^2}, \quad u(x) = \prod_{i=1}^n \sin \frac{m_i \pi x_i}{a_i}, \quad m_i \in \mathbb{N}_{>0}.$$

For Neumann boundary conditions,

$$\mu = \pi^2 \sum_{i=1}^n \frac{m_i^2}{a_i^2}, \quad u(x) = \prod_{i=1}^n \cos \frac{m_i \pi x_i}{a_i}, \quad m_i \in \mathbb{N}_{\geq 0}.$$

When an eigenvalue has multiplicity greater than one, its eigenspace consists of linear combinations of the corresponding product functions.

Example 1.2 (The unit disk). Let $\Omega = \{(x, y) : x^2 + y^2 < 1\}$. Separation of variables in polar coordinates gives Dirichlet eigenvalues $j_{m,k}^2$, where $j_{m,k}$ is the k th positive zero of the Bessel function J_m . For integer m ,

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x \sin t - mt)} dt.$$

The radial factor recorded in the notes is $J_m(j_{m,k}r)$.

Editor's note. The angular factors, not written in the source, are $\cos(m\theta)$ and $\sin(m\theta)$, with only the constant factor for $m = 0$. The radial factor alone is not a full eigenfunction when $m > 0$.

In particular, $\lambda_1 = j_{0,1}^2 \approx (2.40)^2$ and $\lambda_2 = \lambda_3 \approx (3.83)^2$. The nonzero Neumann eigenvalues are $(j'_{m,k})^2$, where $j'_{m,k}$ denotes a positive zero of J'_m ; additionally $\mu_0 = 0$. Thus $\mu_1 = \mu_2 \approx (1.84)^2$.

Scaling. For $a > 0$, let $a\Omega = \{ax : x \in \Omega\}$. Then

$$\lambda_k(a\Omega) = a^{-2}\lambda_k(\Omega), \quad \mu_k(a\Omega) = a^{-2}\mu_k(\Omega).$$

Disjoint unions. If $\Omega = \bigsqcup_{j=1}^m \Omega_j$, its spectrum is the multiset union of the spectra of its components:

$$\begin{aligned} \{\lambda_k(\Omega) : k \geq 1\} &= \bigsqcup_{j=1}^m \{\lambda_\ell(\Omega_j) : \ell \geq 1\}, \\ \{\mu_k(\Omega) : k \geq 0\} &= \bigsqcup_{j=1}^m \{\mu_\ell(\Omega_j) : \ell \geq 0\}. \end{aligned}$$

Products. For $\Omega = \Omega_1 \times \Omega_2$,

$$\{\lambda_k(\Omega) : k \geq 1\} = \{\lambda_\ell(\Omega_1) + \lambda_j(\Omega_2) : \ell, j \geq 1\},$$

with multiplicities; the analogous formula holds for Neumann eigenvalues with $\ell, j \geq 0$.

1.3. Weak formulation. In the form-theoretic (Friedrichs) construction recorded in the notes, the two Laplacians are associated with densely defined, nonnegative closed quadratic forms

$$q(u, v) = \int_{\Omega} \nabla u \cdot \overline{\nabla v} \, dx.$$

For Dirichlet boundary conditions,

$$\text{Dom}(q_D) = H_0^1(\Omega), \quad \text{Dom}(-\Delta^D) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\},$$

and $\langle -\Delta^D u, v \rangle = q_D(u, v)$.

Editor's note. It also equates this domain with $H_0^1(\Omega) \cap H^2(\Omega)$. That equality requires additional regularity, for example $C^{1,1}$ boundary; it does not hold for general Lipschitz domains.

For Neumann boundary conditions,

$$\text{Dom}(q_N) = H^1(\Omega),$$

$$\text{Dom}(-\Delta^N) = \{u \in H^1(\Omega) : \exists f \in L^2(\Omega) \text{ such that } q_N(u, v) = \langle f, v \rangle \text{ for every } v \in H^1(\Omega)\}.$$

Recall the weak Green identity

$$\int_{\Omega} (-\Delta u) \bar{v} \, dx = q_N(u, v) - \langle \partial_\nu u, v \rangle_{\partial\Omega}.$$

The operator-domain condition implies $\partial_\nu u = 0$ in $H^{-1/2}(\partial\Omega)$.

Theorem 1.1. *On a bounded Lipschitz domain, $-\Delta^D$ and $-\Delta^N$ are self-adjoint and have compact resolvent. Their eigenvalues, counted with multiplicity, satisfy*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \longrightarrow \infty, \quad 0 = \mu_0 \leq \mu_1 \leq \cdots \longrightarrow \infty.$$

The eigenfunctions can be chosen to form orthonormal bases of $L^2(\Omega)$.

The Dirichlet compactness follows for every bounded open set; the Neumann argument uses the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

2. EIGENVALUE COMPARISON

2.1. The min-max principle. The Rayleigh quotient is

$$\mathcal{R}[u] = \frac{q(u, u)}{\|u\|_{L^2}^2} = \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} |u|^2 dx}, \quad u \neq 0.$$

For an orthonormal Dirichlet eigenbasis $\{u_i\}$, write $u = \sum_i c_i u_i$ with $c_i = \langle u, u_i \rangle$. Then

$$\mathcal{R}[u] = \frac{\sum_i |c_i|^2 \lambda_i}{\sum_i |c_i|^2}.$$

In particular, $\mathcal{R}[u_k] = \lambda_k$. Set $V_k = \text{span}\{u_1, \dots, u_k\}$. Then

$$u \in V_k \setminus \{0\} \implies \mathcal{R}[u] \leq \lambda_k, \quad u \in V_{k-1}^\perp \setminus \{0\} \implies \mathcal{R}[u] \geq \lambda_k.$$

Equality holds precisely for vectors in the λ_k -eigenspace within the indicated subspace.

Theorem 2.1 (Min-max). *For $k \geq 1$,*

$$\begin{aligned} \lambda_k &= \min_{\substack{V \subset H_0^1(\Omega) \\ \dim V = k}} \max_{0 \neq u \in V} \mathcal{R}[u] \\ &= \max_{\substack{V \subset H_0^1(\Omega) \\ \dim V = k-1}} \min_{\substack{0 \neq u \in H_0^1(\Omega) \\ u \perp V}} \mathcal{R}[u]. \end{aligned}$$

Editor's note. The Neumann max-min expression below expands the ellipsis on source page. It uses the same variational argument.

For $k \geq 0$,

$$\mu_k = \min_{\substack{V \subset H^1(\Omega) \\ \dim V = k+1}} \max_{0 \neq u \in V} \mathcal{R}[u] = \max_{\substack{V \subset H^1(\Omega) \\ \dim V = k}} \inf_{\substack{0 \neq u \in H^1(\Omega) \\ u \perp V}} \mathcal{R}[u].$$

Proof. If $\dim V = k$, then $V \cap V_{k-1}^\perp$ contains a nonzero vector, so $\max_{u \in V \setminus \{0\}} \mathcal{R}[u] \geq \lambda_k$. Taking $V = V_k$ gives the reverse inequality. The other formulas follow similarly. $\square$

Remark 2.1.

$$\lambda_1 = \min_{0 \neq u \in H_0^1(\Omega)} \mathcal{R}[u] = \inf_{0 \neq u \in C_c^\infty(\Omega)} \mathcal{R}[u].$$

Editor's note. For the following formula the connectedness of Ω is assumed, so that the zero eigenspace consists of constants. This hypothesis was implicit in source page 4.

For connected Ω ,

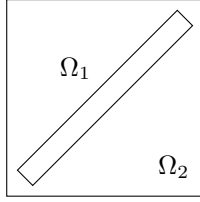
$$\mu_1 = \min_{\substack{0 \neq u \in H^1(\Omega) \\ \int_{\Omega} u dx = 0}} \mathcal{R}[u].$$

2.2. Dirichlet–Neumann comparison.

Domain monotonicity. If $\Omega_1 \subset \Omega_2$, then $\lambda_k(\Omega_1) \geq \lambda_k(\Omega_2)$. Indeed, zero extension embeds $H_0^1(\Omega_1)$ into $H_0^1(\Omega_2)$, so min–max applies. If $\Omega_2 \setminus \Omega_1$ contains an open set, the notes state that unique continuation gives strict inequality.

Editor's note. This strict domain-monotonicity statement is used with Ω_2 connected. The source does not explicitly record that assumption.

Domain monotonicity fails for Neumann eigenvalues. For instance, a long, thin diagonal rectangle inside the unit square has first positive Neumann eigenvalue approximately $\pi^2/2$, whereas the square has $\mu_1 = \pi^2$.



Theorem 2.2 (Dirichlet–Neumann bracketing). *Suppose disjoint Lipschitz pieces Ω_i partition Ω up to interfaces of measure zero. Let $0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$ be all the Dirichlet eigenvalues of the pieces, rearranged with multiplicity, and let $0 \leq \tilde{\mu}_0 \leq \tilde{\mu}_1 \leq \dots$ be the corresponding Neumann list. Then*

$$\tilde{\mu}_{k-1} \leq \mu_{k-1}(\Omega) \leq \lambda_k(\Omega) \leq \tilde{\lambda}_k.$$

Proof. We prove the three inequalities in the order used in the handwritten notes.

