INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATION

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Abstract. These notes were originally taken while I was studying Professor Jianchun Chu's "Introduction to Second-Order Elliptic Partial Differential Equations", and the digital version has now been completed with the assistance of GTP 5.6 Sol. Please email me (yijiemiao0420@gmail.com or yijiemiao@seu.edu.cn) if you spot any typos or errors. Thank you!

1. Introduction and Laplace's Equation

1.1. **What is a PDE?.** A partial differential equation (PDE) is an equation for an unknown multi-variable function that involves partial derivatives. The order of a PDE is the order of the highest derivative that occurs in it.

Let u be a function of x_1 and x_2. Examples:

$$\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} + u = 0 \qquad \text{(first order)},$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \qquad \text{(second order)},$$

$$\frac{\partial^3 u}{\partial x_1^3} - \frac{\partial^2 u}{\partial x_1 \partial x_2} + 5\frac{\partial u}{\partial x_1} = 10 \qquad \text{(third order)}.$$

Remark 1.1. An ordinary differential equation (ODE) is an equation for an unknown single-variable function that involves ordinary derivatives. For PDEs and ODEs, the methods are totally different.

1.2. **Some geometric motivations.** Many geometric problems can be transformed into PDEs. Examples:

Calabi's conjecture, Yamabe problem, Minkowski problem,

minimal surface problem, prescribed curvature problem.

For these geometric problems, the corresponding PDE questions are: existence, uniqueness, and regularity.

1.3. **Main contents in this mini-course.** The topic is the theory of linear second order elliptic PDEs. The main goal is the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

The route of the course is

$$\text{Laplace's equation} \longrightarrow \text{Poisson's equation} \longrightarrow \text{general equation}.$$

The method is the continuity method. The main tools are the maximum principle and Schauder estimates.

Remark 1.2. The above method and tools are still widely used in the study of nonlinear second order elliptic PDEs.

1.4. **Laplace's Equation.**

Notations. $\mathbb{R}^n$ denotes n-dimensional Euclidean space. If $x \in \mathbb{R}^n$, then

$$x = (x_1, \ldots, x_n)^T, \qquad |x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

For $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \qquad B_r = B_r(0).$$

Write

$$\partial_i = \frac{\partial}{\partial x_i}, \qquad \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \qquad \Delta = \sum_{i=1}^n \partial_{ii}.$$

Thus Δu is the trace of the Hessian matrix $(\partial_{ij} u)$.

Definition 1.1. (1) Laplace's equation is $\Delta u = 0$.

(2) Let u be a C^2 function.
- u is subharmonic if $\Delta u \geq 0$.
- u is superharmonic if $\Delta u \leq 0$.
- u is harmonic if $\Delta u = 0$.

Example 1.1 (Second order polynomials)**.** Let

$$u(x) = \sum_{i,j=1}^n a_{ij} x_i x_j + \sum_{i=1}^n b_i x_i + c, \qquad a_{ij} = a_{ji}.$$

Then

$$\partial_{ij} u = 2a_{ij}.$$

Writing $A = (a_{ij})$, we have

$$\Delta u = 2 \operatorname{Tr} A.$$

Hence

$$\operatorname{Tr} A \geq 0 \iff u \text{ is subharmonic},$$

$$\operatorname{Tr} A \leq 0 \iff u \text{ is superharmonic},$$

$$\operatorname{Tr} A = 0 \iff u \text{ is harmonic}.$$

Example 1.2 (Holomorphic functions)**.** Let f be a holomorphic function on $\Omega \subset \mathbb{C}$. Then $\Re f$ and $\Im f$ are both harmonic. Write

$$f = u + \sqrt{-1}\, v.$$

The Cauchy–Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Therefore

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

Similarly, $\Delta v = 0$.

1.5. **Fundamental solutions.** Laplace's equation is invariant under orthogonal transformations.

Lemma 1.1. *Suppose u is harmonic and $A = (a_{ij})$ is an orthogonal matrix ($AA^T = I_n$). Then*
$$v(x) := u(Ax)$$
is harmonic.