- (1) Let $\tilde{\Omega} = \bigsqcup_{i=1}^m \Omega_i \subset \Omega$. By domain monotonicity and the spectrum of a disjoint union,

$$\lambda_k(\Omega) \leq \lambda_k(\tilde{\Omega}) = \tilde{\lambda}_k.$$

- (2) Since $H_0^1(\Omega) \subset H^1(\Omega)$, the min–max principle gives

$$\mu_{k-1}(\Omega) \leq \lambda_k(\Omega).$$

- (3) We show $\tilde{\mu}_k \leq \mu_k(\Omega)$ for $k \geq 0$. Let $\tilde{u}_i$ be an L^2 -normalized Neumann eigenfunction associated with $\tilde{\mu}_i$ on a piece $\Omega_{c(i)}$, regarded as a function on the disjoint union by zero extension to the other pieces. Choose these functions to form an orthonormal eigenbasis of the disjoint union. Let u_j be orthonormal Neumann eigenfunctions on Ω associated with $\mu_j(\Omega)$.

There exist $\alpha_0, \dots, \alpha_k$, not all zero, such that

$$\sum_{j=0}^k \alpha_j \int_{\Omega_{c(i)}} u_j \tilde{u}_i dx = 0, \quad 0 \leq i \leq k-1.$$

Indeed, these are k homogeneous linear equations in $k+1$ unknowns. Set

$$u = \sum_{j=0}^k \alpha_j u_j \neq 0.$$

Thus u is orthogonal to $\tilde{u}_0, \dots, \tilde{u}_{k-1}$ on the disjoint union. The Rayleigh quotient characterization on that union gives

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx &= \sum_{i=1}^m \int_{\Omega_i} |\nabla u|^2 dx \\ &\geq \tilde{\mu}_k \sum_{i=1}^m \int_{\Omega_i} |u|^2 dx = \tilde{\mu}_k \int_{\Omega} |u|^2 dx. \end{aligned}$$

On the other hand, $u \in \text{span}\{u_0, \dots, u_k\}$, so

$$\int_{\Omega} |\nabla u|^2 dx = \sum_{j=0}^k |\alpha_j|^2 \mu_j(\Omega) \leq \mu_k(\Omega) \sum_{j=0}^k |\alpha_j|^2 = \mu_k(\Omega) \int_{\Omega} |u|^2 dx.$$

Consequently $\tilde{\mu}_k \leq \mu_k(\Omega)$; shifting the index gives the first inequality in the theorem. $\square$

2.3. Pólya's conjecture. The Friedlander–Filonov inequality strengthens the elementary comparison to $\mu_k \leq \lambda_k$. Put

$$\omega_n = |B_1(0)|, \quad C_n = \frac{\omega_n}{(2\pi)^n}.$$

Weyl's law says

$$\lambda_k, \mu_k \sim \frac{4\pi^2}{(\omega_n |\Omega|)^{2/n}} k^{2/n} = \left(\frac{k}{C_n |\Omega|} \right)^{2/n}.$$

The two-term Weyl conjecture takes the form (in dimensions $n \geq 2$)

Editor's note. Source page mixes ω_n and C_n in its second term. The following normalization uses $C_n = \omega_n/(2\pi)^n$ consistently and agrees with the counting-function formula later on that page. These are conjectural two-term formulas in the general setting, not a consequence of the leading Weyl law alone.

$$\begin{aligned} \lambda_k &= \left(\frac{k}{C_n |\Omega|} \right)^{2/n} + \frac{C_{n-1} |\partial\Omega|}{2nC_n |\Omega|} \left(\frac{k}{C_n |\Omega|} \right)^{1/n} + o(k^{1/n}), \\ \mu_k &= \left(\frac{k}{C_n |\Omega|} \right)^{2/n} - \frac{C_{n-1} |\partial\Omega|}{2nC_n |\Omega|} \left(\frac{k}{C_n |\Omega|} \right)^{1/n} + o(k^{1/n}). \end{aligned}$$

Pólya's conjecture asserts, for every $k \geq 1$,

$$\boxed{\mu_k(\Omega) \leq \left(\frac{k}{C_n |\Omega|} \right)^{2/n} \leq \lambda_k(\Omega).}$$

3. WEYL'S LAW

3.1. Counting functions. Define, using strict inequalities,

$$N_{\Omega}^D(E) = \#\{k \geq 1 : \lambda_k(\Omega) < E\}, \quad N_{\Omega}^N(E) = \#\{k \geq 0 : \mu_k(\Omega) < E\}.$$

Weyl's law is equivalent to

$$N_{\Omega}^D(E) \sim C_n |\Omega| E^{n/2}, \quad N_{\Omega}^N(E) \sim C_n |\Omega| E^{n/2}.$$

The leading term is the phase-space volume

$$\frac{1}{(2\pi)^n} \text{Vol}\{(x, \xi) \in T^*\Omega : |\xi|^2 < E\}.$$

This relates the quantum counting function to classical mechanics.

Lemma 3.1 (Asymptotic inversion). *Let $p > q > 0$, $a > 0$, $c \in \mathbb{R}$, and $0 < \theta = p - q < 1$. Then*

$$N(E) = aE^p + cE^q + o(E^q)$$

corresponds to

$$\lambda_k = \left(\frac{k}{a}\right)^{1/p} - \frac{c}{ap} \left(\frac{k}{a}\right)^{(1-\theta)/p} + o(k^{(1-\theta)/p}).$$

Thus the two-term Weyl conjecture can be written as

$$\begin{aligned} N_\Omega^D(E) &= C_n |\Omega| E^{n/2} - \frac{1}{4} C_{n-1} |\partial\Omega| E^{(n-1)/2} + o(E^{(n-1)/2}), \\ N_\Omega^N(E) &= C_n |\Omega| E^{n/2} + \frac{1}{4} C_{n-1} |\partial\Omega| E^{(n-1)/2} + o(E^{(n-1)/2}). \end{aligned}$$

Pólya's conjecture is equivalent to

$$N_\Omega^D(E) \leq C_n |\Omega| E^{n/2}, \quad N_\Omega^N(E) \geq C_n |\Omega| E^{n/2} \quad (E > 0).$$

3.2. Boxes and lattice points. For $\Omega = \prod_{i=1}^n (0, a_i)$,

$$N_\Omega^D(E) = \# \left\{ m \in \mathbb{N}_{>0}^n : \sum_i \frac{m_i^2}{a_i^2} < \frac{E}{\pi^2} \right\}.$$

This equals 2^{-n} times the number of lattice points in the full ellipsoid after removing points on coordinate hyperplanes. The lattice-point estimate recorded in the notes is

$$\#(\mathbb{Z}^n \cap aD) = a^n |D| + O(a^{n-1-1/3}), \quad a \rightarrow \infty.$$

Remark 3.1. The handwritten statement says “strictly convex D ”. The displayed improved remainder requires additional smoothness and curvature assumptions; it is used here for ellipsoids. The one-dimensional case is treated directly.

The coordinate-hyperplane terms give, for $n \geq 2$,

$$\begin{aligned} N_\Omega^D(E) &= \frac{\omega_n}{2^n} \left(\frac{\sqrt{E}}{\pi} \right)^n \prod_i a_i \\ &\quad - \frac{\omega_{n-1}}{2^n} \left(\frac{\sqrt{E}}{\pi} \right)^{n-1} \sum_i \prod_{j \neq i} a_j + o(E^{(n-1)/2}) \\ &= C_n |\Omega| E^{n/2} - \frac{1}{4} C_{n-1} |\partial\Omega| E^{(n-1)/2} + o(E^{(n-1)/2}). \end{aligned}$$

3.3. Proof of the Dirichlet Weyl law. Choose grid cubes Ω_i contained in Ω ($1 \leq i \leq N$), and add the cubes meeting the boundary ($N+1 \leq i \leq M$). Let U be the interior union and V the interior of the union of all closed cubes. Arrange that

$$|\Omega \setminus U| < \varepsilon, \quad |V \setminus \Omega| < \varepsilon.$$

Domain monotonicity and bracketing give

$$\sum_{i=1}^N N_{\Omega_i}^D(E) \leq N_\Omega^D(E) \leq N_V^D(E) \leq N_V^N(E) \leq \sum_{i=1}^M N_{\Omega_i}^N(E).$$

Editor's note. The limiting step indicated by arrows on source pages 7–8 is written out here.

Divide by $E^{n/2}$ and send $E \rightarrow \infty$. The box Weyl law yields

$$C_n(|\Omega| - \varepsilon) \leq \liminf \frac{N_\Omega^D(E)}{E^{n/2}} \leq \limsup \frac{N_\Omega^D(E)}{E^{n/2}} \leq C_n(|\Omega| + \varepsilon).$$

Finally let $\varepsilon \downarrow 0$.

3.4. Proof of the Neumann Weyl law. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We prove

$$\lim_{E \rightarrow \infty} \frac{N_\Omega^N(E)}{E^{n/2}} = C_n |\Omega|.$$

The lower bound follows immediately from the Dirichlet Weyl law and $N_\Omega^N(E) \geq N_\Omega^D(E)$:

$$\liminf_{E \rightarrow \infty} \frac{N_\Omega^N(E)}{E^{n/2}} \geq C_n |\Omega|.$$

For the upper bound, Neumann eigenvalues have no domain monotonicity. We instead use the two lemmas recorded in the handwritten notes and apply Neumann bracketing to a decomposition of Ω .

Lemma 3.2 (Boundary decomposition). *There exist constants $h_0, C, L > 0$, depending on Ω , such that for every $0 < h < h_0$, the domain admits, up to a null set, a disjoint decomposition*

$$\Omega = \left(\bigsqcup_{j=1}^{n_h} Q_{h,j} \right) \sqcup \left(\bigsqcup_{\ell=1}^{m_h} D_{h,\ell} \right)$$

with the following properties:

- (1) Each $Q_{h,j}$ is a cube of side length h and $\overline{Q_{h,j}} \subset \Omega$.
- (2) The number of boundary cells satisfies $m_h \leq Ch^{1-n}$.
- (3) Each $D_{h,\ell}$ is the image of a box

$$P_{h,\ell} = (0, h)^{n-1} \times (0, \alpha_{h,\ell} h), \quad L^{-1} \leq \alpha_{h,\ell} \leq L,$$

under an L -bi-Lipschitz map, with L independent of h and ℓ .

- (4) The boundary cells lie in a collar of $\partial\Omega$ of width $O(h)$.

In particular,

$$0 \leq |\Omega| - n_h h^n = \sum_{\ell=1}^{m_h} |D_{h,\ell}| \leq C_\Omega h.$$

This geometric decomposition is used as an input, as in the handwritten notes; its construction is not proved there.

Lemma 3.3 (Uniform estimate for boundary cells). *There is a constant $C_1 = C_1(n, L)$ such that, for every boundary cell and every $E > 0$,*

$$N_{D_{h,\ell}}^N(E) \leq C_1 (1 + h^n E^{n/2}).$$

Supplementary proof of the estimate stated in the notes. Write $D = F(P)$, where $P = P_{h,\ell}$ and F is L -bi-Lipschitz. For $u \in H^1(D)$, set $v = u \circ F$. The change-of-variables formula and the Sobolev chain rule give

$$\int_D |u|^2 dx \leq L^n \int_P |v|^2 dx, \quad \int_D |\nabla u|^2 dx \geq L^{-n-2} \int_P |\nabla v|^2 dx.$$

Thus $\mathcal{R}_D[u] \geq L^{-2n-2}\mathcal{R}_P[v]$. Since pullback is a linear isomorphism between the two H^1 spaces, min-max yields

$$\mu_k(D) \geq L^{-2n-2}\mu_k(P), \quad N_D^N(E) \leq N_P^N(L^{2n+2}E).$$

The Neumann eigenvalues of P are

$$\frac{\pi^2}{h^2} \left(m_1^2 + \cdots + m_{n-1}^2 + \frac{m_n^2}{\alpha_{h,\ell}^2} \right), \quad m \in \mathbb{N}_{\geq 0}^n.$$

Counting the possible values of each m_i separately, and using $\alpha_{h,\ell} \leq L$, gives

$$N_P^N(L^{2n+2}E) \leq (1 + C(n, L)h\sqrt{E})^n \leq C_1(1 + h^n E^{n/2}),$$

as required. The constant is independent of h , ℓ , and E . $\square$

Proof of the upper bound. Fix $0 < h < \min\{h_0, 1\}$. Neumann bracketing gives

$$\begin{aligned} N_\Omega^N(E) &\leq \sum_{j=1}^{n_h} N_{Q_{h,j}}^N(E) + \sum_{\ell=1}^{m_h} N_{D_{h,\ell}}^N(E) \\ &\leq \sum_{j=1}^{n_h} N_{Q_{h,j}}^N(E) + C_1 m_h (1 + h^n E^{n/2}). \end{aligned}$$

After division by $E^{n/2}$,

$$\frac{N_\Omega^N(E)}{E^{n/2}} \leq \sum_{j=1}^{n_h} \frac{N_{Q_{h,j}}^N(E)}{E^{n/2}} + C_1 m_h E^{-n/2} + C_1 m_h h^n.$$

For this fixed h , there are finitely many interior cubes, and their explicit spectra give

$$\lim_{E \rightarrow \infty} \sum_{j=1}^{n_h} \frac{N_{Q_{h,j}}^N(E)}{E^{n/2}} = C_n n_h h^n.$$

Also $m_h E^{-n/2} \rightarrow 0$, while $m_h h^n \leq Ch$. Hence

$$\begin{aligned} \limsup_{E \rightarrow \infty} \frac{N_\Omega^N(E)}{E^{n/2}} &\leq C_n n_h h^n + C_1 m_h h^n \\ &\leq C_n |\Omega| + C_2 h, \end{aligned}$$

where C_2 does not depend on h . Letting $h \downarrow 0$ proves the upper bound. Together with the lower bound, this establishes the Neumann Weyl law. $\square$

Editor's note. The argument follows source page 8. The supplementary proof of the boundary-cell estimate and the explicit lim inf/lim sup calculations expand steps that were only stated or indicated by arrows in the handwritten notes. The order of limits is essential: first $E \rightarrow \infty$ with h fixed, then $h \downarrow 0$.

4. PÓLYA'S CONJECTURE FOR TILING DOMAINS

Scope and provenance. Chapters 4–6 follow the handwritten notes, with missing arguments expanded and hypotheses checked against the references. Proofs added during this revision are identified where appropriate. Unless a disjoint union is explicitly mentioned, a domain means a connected open set. Throughout these chapters the domains are bounded; whenever Neumann eigenvalues are used we assume compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$, in particular for Lipschitz domains. The eigenvalues are counted with multiplicity, $\lambda_1, \lambda_2, \dots$ and $\mu_0, \mu_1, \dots$, and

$$N_\Omega^D(E) = \#\{j : \lambda_j < E\}, \quad N_\Omega^N(E) = \#\{j : \mu_j < E\}, \quad E > 0.$$

We set both counting functions equal to zero at nonpositive arguments. We also use $N_{\Omega, \leq}^D(E) = \#\{j : \lambda_j \leq E\}$ when an endpoint must be included.

A Euclidean tiling by Ω is a locally finite family of congruent copies with disjoint interiors covering $\mathbb{R}^n$ up to their boundaries. We assume the tile boundaries have measure zero and that the variational restrictions and zero extensions used below are legitimate; bounded Lipschitz tiles suffice.

Theorem 4.1 (Pólya–Kellner tiling argument). *For such a bounded tiling domain,*

$$\mu_k(\Omega) \leq \left(\frac{k}{C_n |\Omega|} \right)^{2/n} \leq \lambda_k(\Omega), \quad k \geq 1.$$

The classical planar results are due to Pólya and Kellner [1, 2]. The arguments below are dimension-independent.

4.1. Pólya's Dirichlet argument. Let $Q = (0, 1)^n$, and scale a tiling by the factor $r > 0$. Let Q_r be the disjoint union of all scaled tiles contained in Q , and let n_r be their number. Every tile has diameter $r \operatorname{diam} \Omega$. The part of Q not covered by these interior tiles lies in a collar of ∂Q of width $O(r)$, so

$$n_r r^n |\Omega| = 1 + O(r), \quad n_r = \frac{1 + O(r)}{r^n |\Omega|}.$$

Fix $k \geq 1$. Since the spectrum of Q_r repeats each eigenvalue of $r\Omega$ exactly n_r times,

$$\begin{aligned} \lambda_k(\Omega) &= r^2 \lambda_{kn_r}(Q_r) \geq r^2 \lambda_{kn_r}(Q) \\ &= r^2 \left[\left(\frac{kn_r}{C_n} \right)^{2/n} + o(n_r^{2/n}) \right] = \left(\frac{k}{C_n |\Omega|} \right)^{2/n} + o(1). \end{aligned}$$

Here domain monotonicity gives the inequality and the cube Weyl law gives the asymptotic. Let $r \downarrow 0$.

4.2. Kellner's Neumann argument. Let $\tilde{Q}_r$ be the disjoint union of all scaled tiles whose closures meet $\bar{Q}$, with number M_r . The union covers Q up to a null set and lies in an $O(r)$ neighborhood of $\bar{Q}$. Consequently

$$M_r r^n |\Omega| = 1 + O(r).$$

For $u \in H_0^1(Q)$, extend u by zero to $\mathbb{R}^n$ and restrict it to the tiles in $\tilde{Q}_r$. This defines an injective linear map into their direct sum of H^1 spaces, preserving both the L^2 norm and the Dirichlet energy. Min–max therefore gives

$$\mu_j(\tilde{Q}_r) \leq \lambda_{j+1}(Q), \quad j \geq 0.$$

Taking $j = kM_r$ and using the repeated Neumann spectrum,

$$\begin{aligned}\mu_k(\Omega) &= r^2 \mu_{kM_r}(\tilde{Q}_r) \leq r^2 \lambda_{kM_r+1}(Q) \\ &= \left(\frac{k}{C_n |\Omega|} \right)^{2/n} + o(1).\end{aligned}$$

Again let $r \downarrow 0$.