Proof. Write $y = Ax$. By the chain rule,
$$\frac{\partial v}{\partial x_i} = \sum_{k=1}^n \frac{\partial u}{\partial y_k} a_{ki},$$
and
$$\frac{\partial^2 v}{\partial x_i \partial x_j} = \sum_{k,l=1}^n \frac{\partial^2 u}{\partial y_k \partial y_l} a_{ki} a_{lj}.$$
In matrix form,
$$\left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right) = A^T \left(\frac{\partial^2 u}{\partial y_k \partial y_l} \right) A.$$
Thus
$$\Delta v = \operatorname{Tr}\left(A^T (D_y^2 u) A\right) = \operatorname{Tr}\left(D_y^2 u \, AA^T\right) = \Delta u = 0.$$
$\square$

Motivated by the above lemma, let us find radial solutions. Suppose u depends only on $r = |x|$, i.e.
$$u(x) = f(r),$$
where f is a single-variable function. The goal is to apply $\Delta u = 0$ to solve for f. We have
$$\partial_{ii} u = \partial_i \left(f'(r) \partial_i r \right) = f''(r)(\partial_i r)^2 + f'(r) \partial_{ii} r.$$
Since
$$r = \sqrt{x_1^2 + \cdots + x_n^2}, \qquad \partial_i r = \frac{x_i}{r}, \qquad \partial_{ii} r = \frac{1}{r} - \frac{x_i^2}{r^3},$$
we obtain
$$\partial_{ii} u = f''(r) \frac{x_i^2}{r^2} + f'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right).$$
Therefore
$$0 = \Delta u = f''(r) + f'(r) \left(\frac{n}{r} - \frac{1}{r} \right),$$
or equivalently
$$f''(r) + \frac{n-1}{r} f'(r) = 0.$$
Multiplying by r^{n-1}, we get
$$r^{n-1} f''(r) + (n-1) r^{n-2} f'(r) = 0,$$
so
$$\left(r^{n-1} f'(r) \right)' = 0, \qquad r^{n-1} f'(r) = C.$$
Thus
$$f(r) = \begin{cases} C_1 \log r + C_2, & n = 2, \\ C_1 r^{2-n} + C_2, & n \geq 3. \end{cases}$$

Definition 1.2 (Fundamental solution)**.**

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \log |x| \,, & n = 2, \\ \dfrac{1}{n(2-n)\omega_n} |x|^{2-n} \,, & n \geq 3, \end{cases}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, i.e. $\omega_n = |B_1|$.

Remark 1.3.

$$|B_r| = \omega_n r^n, \qquad |\partial B_r| = |\partial B_1| \, r^{n-1}.$$

Also

$$|B_1| = \int_0^1 |\partial B_r| \; dr = |\partial B_1| \int_0^1 r^{n-1} \, dr,$$

hence

$$|B_1| = \frac{1}{n} |\partial B_1| \,, \qquad |\partial B_1| = n\omega_n.$$

Lemma 1.2. *(1) Γ is defined on $\mathbb{R}^n \setminus \{0\}$.*

(2) Γ has a singularity at 0. However, Γ is L^1-integrable near 0, i.e. $\Gamma \in L^1(B_1)$.

(3) Γ is harmonic in $\mathbb{R}^n \setminus \{0\}$, i.e.

$$\Delta\Gamma = 0 \qquad \text{in } \mathbb{R}^n \setminus \{0\} \,.$$

(4) For any $r > 0$,

$$\int_{\partial B_r} \frac{\partial \Gamma}{\partial \nu} \, dS = 1,$$

where ν denotes the unit outward normal to ∂B_r.

Proof. The function is defined away from the origin by its formula. To see harmonicity, use the radial identity

$$\Delta r^p = p(p+n-2) r^{p-2} \qquad (r = |x| > 0).$$

For $n \geq 3$, $p = 2-n$, so $p(p+n-2) = 0$, and hence $\Delta\Gamma = 0$ on $\mathbb{R}^n \setminus \{0\}$. For $n = 2$, the same calculation gives $\Delta \log r = 0$ for $r > 0$.