Editor's note. This writes out the variational comparison underlying the arrows on source page 9. The +1 comes from the different starting indices of the Neumann and Dirichlet lists. It avoids needing any boundary regularity of the assembled union of tiles. No Neumann domain monotonicity is used.

Proposition 4.2 (Inheritance under finite tiling). *If Ω_1 tiles Ω by ℓ congruent copies, then*

$$\ell N_{\Omega_1}^D(E) \leq N_{\Omega}^D(E), \quad N_{\Omega}^N(E) \leq \ell N_{\Omega_1}^N(E), \quad |\Omega| = \ell |\Omega_1|.$$

Thus either Pólya inequality for Ω passes to Ω_1 .

Proof. Impose additional Dirichlet, respectively Neumann, conditions on the interfaces and apply bracketing. Divide by ℓ . $\square$

4.3. Regular tiling domains and strict inequalities. A tiling is *regular* or *periodic* if it is invariant under a full-rank lattice Γ and has finitely many tile orbits. Select one tile from each orbit, and let C be the interior of the union of their closures. This supertile is a fundamental domain for Γ up to null sets, and

$$|C| = \text{covol}(\Gamma) = \ell |\Omega|.$$

The dual lattice is

$$\Gamma^* = \{\xi \in \mathbb{R}^n : \xi \cdot \gamma \in 2\pi\mathbb{Z} \text{ for every } \gamma \in \Gamma\}, \quad \text{covol}(\Gamma^*) = \frac{(2\pi)^n}{|C|}.$$

Lemma 4.1 (Strict Blichfeldt inequality for balls). *For a full-rank lattice $\Lambda \subset \mathbb{R}^n$, $n \geq 2$, and $r > 0$, there exist shifts θ_+, θ_- such that*

$$\begin{aligned}\#((\Lambda + \theta_+) \cap B_r) &\geq \left\lfloor \frac{|B_r|}{\text{covol}(\Lambda)} \right\rfloor + 1, \\ \#((\Lambda + \theta_-) \cap \overline{B}_r) &\leq \left\lceil \frac{|B_r|}{\text{covol}(\Lambda)} \right\rceil - 1.\end{aligned}$$

Expanded proof of the lemma in the notes. For a fundamental cell F , define

$$f(\theta) = \sum_{\xi \in \Lambda} \mathbf{1}_{B_r}(\xi + \theta).$$

Unfolding the integral gives

$$\frac{1}{|F|} \int_F f(\theta) d\theta = \frac{|B_r|}{|F|} =: a.$$

We show that f is not almost everywhere constant. On the sphere ∂B_r , only finitely many other spheres $\partial B_r - \xi$, $\xi \neq 0$, can intersect it. Each intersection has empty relative interior in ∂B_r when $n \geq 2$. Choose a point of ∂B_r on none of those spheres. In a small neighborhood of that point, crossing ∂B_r changes exactly one summand of f . Thus f takes different integer values on two open sets.

An integer-valued, nonconstant function of mean a takes a value at least $\lfloor a \rfloor + 1$ and a value at most $\lceil a \rceil - 1$ on sets of positive measure. Choose the shifts off all translated sphere boundaries, which form a null set. At these shifts the open-ball and closed-ball counts coincide, proving both assertions. $\square$

Theorem 4.3 (Strict Pólya inequalities for regular tilings). *If $n \geq 2$ and Ω is a bounded regular tiling domain as above, then, for every $E > 0$,*

$$N_{\Omega, \leq}^D(E) < C_n |\Omega| E^{n/2} < N_{\Omega}^N(E).$$

Equivalently, for every $k \geq 1$,

$$\mu_k(\Omega) < \left(\frac{k}{C_n |\Omega|} \right)^{2/n} < \lambda_k(\Omega).$$

Proof. For every shift θ , the functions

$$e_{\xi, \theta}(x) = |C|^{-1/2} e^{i(\xi + \theta) \cdot x}, \quad \xi \in \Gamma^*,$$

form an orthonormal basis of $L^2(C)$, since multiplication by $e^{i\theta \cdot x}$ is unitary and the characters of the torus form a basis on any fundamental domain.

Dirichlet case. Let V_E be the span of the Dirichlet eigenfunctions of C with eigenvalues at most E . For $u \in H_0^1(C)$, integration by parts against each plane wave (using zero trace), followed by Parseval for each derivative, gives

$$\int_C |\nabla u|^2 dx = \sum_{\xi \in \Gamma^*} |\xi + \theta|^2 |\langle u, e_{\xi, \theta} \rangle|^2.$$

Hence the coefficient map from V_E to the coordinates with $|\xi + \theta|^2 \leq E$ is injective: if all those coordinates vanish, then

$$0 \geq \int_C |\nabla u|^2 - E \int_C |u|^2 = \sum_{|\xi + \theta|^2 > E} (|\xi + \theta|^2 - E) |\langle u, e_{\xi, \theta} \rangle|^2 \geq 0,$$

and every remaining coordinate vanishes too. Bracketing and Lemma 4.1, applied to Γ^* , imply

$$\begin{aligned} \ell N_{\Omega, \leq}^D(E) &\leq N_{C, \leq}^D(E) \leq \#\{\xi \in \Gamma^* : |\xi + \theta_-|^2 \leq E\} \\ &\leq \lceil C_n |C| E^{n/2} \rceil - 1 < C_n |C| E^{n/2}. \end{aligned}$$

Neumann case. The plane waves with $|\xi + \theta|^2 < E$ span a trial space on C . Both their L^2 inner products and their energy inner products are diagonal, with Rayleigh values $|\xi + \theta|^2$. Thus min-max and bracketing give

$$\#\{\xi \in \Gamma^* : |\xi + \theta|^2 < E\} \leq N_C^N(E) \leq \ell N_{\Omega}^N(E).$$

Choose $\theta = \theta_+$ in Lemma 4.1 to obtain

$$\ell N_{\Omega}^N(E) \geq \lfloor C_n |C| E^{n/2} \rfloor + 1 > C_n |C| E^{n/2}.$$

Divide by ℓ . For the eigenvalue formulation, evaluate the two counting inequalities at $E = (k/(C_n |\Omega|))^{2/n}$. $\square$

Editor's note. The Dirichlet coefficient argument is from source pages 10–11. The nonconstancy argument and the Neumann trial-space proof fill gaps left in the notes; the Neumann comparison also agrees with [8, Section 5.1].

Corollary 4.1. *Writing $A = C_n|\Omega|E^{n/2}$, integrality gives*

$$A - N_{\Omega, \leq}^D(E) \geq A + 1 - \lceil A \rceil, \quad N_{\Omega}^N(E) - A \geq \lfloor A \rfloor + 1 - A.$$

These pointwise gaps alone do not supply a uniform correction of order $E^{(n-1)/2}$; that stronger question is addressed in Chapter 6.

5. PÓLYA'S CONJECTURE FOR PRODUCTS

5.1. Riesz means and semiclassical inequalities. For $\gamma > 0$, define

$$R_{\gamma, \Omega}^D(E) = \sum_{j \geq 1} (E - \lambda_j)_+^\gamma, \quad R_{\gamma, \Omega}^N(E) = \sum_{j \geq 0} (E - \mu_j)_+^\gamma.$$

At $\gamma = 0$ we use $x_+^0 = \mathbf{1}_{\{x > 0\}}$, so the Riesz means agree with the strict counting functions. The constants are

$$L_{\gamma, n} = \frac{\Gamma(\gamma + 1)}{(4\pi)^{n/2} \Gamma(\gamma + 1 + n/2)}.$$

In particular,

$$L_{0, n} = C_n, \quad L_{1, n} = \frac{2}{n+2} C_n, \quad C_p L_{p/2, q} = C_{p+q}.$$

Lemma 5.1 (Aizenman–Lieb identity). *For $\gamma > \sigma \geq 0$,*

$$x_+^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\sigma + 1) \Gamma(\gamma - \sigma)} \int_0^\infty t^{\gamma - \sigma - 1} (x - t)_+^\sigma dt.$$

Proof. For $x > 0$, substitute $t = xs$ and evaluate the beta integral. For $x \leq 0$, both sides vanish. Summing this nonnegative identity permits the same passage from one Riesz order to another. $\square$

Theorem 5.1 (Berezin–Li–Yau and Kröger). *For every $E > 0$ and $\gamma \geq 1$,*

$$R_{\gamma, \Omega}^D(E) \leq L_{\gamma, n} |\Omega| E^{\gamma + n/2} \leq R_{\gamma, \Omega}^N(E).$$

For bounded domains in the setting above both inequalities are strict.

Proof. It suffices to prove the result at order one and apply Aizenman–Lieb. Let u_j be normalized Dirichlet eigenfunctions extended by zero, and use the unitary Fourier transform. For $\lambda_j < E$,

$$E - \lambda_j = \int_{\mathbb{R}^n} (E - |\xi|^2) |\widehat{u}_j(\xi)|^2 d\xi \leq \int_{|\xi|^2 < E} (E - |\xi|^2) |\widehat{u}_j(\xi)|^2 d\xi.$$

Bessel's inequality for the plane wave restricted to Ω gives $\sum_j |\widehat{u}_j(\xi)|^2 \leq (2\pi)^{-n} |\Omega|$. Summing and integrating yields the Dirichlet bound. It is strict if $E \leq \lambda_1$, because the left side is zero. If $E > \lambda_1$, discarding the integral over $|\xi|^2 > E$ is already strict for u_1 : the Fourier transform of a nonzero compactly supported function cannot vanish on that open set.