Suppose $n \geq 3$; the case $n = 2$ is similar. For (2), For (2),

$$\int_{B_1} \Gamma(x) \, dx = \int_0^1 \int_{\partial B_r} \frac{1}{n(2-n)\omega_n} |x|^{2-n} \; dx \, dr.$$

Using $|\partial B_r| = |\partial B_1| \, r^{n-1} = n\omega_n r^{n-1}$,

$$\int_{B_1} |\Gamma(x)| \; dx = \int_0^1 \frac{1}{n(n-2)\omega_n} r^{2-n} \, |\partial B_r| \; dr = \int_0^1 \frac{1}{n-2} r \, dr = \frac{1}{2(n-2)} < \infty.$$

For (4), fix $x \in \partial B_r$. Then $\nu = x / |x|$, and

$$\frac{\partial \Gamma}{\partial \nu} = \nabla \Gamma \cdot \nu = \sum_{i=1}^n \frac{1}{n\omega_n} |x|^{1-n} \frac{x_i}{|x|} \frac{x_i}{|x|} = \frac{|x|^{1-n}}{n\omega_n}.$$

Therefore

$$\int_{\partial B_r} \frac{\partial \Gamma}{\partial \nu} \, dS = \frac{r^{1-n}}{n\omega_n} |\partial B_r| = 1.$$

$\square$

1.6. **Integral formulas.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Let $\nu = (\nu_1, \ldots, \nu_n)$ denote the unit outward normal to $\partial\Omega$. The divergence theorem says that if

$$\vec{F} = (F_1, \ldots, F_n)$$

is a C^1 vector field on $\overline{\Omega}$, then

$$\int_\Omega \operatorname{div} \vec{F}\, dx = \int_{\partial\Omega} \vec{F} \cdot \nu \, dS, \qquad \operatorname{div} \vec{F} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}.$$

Remark 1.4. If $u \in C^1(\overline{\Omega})$, choose

$$\vec{F} = (0, \ldots, u, \ldots, 0),$$

where u is in the i-th component. Then the divergence theorem gives

$$\int_\Omega \partial_i u \, dx = \int_{\partial\Omega} u \nu_i \, dS.$$

This is integration by parts.

Green's formula. Let $u, v \in C^2(\overline{\Omega})$. Take $\vec{F} = u\nabla v$. Then

$$\int_\Omega \operatorname{div}(u\nabla v)\, dx = \int_{\partial\Omega} u\nabla v \cdot \nu \, dS.$$

Since

$$\operatorname{div}(u\nabla v) = u\Delta v + \nabla u \cdot \nabla v,$$

the first version of Green's formula is

$$\int_\Omega (u\Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\, dS.$$

Interchanging u and v,

$$\int_\Omega (v\Delta u + \nabla u \cdot \nabla v)\, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, dS.$$

Subtracting gives the second version of Green's formula:

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS.$$

Taking $u = 1$, one obtains the third version:

$$\int_\Omega \Delta v \, dx = \int_{\partial\Omega} \frac{\partial v}{\partial \nu}\, dS.$$

1.7. **Green's identity.**

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. If $u \in C^2(\overline{\Omega})$, then for any $x \in \Omega$,*

$$u(x) = \int_\Omega \Gamma(x-y)\Delta_y u(y)\, dy - \int_{\partial\Omega} \left[\Gamma(x-y)\frac{\partial u}{\partial \nu_y}(y) - u(y)\frac{\partial}{\partial \nu_y}\Gamma(x-y) \right] dS_y.$$

Proof. Fix $x \in \Omega$. Write $\Gamma(y) = \Gamma(x-y)$. Choose $r > 0$ sufficiently small such that $B_r(x) \subset \Omega$. Then Γ is smooth in $\Omega \setminus B_r(x)$.