For the Neumann case let v_j be an orthonormal eigenbasis and put $e_\xi(x) = (2\pi)^{-n/2} e^{i\xi \cdot x}$ on Ω and $S = \|e_\xi\|_2^2 = (2\pi)^{-n} |\Omega|$. The form-domain spectral expansion gives

$$\sum_j |\langle e_\xi, v_j \rangle|^2 = S, \quad \sum_j \mu_j |\langle e_\xi, v_j \rangle|^2 = S |\xi|^2.$$

Jensen's inequality for the convex function $t \mapsto (E - t)_+$ gives

$$\sum_j (E - \mu_j)_+ |\langle e_\xi, v_j \rangle|^2 \geq S(E - |\xi|^2)_+.$$

Integrate in ξ and use Plancherel, $\int |\langle e_\xi, v_j \rangle|^2 d\xi = 1$. This proves the Neumann bound.

For strictness, near $\xi = 0$ the coefficient of the constant zero mode is nonzero. The waves in any open set of frequencies cannot all belong to the finite-dimensional span of the modes with $\mu_j \leq E$: arbitrarily many distinct exponential functions are linearly independent on an open domain. Thus, on a nonempty open set near zero, weights occur both below and above E , making Jensen strict. Aizenman–Lieb then preserves strictness for $\gamma > 1$. $\square$

Editor's note. The Dirichlet calculation is in source page 12. The Neumann proof and the strictness details have been supplied to justify the later product arguments; compare [3, 4]. The range $\gamma \geq 1$ is essential.

Theorem 5.2 (Li–Yau and Kröger sum bounds). *For $k \geq 1$,*

$$\begin{aligned} \sum_{j=1}^k \lambda_j &\geq \frac{n}{n+2} \left(\frac{k}{C_n |\Omega|} \right)^{2/n} k, \\ \sum_{j=0}^{k-1} \mu_j &\leq \frac{n}{n+2} \left(\frac{k}{C_n |\Omega|} \right)^{2/n} k. \end{aligned}$$

Consequently,

$$\lambda_k \geq \frac{n}{n+2} \left(\frac{k}{C_n |\Omega|} \right)^{2/n}, \quad \mu_k \leq \left(\frac{n+2}{2} \right)^{2/n} \left(\frac{k}{C_n |\Omega|} \right)^{2/n}.$$

Proof. For any increasing eigenvalue list ν_j , its first k terms have sum

$$S_k = \sup_{E \geq 0} \left(kE - \sum_j (E - \nu_j)_+ \right).$$

Apply the order-one bounds in Theorem 5.1 and optimize at $E = (k/(C_n |\Omega|))^{2/n}$. This proves both sum inequalities. Monotonicity of λ_j gives the individual Dirichlet bound. For the Neumann bound, evaluate the Riesz inequality at $E = \mu_k > 0$:

$$L_{1,n} |\Omega| \mu_k^{1+n/2} \leq R_{1,\Omega}^N(\mu_k) \leq k \mu_k.$$

If $\mu_k = 0$, the conclusion is immediate. $\square$

5.2. Laptev's product theorem.

Theorem 5.3 (Laptev, 1997; [3]). *Let $\Omega_1 \subset \mathbb{R}^{d_1}$, $d_1 \geq 2$, and $\Omega_2 \subset \mathbb{R}^{d_2}$ be bounded domains. If Ω_1 satisfies the Dirichlet part of Pólya's conjecture, so does $\Omega_1 \times \Omega_2$. The corresponding implication holds for the Neumann part when the Neumann spectra are discrete.*

Proof. The product spectrum and the counting inequality for Ω_1 give

$$\begin{aligned} N_{\Omega_1 \times \Omega_2}^D(E) &= \sum_{\lambda_\ell(\Omega_2) < E} N_{\Omega_1}^D(E - \lambda_\ell(\Omega_2)) \\ &\leq C_{d_1} |\Omega_1| R_{d_1/2, \Omega_2}^D(E) \\ &\leq C_{d_1} |\Omega_1| L_{d_1/2, d_2} |\Omega_2| E^{(d_1+d_2)/2} \\ &= C_{d_1+d_2} |\Omega_1 \times \Omega_2| E^{(d_1+d_2)/2}. \end{aligned}$$

For the Neumann case, sum over $\mu_\ell(\Omega_2) < E$ instead; both inequalities reverse. The restriction $d_1 \geq 2$ ensures that $d_1/2 \geq 1$, precisely the range available in Theorem 5.1. $\square$

Because the Riesz bounds are strict, the same proof gives strict counting inequalities for the product for all $E > 0$. The Dirichlet counting bound also holds with the endpoint included, by summing over $\lambda_\ell \leq E$ and taking the right-hand limit in the counting estimate for Ω_1 ; the order $d_1/2 > 0$ makes the zero terms harmless.

Proposition 5.4 (The interval factor and order 1/2). *For $\Omega \subset \mathbb{R}^n$, the following are equivalent:*

- (1) $(0, 1) \times a\Omega$ satisfies Dirichlet Pólya for every $a > 0$;
- (2) $R_{1/2, \Omega}^D(E) \leq L_{1/2, n} |\Omega| E^{(n+1)/2}$ for every $E > 0$.

Proof. Scaling turns (1) into the same assertion for $(0, L) \times \Omega$ for every $L > 0$. Write

$$N_{(0, L) \times \Omega}^D(E) = \sum_{\lambda_j < E} \# \left\{ m \geq 1 : m < \frac{L}{\pi} \sqrt{E - \lambda_j} \right\}.$$

The cardinality on the right is at most $L\sqrt{E - \lambda_j}/\pi$, proving (2) $\Rightarrow$ (1) since $L_{1/2, n} = \pi C_{n+1}$. For fixed E there are only finitely many summands, and each cardinality differs from $L\sqrt{E - \lambda_j}/\pi$ by at most one. Thus

$$\lim_{L \rightarrow \infty} \frac{1}{L} N_{(0, L) \times \Omega}^D(E) = \frac{1}{\pi} R_{1/2, \Omega}^D(E).$$

Dividing the conjectured product bound by L and taking this limit proves the converse. $\square$

5.3. Thin products with an interval factor.

Theorem 5.5 (He–Wang; [4, Theorem 1.2]). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with piecewise smooth boundary. There exists $a_0 > 0$ such that $(0, a) \times \Omega$ satisfies both Pólya inequalities whenever $0 < a < a_0$.*

We use the Seeley remainder estimate recorded in the notes,

$$N_{\Omega}^{D/N}(E) = C_d |\Omega| E^{d/2} + O_{\Omega}(E^{(d-1)/2}), \quad E \rightarrow \infty.$$

Its all-energy consequences, after increasing $A > 0$, are

$$(5.1) \quad N_{\Omega}^D(E) \leq C_d |\Omega| E^{d/2} + A E^{(d-1)/2},$$

$$(5.2) \quad N_{\Omega}^N(E) \geq C_d |\Omega| E^{d/2} - A E^{(d-1)/2},$$

for every $E > 0$. The Dirichlet first eigenvalue and the Neumann zero mode justify extending these one-sided inequalities down to small E .

Lemma 5.2 (Uniform gaps in the one-dimensional sums). *Fix $p > 0$ and define*

$$g_p(t) = (1 - t^2)_+^p, \quad D_p(M) = M \int_0^1 g_p(t) dt - \sum_{\ell \geq 1} g_p(\ell/M).$$

For all $M \geq 1$, there are constants $\delta_p^D, \delta_p^N > 0$, independent of M , such that

$$\delta_p^D \leq D_p(M) \leq 1 - \delta_p^N.$$

Consequently, for $M = a\sqrt{E}/\pi \geq 1$ and $f_s(x) = (E - \pi^2 x^2/a^2)_+^{s/2}$,

$$\begin{aligned} \sum_{\ell \geq 1} f_s(\ell) &\leq aL_{s/2,1} E^{(s+1)/2} - \delta_{s/2}^D E^{s/2}, \\ \sum_{\ell \geq 0} f_s(\ell) &\geq aL_{s/2,1} E^{(s+1)/2} + \delta_{s/2}^N E^{s/2}. \end{aligned}$$

Supplementary completion of the sum estimates. Since g_p is strictly decreasing on $(0, 1)$ and vanishes at one, comparison with the left and right Riemann sums gives $0 < D_p(M) < 1$. These inequalities are strict for every finite $M > 0$. For $p > 0$, g_p is absolutely continuous on $[0, 1]$, with $g'_p \in L^1(0, 1)$. Integration against the step function $\lfloor Mt \rfloor$ gives

$$D_p(M) = \int_0^1 \{Mt\} [-g'_p(t)] dt,$$

where $\{x\} = x - \lfloor x \rfloor$. As $M \rightarrow \infty$, the fractional-part functions average to $1/2$ against any $L^1(0, 1)$ function. This follows first for step functions by integrating over periods, then for L^1 functions by approximation and $0 \leq \{Mt\} < 1$. Hence

$$D_p(M) \rightarrow \frac{1}{2} \int_0^1 [-g'_p(t)] dt = \frac{1}{2}.$$

The function D_p is continuous for $M > 0$: only finitely many summands occur locally, and a newly appearing summand is zero at its first endpoint. On a compact interval $[1, M_0]$ the strict inequalities therefore have positive uniform gaps; the limit $1/2$ handles $M \geq M_0$. Rescaling by $E^{s/2}$ proves the conclusion, using

$$\int_0^M f_s(x) dx = aL_{s/2,1} E^{(s+1)/2}.$$

□

Editor's note. Source pages 14–15 obtain the same type of favorable term by a concave/convex split, but leave an error term blank. Lemma 5.2 supplies a complete alternative for that missing step. It gives positive dimension-dependent constants rather than the explicit constants pursued in [4].

Proof of Theorem 5.5. Write $V = |\Omega|$, $p = d/2$, $q = (d-1)/2$, and $M = a\sqrt{E}/\pi$. First suppose $M \geq 1$.

Dirichlet case. The product formula and (5.1) give

$$N_{(0,a) \times \Omega}^D(E) \leq C_d V \sum_{\ell \geq 1} f_d(\ell) + A \sum_{\ell \geq 1} f_{d-1}(\ell).$$

The second sum is at most its integral. By Lemma 5.2 and $C_d L_{d/2,1} = C_{d+1}$,

$$N_{(0,a) \times \Omega}^D(E) \leq C_{d+1} a V E^{(d+1)/2} - (C_d V \delta_p^D - A a L_{q,1}) E^{d/2}.$$

Choose a small enough that the coefficient in parentheses is positive. For $M < 1$ the Dirichlet counting function is zero, completing this case.

Neumann case. The index $\ell = 0$ must be included. Monotonicity gives

$$\sum_{\ell \geq 0} f_{d-1}(\ell) \leq E^q + \int_0^M f_{d-1}(x) dx = E^q + aL_{q,1}E^{d/2}.$$

Thus (5.2) and Lemma 5.2 yield

$$N_{(0,a) \times \Omega}^N(E) \geq C_{d+1}aVE^{(d+1)/2} + (C_dV\delta_p^N - AaL_{q,1})E^{d/2} - AE^q.$$

Since $M \geq 1$ implies $E^{-1/2} \leq a/\pi$, it follows that

$$N_{(0,a) \times \Omega}^N(E) \geq C_{d+1}aVE^{(d+1)/2} + \left[C_dV\delta_p^N - Aa \left(L_{q,1} + \frac{1}{\pi} \right) \right] E^{d/2}.$$

This proves the required inequality for $M \geq 1$ after decreasing a .

It remains to consider $0 < E < \pi^2/a^2$. Only the interval zero mode contributes, so the counting function is $N_{\Omega}^N(E)$. Since $C_d > \pi C_{d+1}$ (equivalently $|B_1^d| > |B_1^{d+1}|/2$), Weyl's law gives E_0 such that

$$N_{\Omega}^N(E) \geq \pi C_{d+1}VE^{d/2} > C_{d+1}aVE^{(d+1)/2} \quad (E_0 < E < \pi^2/a^2).$$

For $0 < E \leq E_0$, the zero mode gives $N_{\Omega}^N(E) \geq 1$; impose also $C_{d+1}aVE_0^{(d+1)/2} < 1$. All conditions on a involve fixed positive constants, so a single $a_0 > 0$ works in every energy range. $\square$

For comparison with source page 14, when $d = 2$ the elementary concavity argument gives the more explicit Dirichlet gap $E/8$. The proof above retains the handwritten method's essential distinction between a favorable leading-sum gap and the Seeley error, and treats all $d \geq 2$ at once.

5.4. General thin products.

Theorem 5.6 (He–Wang; [4, Theorem 1.1, v3]). *Let $\Omega_i \subset \mathbb{R}^{d_i}$ be bounded domains, $d_1, d_2 \geq 2$. Assume that Ω_1 is Lipschitz and that Ω_2 has piecewise smooth boundary. Then $a\Omega_1 \times \Omega_2$ satisfies both Pólya inequalities for all sufficiently small $a > 0$.*

Editor's note. The v3 version of He–Wang uses the 2025 Riesz asymptotics and permits Lipschitz Ω_1 in both boundary conditions. The earlier transcription used v2 and imposed the stronger sufficient C^1 assumption for Neumann. The dimension restrictions are still necessary for this stated argument.

Theorem 5.7 (Frank–Larson, 2025; [5]). *For a bounded Lipschitz $U \subset \mathbb{R}^d$, $d \geq 2$, and $\gamma > 0$,*

$$\begin{aligned} R_{\gamma,U}^D(t) &= L_{\gamma,d}|U|t^{\gamma+d/2} - \frac{1}{4}L_{\gamma,d-1}|\partial U|t^{\gamma+(d-1)/2} + o(t^{\gamma+(d-1)/2}), \\ R_{\gamma,U}^N(t) &= L_{\gamma,d}|U|t^{\gamma+d/2} + \frac{1}{4}L_{\gamma,d-1}|\partial U|t^{\gamma+(d-1)/2} + o(t^{\gamma+(d-1)/2}). \end{aligned}$$

In particular, for each fixed order, the corresponding one-sided estimates hold with $1/5$ in place of $1/4$ for all sufficiently large t .

Lemma 5.3 (Uniform Riesz gaps and remainder bounds). *For $U \subset \mathbb{R}^d$ bounded Lipschitz, $p \geq 1$, $q = p - 1/2 > 0$, and $s = p + (d - 1)/2 = q + d/2$, define*

$$G_D(t) = L_{p,d}|U|t^{p+d/2} - R_{p,U}^D(t), \quad G_N(t) = R_{p,U}^N(t) - L_{p,d}|U|t^{p+d/2}.$$

For each $t_* > 0$, there are $\delta, K > 0$ such that, for $t \geq t_*$,

$$G_D(t) \geq \delta t^s, \quad G_N(t) \geq \delta t^s, \quad R_{q,U}^{D/N}(t) \leq K t^s.$$

Proof. The strict bounds in Theorem 5.1 make both gaps positive for $t > 0$. The two-term asymptotics give a positive lower bound for $G_{D/N}(t)/t^s$ at large t ; continuity gives one on each remaining compact interval. For the upper bounds, the leading Riesz asymptotics give $R_q(t) = O(t^{q+d/2})$ at infinity, while continuity handles $t \geq t_*$. For Dirichlet one can instead use the all-energy bound from Li–Yau,

$$R_{q,U}^D(t) \leq \left(\frac{d+2}{d} \right)^{d/2} L_{q,d} |U| t^{q+d/2},$$

obtained by summing the individual lower eigenvalue bound and comparing the resulting decreasing sum with its integral. This argument is valid also when $q < 1$. $\square$

Proof of Theorem 5.6. Put $p = d_2/2$, $q = (d_2 - 1)/2$, $s = (d_1 + d_2 - 1)/2$, $A = C_{d_2} |\Omega_2|$, and $t = a^2 E$. With B a Seeley constant for Ω_2 , the product counting formula gives

$$\begin{aligned} N_{a\Omega_1 \times \Omega_2}^D(E) &\leq a^{-d_2} [AR_{p,\Omega_1}^D(t) + BaR_{q,\Omega_1}^D(t)], \\ N_{a\Omega_1 \times \Omega_2}^N(E) &\geq a^{-d_2} [AR_{p,\Omega_1}^N(t) - BaR_{q,\Omega_1}^N(t)]. \end{aligned}$$

The common semiclassical target is

$$a^{-d_2} AL_{p,d_1} |\Omega_1| t^{p+d_1/2} = C_{d_1+d_2} |a\Omega_1 \times \Omega_2| E^{(d_1+d_2)/2}.$$

For Dirichlet, when $t \leq \lambda_1(\Omega_1)$ the product count is zero. For $t \geq \lambda_1(\Omega_1) > 0$, Lemma 5.3 makes the difference from the target at most $a^{-d_2}(-A\delta + BaK)t^s$, which is negative for small a .

For Neumann, the same lemma controls $t \geq t_*$, for any fixed $t_* > 0$. Choose t_* so small that

$$t_* < \mu_1(\Omega_1), \quad C_{d_1+d_2} |\Omega_1| t_*^{d_1/2} < \frac{1}{2} C_{d_2}.$$

Here domains are connected, so $\mu_1(\Omega_1) > 0$. For $0 < t < t_*$ only its constant mode contributes, and the product count equals $N_{\Omega_2}^N(E)$. Weyl's law gives $N_{\Omega_2}^N(E) \geq \frac{1}{2} C_{d_2} |\Omega_2| E^{d_2/2}$ for $E \geq E_0$, which dominates the target by the choice of t_* . For $0 < E \leq E_0$, the product has a zero mode and its count is at least one; decrease a so that the target at E_0 is less than one. This covers the remaining energies and proves the theorem. $\square$

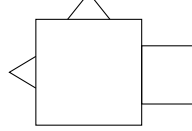
Editor's note. This proof expands the large-, middle-, and small-energy strategy on source pages 15–16. The compact-energy gaps and the error estimates are explicit enough to justify choosing a single a_0 ; it is not an explicit numerical threshold.