Applying the second version of Green's formula to u and Γ in $\Omega \setminus B_r(x)$,

$$\int_{\Omega\setminus B_r(x)} (\Gamma\Delta u - u\Delta_y \Gamma)\, dy = \int_{\partial\Omega} \left(\Gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu_y} \right) dS_y + \int_{\partial B_r(x)} \left(\Gamma\frac{\partial u}{\partial \nu} - u\frac{\partial \Gamma}{\partial \nu_y} \right) dS_y.$$

Recall that $\Delta_y \Gamma = 0$ in $\Omega \setminus B_r(x)$. Hence

$$\int_{\Omega\setminus B_r(x)} \Gamma \Delta u \, dy = \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu_y} \right) dS_y + \int_{\partial B_r(x)} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu_y} \right) dS_y.$$

Let $r \to 0$. Since $u \in C^2(\overline{\Omega})$ and $\Gamma \in L^1(B_1(x))$,

$$\int_{\Omega\setminus B_r(x)} \Gamma \Delta u \, dy \longrightarrow \int_{\Omega} \Gamma \Delta u \, dy.$$

Step 1: compute Γ and $\partial\Gamma/\partial\nu_y$ on $\partial B_r(x)$. For $n \geq 3$,

$$\Gamma = \Gamma(x-y) = \frac{1}{n(2-n)\omega_n} |y-x|^{2-n} = \frac{r^{2-n}}{n(2-n)\omega_n} \qquad \text{on } \partial B_r(x).$$

Also

$$\frac{\partial \Gamma}{\partial y_i} = \frac{1}{n\omega_n} \frac{y_i - x_i}{|y-x|^n}.$$

On $\partial B_r(x)$, the outward normal of $\Omega \setminus B_r(x)$ along the inner boundary is

$$\nu_y = -\frac{y-x}{r}, \qquad \nu_i = -\frac{y_i - x_i}{r}.$$

Thus

$$\frac{\partial \Gamma}{\partial \nu_y} = \sum_{i=1}^{n} \frac{\partial \Gamma}{\partial y_i} \nu_i = -\sum_{i=1}^{n} \frac{1}{n\omega_n r} \frac{|y_i - x_i|^2}{|y-x|^n} = -\frac{1}{n\omega_n r^{n-1}} \qquad \text{on } \partial B_r(x).$$

Step 2: compute

$$\lim_{r\to 0} \int_{\partial B_r(x)} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) dS.$$

First,

$$\left| \int_{\partial B_r(x)} \Gamma \frac{\partial u}{\partial \nu} \, dS \right| = \left| \frac{r^{2-n}}{n(2-n)\omega_n} \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu} \, dS \right|$$

and

$$\left| \frac{\partial u}{\partial \nu} \right| \leq |\nabla u| \, .$$

Therefore

$$\left| \int_{\partial B_r(x)} \Gamma \frac{\partial u}{\partial \nu} \, dS \right| \leq \frac{r^{2-n}}{n(2-n)\omega_n} |\partial B_r(x)| \max_{\partial B_r(x)} |\nabla u| = \frac{r}{n-2} \max_{\partial B_r(x)} |\nabla u| \longrightarrow 0.$$

Next,

$$-\int_{\partial B_r(x)} u \frac{\partial \Gamma}{\partial \nu} \, dS = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y) \, dS_y.$$

By Taylor's formula,

$$\frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} \big(u(x) + O(r) \big) \, dS_y = u(x) + O(r) \longrightarrow u(x).$$

Recalling the identity before taking the limit, the small boundary term contributes $0 + u(x)$, hence

$$u(x) = \int_{\Omega} \Gamma \Delta u \, dy - \int_{\partial\Omega} \left(\Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} \right) dS.$$

□

Remark 1.5. Taking $u = 1$ in Green's identity gives, for every $x \in \Omega$,

$$\int_{\partial\Omega} \frac{\partial\Gamma}{\partial\nu}(x-y)\, dS = 1.$$

2. Dirichlet Problem in Balls for Laplace's Equation

2.1. **Recall.** Laplace's equation is

$$\Delta u = 0.$$

The fundamental solution is

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \log|x|, & n = 2, \\ \dfrac{1}{n(2-n)\omega_n} |x|^{2-n}, & n \geq 3. \end{cases}$$

Green's identity says

$$u(x) = \int_\Omega \Gamma(x-y)\Delta_y u(y)\, dy - \int_{\partial\Omega} \left[\Gamma(x-y)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu_y}(x-y) \right] dS_y.$$

The goal of this section is to solve the Dirichlet problem in

$$B_R = B_R(0): \qquad \begin{cases} \Delta u = 0 & \text{in } B_R, \\ u = \varphi & \text{on } \partial B_R. \end{cases}$$

2.2. **Green's function.**

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Suppose that $\Phi(x,y)$, defined on $\Omega \times \Omega$, satisfies, for any $x \in \Omega$,

$$\begin{cases} \Delta_y \Phi(x,y) = 0 & \text{in } \Omega, \\ \Phi(x,y) = \Gamma(x-y) & \text{on } \partial\Omega. \end{cases}$$

Then

$$G(x,y) = \Gamma(x-y) - \Phi(x,y)$$

is called the Green function of Ω.

Lemma 2.1. *If $u \in C^2(\overline{\Omega})$ satisfies*

$$\begin{cases} \Delta u = 0 & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega, \end{cases}$$

then

$$u(x) = \int_{\partial\Omega} \varphi(y) \frac{\partial G}{\partial \nu_y}(x,y)\, dS_y.$$

Proof. By Green's identity,

$$u(x) = \int_\Omega \Gamma(x-y)\Delta_y u(y)\, dy - \int_{\partial\Omega} \left[\Gamma(x-y)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Gamma}{\partial \nu_y}(x-y) \right] dS_y.$$

Since $\Delta u = 0$, and $\Gamma = G + \Phi$,

$$u(x) = -\int_{\partial\Omega} \Gamma(x-y)\frac{\partial u}{\partial \nu}(y)\, dS_y + \int_{\partial\Omega} u(y) \left[\frac{\partial G}{\partial \nu_y}(x,y) + \frac{\partial \Phi}{\partial \nu_y}(x,y) \right] dS_y.$$

Using $\Phi = \Gamma$ on $\partial\Omega$, the remaining boundary term can be converted by Green's formula to

$$\int_\Omega \left(u(y)\Delta_y \Phi(x,y) - \Phi(x,y)\Delta u(y) \right) dy = 0.$$

Since $u = \varphi$ on $\partial\Omega$, the claimed formula follows. □

Remark 2.1. Two questions remain: existence of Green's function, and whether the formal solution above is a classical solution.

2.3. Green's function in $B_R = B_R(0)$.

Proposition 2.1. *Define $\Phi(x,y)$ as follows. When $n=2$,*

$$\Phi(x,y) = \begin{cases} \dfrac{1}{2\pi}\log R, & x=0,\ y\in B_R,\\ \dfrac{1}{2\pi}\log\left|\dfrac{|x|}{R}y-\dfrac{R}{|x|}x\right|, & x\neq 0,\ y\in B_R. \end{cases}$$

When $n\ge 3$,

$$\Phi(x,y) = \begin{cases} \dfrac{1}{n(2-n)\omega_n}R^{2-n}, & x=0,\ y\in B_R,\\ \dfrac{1}{n(2-n)\omega_n}\left|\dfrac{|x|}{R}y-\dfrac{R}{|x|}x\right|^{2-n}, & x\neq 0,\ y\in B_R. \end{cases}$$

Then

$$G(x,y) = \Gamma(x-y) - \Phi(x,y)$$

is the Green function of B_R.

Proof. It suffices to verify

$$\begin{cases} \Delta_y\Phi(x,y) = 0 & y\in B_R,\\ \Phi(x,y) = \Gamma(x-y) & y\in\partial B_R. \end{cases}$$

Suppose $n\ge 3$; the case $n=2$ is similar.