The exponential improvement missing from the notes. The schematic expression $1 - e^{\dots}$ on source page 16 can be replaced by the following cited result, with its authorship corrected. Frank–Larson–Pfeiffer [6, Theorem 1] prove that for $n \geq 1$ there are $c_n, c'_n > 0$ such that, for $\gamma \geq 1$ and $E > 0$,

$$\begin{aligned} R_{\gamma,\Omega}^D(E) &\leq L_{\gamma,n} |\Omega| E^{\gamma+n/2} (1 - c_n e^{-c'_n \sqrt{E} |\Omega|^{1/n}}), \\ R_{\gamma,\Omega}^N(E) &\geq L_{\gamma,n} |\Omega| E^{\gamma+n/2} (1 + c_n e^{-c'_n \sqrt{E} |\Omega|^{1/n}}). \end{aligned}$$

We use the bounded Lipschitz setting of these notes; the cited result allows more general finite-measure open sets. This supplies a quantitative compact-energy gap, but is not needed in the preceding proof.

On constants. As noted on source page 16, bracketing for a domain assembled from elementary pieces can sometimes provide an explicit Seeley constant. The thresholds here also involve compact-energy Riesz gaps, so an existence proof for a_0 should not be mistaken for a numerical formula without further estimates.



Abstract sequence viewpoint. The source suggests replacing product spectra by a rearranged multiset $\{b_i + \alpha^2 c_j\}$. The counting identity is then

$$\#\{(i, j) : b_i + \alpha^2 c_j < E\} = \sum_{\alpha^2 c_j < E} N_b(E - \alpha^2 c_j).$$

Arguments above apply if the two sequences have the corresponding counting estimates, strict Riesz gaps, and remainder bounds. These assumptions must be stated; a general sequence need not satisfy them. A precise abstract formulation is given in [4, Theorem 5.1].

6. SMALL HOLES AND ENHANCED PÓLYA INEQUALITIES

6.1. Removing a hole from one factor. For a Lipschitz subdomain K with $\overline{K} \subset \Omega$, Dirichlet–Neumann bracketing gives

$$(6.1) \quad N_{\Omega \setminus \overline{K}}^D(E) \leq N_{\Omega}^D(E) - N_K^D(E),$$

$$(6.2) \quad N_{\Omega \setminus \overline{K}}^N(E) \geq N_{\Omega}^N(E) - N_K^N(E).$$

The Neumann inequality is a bracketing statement for a decomposition, not domain monotonicity. The closed hole is removed so that the remaining set is open.

Theorem 6.1 (He–Wang, a sufficient form of [4, Theorems 6.1–6.2, v3]). *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be bounded Lipschitz domains, $d_1 \geq 1$, $d_2 \geq 2$. Let $U \subset \mathbb{R}^{d_2}$ be a bounded domain with smooth boundary, with $0 \in U$ and $\overline{U} \subset \Omega_2$. For small $a > 0$, put*

$$V_a = \Omega_2 \setminus \overline{aU}, \quad X_a = \Omega_1 \times V_a.$$

If Ω_2 satisfies Dirichlet Pólya, then X_a satisfies Dirichlet Pólya for every sufficiently small a . The analogous implication holds for Neumann Pólya.

The assumptions above are deliberately sufficient: they make the hole interior and all variational spaces and low-eigenvalue comparisons unambiguous. The cited paper allows piecewise smooth holes and gives its precise hypotheses separately for the two boundary conditions.

Expanded product-gap argument. Write $m = d_1$, $d = d_2$, $p = d/2$, $q = (d-1)/2$, $s = (m+d-1)/2$, and $W_a = |\Omega_2| - a^d|U|$. Take a small enough that $\overline{aU} \subset \Omega_2$ and $W_a \geq |\Omega_2|/2$. The lower Dirichlet Seeley estimate for U and (6.1) imply

$$N_{V_a}^D(E) \leq C_d W_a E^{d/2} + A a^{d-1} E^{(d-1)/2}.$$

Summing over the first factor gives

$$N_{X_a}^D(E) \leq C_d W_a R_{p,\Omega_1}^D(E) + A a^{d-1} R_{q,\Omega_1}^D(E).$$

For $E \geq \lambda_1(\Omega_1)$, the gap and remainder estimates of Lemma 5.3 make the right side at most

$$C_{m+d} |\Omega_1| W_a E^{(m+d)/2} - (C_d W_a \delta - A a^{d-1} K) E^s.$$

For small a the coefficient is positive; below $\lambda_1(\Omega_1)$ the product count vanishes. If $m = 1$, Ω_1 is an interval and Lemma 5.2, together with continuity on compact energies, supplies the same Riesz gap of order E^p .

For Neumann, the all-energy *upper* bound for the connected hole is

$$N_U^N(t) \leq C_d |U| t^{d/2} + A t^{(d-1)/2} + 1.$$

The additional one cannot be omitted: $N_U^N(t) \geq 1$ as $t \downarrow 0$. Thus (6.2) yields

$$N_{V_a}^N(E) \geq C_d W_a E^{d/2} - A a^{d-1} E^{(d-1)/2} - 1,$$

and consequently

$$N_{X_a}^N(E) \geq C_d W_a R_{p,\Omega_1}^N(E) - A a^{d-1} R_{q,\Omega_1}^N(E) - N_{\Omega_1}^N(E).$$

At large E , the first Riesz mean has a positive gap δE^s and the second is $O(E^s)$, whereas $N_{\Omega_1}^N(E) = O(E^{m/2}) = o(E^s)$, because $d \geq 2$. Choose a to absorb the second term, then choose a fixed Λ to absorb the last term. This proves the desired bound for $E \geq \Lambda$, uniformly for all smaller a . The interval case uses Lemma 5.2 instead of the m -dimensional two-term Riesz asymptotics.

For bounded energies, put $X = \Omega_1 \times \Omega_2$. Laptev's strict product argument gives $\mu_k(X) < (k/(C_{m+d}|X|))^{2/(m+d)}$ for every $k \geq 1$, since the Pólya factor has dimension $d \geq 2$. Fix a finite number of its eigenfunctions $\varphi_0, \dots, \varphi_K$. They are bounded, and for their span

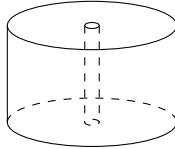
$$\int_{X \setminus X_a} |f|^2 \leq \eta_a \|f\|_{L^2(X)}^2, \quad \eta_a \rightarrow 0.$$

Restriction to X_a is injective for $\eta_a < 1$, and removes nonnegative energy, so min-max gives

$$\mu_k(X_a) \leq \frac{\mu_k(X)}{1 - \eta_a}, \quad 0 \leq k \leq K.$$

Choose K with $(k/(C_{m+d}|X|))^{2/(m+d)} > \Lambda$ for $k > K$. The finitely many strict inequalities for $k \leq K$ persist for small a , while the Weyl thresholds increase when volume is removed. All remaining eigenvalues are controlled by the high-energy bound. This completes the Neumann proof. $\square$

Editor's note. The zero-mode subtraction and the finite-dimensional restriction argument supply the low-energy part omitted in the handwritten sketch. The latter is a one-sided estimate; it does not assume Neumann domain monotonicity or require an unproved spectral-convergence assertion.



6.2. Enhanced inequalities and their stability. Let $n \geq 2$, $p = n/2$, and $q = (n-1)/2$. We say that Ω satisfies the *enhanced Dirichlet Pólya inequality* (EDP) if some $b > 0$ satisfies

$$N_{\Omega}^D(E) \leq (C_n |\Omega| E^p - b E^q)_+, \quad E > 0.$$

It satisfies the *enhanced Neumann Pólya inequality* (ENP) if there are $b, c > 0$ such that

$$N_{\Omega}^N(E) \geq C_n |\Omega| E^p + \max\{b E^q, c\}, \quad E > 0.$$

The normalizations agree with [8] after rescaling the constants. They distinguish a high-energy surface-order correction from the low-energy positive gap.

Lemma 6.1 (From strictness and a large-energy correction to enhancement). *Suppose, for all $E > 0$, $N_{\Omega, \leq}^D(E) < C_n |\Omega| E^p$, and for large E there is an upper bound $N_{\Omega}^D(E) \leq C_n |\Omega| E^p - \beta E^q$ with $\beta > 0$. Then Ω satisfies EDP. If $N_{\Omega}^N(E) > C_n |\Omega| E^p$ for all $E > 0$ and a lower bound with $+\beta E^q$ holds at large E , then Ω satisfies ENP.*

Proof. On a compact positive-energy interval, each counting function has finitely many jumps. In the Dirichlet case the endpoint-included strict inequality gives a positive lower bound for the deficit, including its right limits. Choose b small enough to dominate $b E^q$ there and ensure the positive-part cutoff is below λ_1 . At large energies use $b \leq \beta$; below λ_1 the count is zero. For Neumann, the strict inequality at each jump gives a positive minimum gap on compact intervals. Near zero the constant mode gives $N_{\Omega}^N(E) \geq 1$. Combine these with the large-energy correction, and decrease b, c to obtain the maximum of the two terms. $\square$

Theorem 6.2 (Small interior holes; a Lipschitz sufficient formulation). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain satisfying EDP, respectively ENP, and let U be a fixed bounded smooth domain. There is $a_0 > 0$ such that, whenever $0 < a < a_0$ and $x + a\bar{U} \subset \Omega$, the punctured domain*

$$\Omega_a = \Omega \setminus \overline{x + a\bar{U}}$$

satisfies the same enhanced inequality, with possibly different positive constants.