Case 1: $x=0$. Then

$$\Phi(0,y) = \frac{1}{n(2-n)\omega_n}R^{2-n},$$

so $\Delta_y\Phi(0,y)=0$ in B_R, and for $y\in\partial B_R$,

$$\Gamma(0-y) = \frac{1}{n(2-n)\omega_n}|y|^{2-n} = \Phi(0,y).$$

Case 2: $x\neq 0$. Put

$$x^* = \frac{R^2}{|x|^2}x.$$

Then

$$\Phi(x,y) = \frac{1}{n(2-n)\omega_n}\left|\frac{|x|}{R}y-\frac{R}{|x|}x\right|^{2-n} = \left(\frac{|x|}{R}\right)^{2-n}\Gamma(y-x^*).$$

Since

$$|x^*| = \frac{R^2}{|x|^2}|x| = \frac{R}{|x|}R > R,$$

we have $x^*\notin\overline{B_R}$, hence $\Gamma(y-x^*)$ is harmonic in B_R. For $y\in\partial B_R$, the similar triangles in the figure give

$$\frac{|y-x|}{|y-x^*|} = \frac{|x|}{R},\qquad |y-x| = \frac{|x|}{R}|y-x^*|.$$

Thus for $y\in\partial B_R$,

$$\Phi(x,y) = \left(\frac{|x|}{R}\right)^{2-n}\Gamma(y-x^*) = \left(\frac{|x|}{R}\right)^{2-n}\frac{1}{n(2-n)\omega_n}|y-x^*|^{2-n}.$$

Using $|y-x| = (|x|/R)\,|y-x^*|$,

$$\Phi(x,y) = \left(\frac{|x|}{R}\right)^{2-n} \frac{1}{n(2-n)\omega_n} \left(\frac{R}{|x|}\right)^{2-n} |y-x|^{2-n} = \Gamma(y-x) = \Gamma(x-y).$$

□

2.4. **Poisson kernel in B_R.** Recall the preceding lemma:

$$\begin{cases} \Delta u = 0 & \text{in } B_R, \\ u = \varphi & \text{on } \partial B_R \end{cases} \implies u(x) = \int_{\partial B_R} \varphi(y) \frac{\partial G}{\partial \nu_y}(x,y)\, dS_y.$$

Definition 2.2. The Poisson kernel is

$$K(x,y) = \frac{\partial G}{\partial \nu_y}(x,y), \qquad x \in B_R,\ y \in \partial B_R.$$

Lemma 2.2.

$$K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R\,|x-y|^n}, \qquad x \in B_R,\ y \in \partial B_R.$$

Proof. Suppose $n \geq 3$; the case $n = 2$ is similar. Since

$$G(x,y) = \Gamma(x-y) - \Phi(x,y),$$

we compute $\partial G/\partial \nu_y$.

Case 1: $x = 0$. Then

$$G(0,y) = \frac{1}{n(2-n)\omega_n}\left(|y|^{2-n} - R^{2-n}\right),$$

and

$$\frac{\partial G}{\partial y_i}(0,y) = \frac{1}{n\omega_n}\frac{y_i}{|y|^n}.$$

For $y \in \partial B_R$, $\nu = y/R$, hence

$$\frac{\partial G}{\partial \nu_y}(0,y) = \frac{R^2}{n\omega_n R\,|y|^n}.$$

Case 2: $x \neq 0$. Recall

$$\Phi(x,y) = \left(\frac{|x|}{R}\right)^{2-n}\Gamma(y-x^*), \qquad x^* = \frac{R^2}{|x|^2}x,$$

and

$$G(x,y) = \frac{1}{n(2-n)\omega_n}\left[|y-x|^{2-n} - \left(\frac{|x|}{R}\right)^{2-n}|y-x^*|^{2-n}\right].$$

On ∂B_R,

$$|y-x^*| = \frac{R}{|x|}\,|y-x|.$$

Differentiating,

$$\frac{\partial G}{\partial y_i}(x,y) = \frac{1}{n\omega_n}\left[\frac{y_i - x_i}{|y-x|^n} - \left(\frac{|x|}{R}\right)^{2-n}\frac{y_i - \frac{R^2}{|x|^2}x_i}{\left(\frac{R}{|x|}\right)^n |y-x|^n}\right].$$