This is the stability principle recorded on source page 17; the ENP result and its Dirichlet analogue are stated in [8, Theorem 1.1 and the following remark]. We give a proof in the regular setting used here, so that its high- and low-energy mechanisms are visible.

Proof. Write $V = |\Omega|$, $V_a = V - a^n |U|$. Seeley's one-sided estimates for U , scaled to the hole, and bracketing give the required high-energy comparisons uniformly in x .

For EDP put $E_c = (b/(C_n V))^2$. For $E < E_c$, $N_{\Omega}^D(E) = 0$, hence $N_{\Omega_a}^D(E) = 0$. For $E \geq E_c$, use the untruncated EDP expression and the lower Dirichlet counting estimate for the hole to obtain

$$N_{\Omega_a}^D(E) \leq C_n V_a E^p - (b - A a^{n-1}) E^q.$$

Take a so small that $A a^{n-1} \leq b/2$ and $V_a \geq V/2$. Then the new positive-part cutoff for the constant $b/2$ is no larger than E_c , and

$$N_{\Omega_a}^D(E) \leq (C_n V_a E^p - \frac{b}{2} E^q)_+$$

in both energy ranges.

For ENP, the hole's upper Neumann estimate contains a zero-mode term. Thus

$$N_{\Omega_a}^N(E) \geq C_n V_a E^p + (b - Aa^{n-1})E^q - 1.$$

Choose $Aa^{n-1} \leq b/4$ and a fixed E_* with $1 \leq (b/4)E_*^q$. Then the gap is at least $(b/2)E^q$ for $E \geq E_*$.

For $0 < E \leq E_*$ only finitely many original eigenfunctions are needed. If Φ bounds their L^∞ norms and the span has dimension at most J , then

$$\int_{x+aU} |f|^2 \leq J\Phi^2 a^n |U| \|f\|_2^2.$$

As in the preceding proof, for any fixed $\varepsilon > 0$ and all sufficiently small a , restriction and min-max give

$$N_{\Omega_a}^N(E) \geq N_\Omega^N(E/(1+\varepsilon)), \quad 0 < E \leq E_*.$$

Using the original constant c in ENP and $V_a \leq V$,

$$N_{\Omega_a}^N(E) - C_n V_a E^p \geq c - C_n V E_*^p (1 - (1+\varepsilon)^{-p}).$$

First choose ε so that the last term is at most $c/2$, then choose a to make the restriction estimates valid. The gap is now at least $c/2$ on this compact range. Decreasing the high-energy coefficient and the constant gap yields ENP for all $E > 0$. All restriction estimates use only the volume of the hole, so the choice of a_0 is independent of its admissible location x . $\square$

6.3. Quantitative removal of cubes. The geometric quantity left as dots in the notes is surface-order, not simply the sum of side lengths in all dimensions. For a finite family of cubes Q_i with pairwise disjoint closures contained in Ω , define

$$\mathcal{S} = \sum_{i=1}^M \ell(Q_i)^{n-1}, \quad \mathcal{V} = \sum_{i=1}^M \ell(Q_i)^n, \quad \Omega' = \Omega \setminus \bigcup_{i=1}^M \overline{Q_i}.$$

Proposition 6.3 (An explicit Dirichlet sufficient condition). *If Ω satisfies EDP with constant b , and*

$$\mathcal{S} \leq \frac{b}{n^{3/2}\pi C_n},$$

then Ω' satisfies Dirichlet Pólya.

Proof. For a cube of side ℓ , put $R = \ell\sqrt{E}/\pi$. The unit cubes attached to the positive lattice points in B_R contain the positive-orthant ball of radius $(R - \sqrt{n})_+$. Therefore

$$\begin{aligned} N_Q^D(E) &\geq 2^{-n} \omega_n (R - \sqrt{n})_+^n \\ &\geq C_n \ell^n E^{n/2} - n^{3/2} \pi C_n \ell^{n-1} E^{(n-1)/2}. \end{aligned}$$

Subtract the sum of these bounds using Dirichlet bracketing. Where the original EDP expression is nonnegative, the result is

$$N_{\Omega'}^D(E) \leq C_n |\Omega'| E^{n/2} - (b - n^{3/2} \pi C_n \mathcal{S}) E^{(n-1)/2}.$$

Below the original positive-part cutoff the count is zero. The assumed condition proves the claim. $\square$

In the convention $b = C_n|\Omega|C(\Omega)$, this becomes $\mathcal{S} \leq |\Omega|C(\Omega)/(n^{3/2}\pi)$, the normalization used in [8]. The countably infinite cube constructions in [7] require the stated admissibility hypotheses and care with the volume of the closed union at accumulation points; the finite-family result above does not assert a theorem for arbitrary countable unions.

Theorem 6.4 (Finite-cube Neumann removal; [8]). *If Ω satisfies ENP, then for every fixed M there are $S_0(M, \Omega) > 0$ and $V_0(M, \Omega) > 0$ such that $\mathcal{S} < S_0$ and $\mathcal{V} < V_0$ imply Neumann Pólya on Ω' .*

Mechanism and sufficient thresholds. The explicit cube spectrum yields, with a constant K_n depending only on n ,

$$N_Q^N(E) \leq C_n|Q|E^{n/2} + K_n\ell(Q)^{n-1}E^{(n-1)/2} + 1.$$

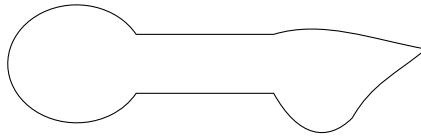
For example, count the nonnegative lattice points by the unit cubes they generate, contained in an orthant ball of radius $R + \sqrt{n}$; for $R \geq 1$ the polynomial error is $O_n(R^{n-1})$, while for $R < 1$ only the zero mode occurs. Subtracting the M estimates gives

$$N_{\Omega'}^N(E) \geq C_n|\Omega'|E^p + (b - K_n\mathcal{S})E^q - M.$$

Choose $S_0 \leq b/(2K_n)$ and $E_* \geq (4M/b)^{1/q}$. Then the high-energy gap is positive for $E \geq E_*$. On $0 < E \leq E_*$ use the finite-dimensional restriction argument in Theorem 6.2, with removed volume $\mathcal{V}$, and choose V_0 so that the ENP gap remains at least $c/2$. This proves the assertion and explains why a surface bound and a volume bound both occur. An infinite number of Neumann zero-mode terms cannot be summed this way. $\square$

6.4. Examples and finite tiling inheritance. For precision, the examples listed on source page 17 can be read in the following ENP versions from [8]: strip-tiling domains; Euclidean balls; periodic tiling domains with smooth or polyhedral boundary; and sufficiently thin products with a Lipschitz first factor and a smooth second factor of dimension at least two. A further example is a Lipschitz domain tiling $U \times \mathbb{R}^{n_2}$, where $U \subset \mathbb{R}^{n_1}$ is Lipschitz and $n_1, n_2 \geq 2$. A strip-tiling domain here tiles $\mathbb{R}^{n-1} \times (0, w)$ by isometries preserving the vertical direction. These hypotheses are part of the results, not consequences of the word “tiling” alone.

If Ω_1 tiles an EDP or ENP domain Ω by ℓ copies, divide the bracketing bounds by ℓ . Since $|\Omega|/\ell = |\Omega_1|$, the enhanced property passes to Ω_1 with correction constants divided by ℓ . For smooth or polyhedral periodic tiles, the strict inequalities in Chapter 4 supply the compact-energy gap; the large-energy surface-order Neumann lower estimate cited in [8, Section 5.1] supplies the remaining input to Lemma 6.1. Strict Pólya alone does not imply ENP or EDP. The unlabelled room-and-passage sketch in the handwritten notes is retained as a geometric illustration only; it does not specify an additional theorem or a width threshold.



6.5. An extremal formulation and the conjecture attributed to Lin. Let $n \geq 2$, and define, over bounded open sets of unit volume,

$$\Lambda^D(k, n) = \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^n, |\Omega| = 1\}.$$

Pólya's conjecture would give $\Lambda^D(k, n) \geq (k/C_n)^{2/n}$. The stronger conjecture recorded at the end of the course asks for $\delta(k, n) > 0$ such that

$$\Lambda^D(k, n) \geq \left((1 + \delta(k, n)) \frac{k}{C_n} \right)^{2/n}.$$

The constant must be uniform over the admissible domains for a fixed pair (k, n) . Strict inequality for each individual domain does not give this uniform positive gap after taking an infimum. In dimension one the unit interval has $\lambda_k = \pi^2 k^2 = (k/C_1)^2$, explaining the restriction $n \geq 2$.

Editor's note. The course attributes this formulation to Lin. His survey [9] concerns the relevant extremal problems, but the exact displayed uniform-gap formulation has not been independently located in the accessible primary material consulted here. It is therefore retained as a conjecture recorded in the course, not as a verified published theorem or as a proven consequence of the preceding results.

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