Thus

$$\frac{\partial G}{\partial y_i}(x,y) = \frac{1}{n\omega_n}\left[\frac{y_i - x_i}{|y-x|^n} - \frac{|x|^2 y_i - R^2 x_i}{R^2 |y-x|^n}\right] = \frac{R^2 - |x|^2}{n\omega_n R^2 |y-x|^n}\, y_i.$$

Since $\nu = y/R$ on ∂B_R,

$$\frac{\partial G}{\partial \nu_y}(x,y) = \sum_i \frac{\partial G}{\partial y_i}\frac{y_i}{R} = \frac{R^2 - |x|^2}{n\omega_n R\,|x-y|^n}.$$

□

Lemma 2.3. *For $x \in B_R$, $y \in \partial B_R$, the Poisson kernel*

$$K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R\,|x-y|^n}$$

satisfies:

(1) *$K(x,y)$ is smooth and positive.*
(2) *$\int_{\partial B_R} K(x,y)\, dS_y = 1$.*
(3) *$\Delta_x K(x,y) = 0$.*

Proof. For (1), if $x \in B_R$ and $y \in \partial B_R$, then $|x-y| > 0$ and $R^2 - |x|^2 > 0$. Thus $K(x,y) > 0$, and the formula shows smoothness in x. For (2), recall that if

$$\begin{cases} \Delta u = 0 & \text{in } B_R, \\ u = \varphi & \text{on } \partial B_R, \end{cases}$$

then

$$u(x) = \int_{\partial B_R} \varphi(y) K(x,y)\, dS_y.$$

Choose $u = 1$. Then

$$\int_{\partial B_R} K(x,y)\, dS_y = 1.$$

For (3), one method is to use the fact that $G(x,y) = G(y,x)$. Since

$$G(x,y) = \Gamma(x-y) - \Phi(x,y),$$

we know $\Delta_y G(x,y) = 0$ away from the pole; by symmetry $\Delta_x G(x,y) = 0$, hence

$$\Delta_x \frac{\partial G}{\partial \nu_y}(x,y) = 0, \qquad \Delta_x K(x,y) = 0.$$

Alternatively, one can verify this by direct calculation. The notes compute

$$\frac{\partial K}{\partial x_i}(x,y) = -\frac{1}{n\omega_n R}\left[2x_i\,|x-y|^{-n} + n(R^2 - |x|^2)\,|x-y|^{-n-2}\,(x_i - y_i)\right],$$

and

$$\frac{\partial^2 K}{\partial x_i \partial x_i}(x,y) = -\frac{1}{n\omega_n R}\Big[2\,|x-y|^{-n} - 4n\,|x-y|^{-n-2}\,(x_i^2 - x_i y_i)$$
$$-n(n+2)(R^2 - |x|^2)\,|x-y|^{-n-4}\,(x_i - y_i)^2 + n(R^2 - |x|^2)\,|x-y|^{-n-2}\Big].$$

Summing over i,

$$\Delta_x K(x,y) = -\frac{1}{n\omega_n R}\left[2n\,|x-y|^{-n} - 4n\,|x-y|^{-n-2}\,(|x|^2 - x\cdot y) - 2n(R^2 - |x|^2)\,|x-y|^{-n-2}\right].$$

Since $y \in \partial B_R$,

$$|x-y|^2 - 2(|x|^2 - x\cdot y) - (R^2 - |x|^2) = 0,$$

and hence $\Delta_x K(x,y) = 0$. □

2.5. **Dirichlet problem in B_R.**

Theorem 2.2. *For $\varphi \in C(\partial B_R)$, define*

$$u(x) = \int_{\partial B_R} \varphi(y) K(x,y)\, dS_y.$$

Then

(1) *u is smooth and $\Delta u = 0$ in B_R.*
(2) *For any $y_0 \in \partial B_R$,*

$$\lim_{x\to y_0} u(x) = \varphi(y_0).$$

Proof. Part (1) follows from smoothness of K and $\Delta_x K = 0$. For (2), the goal is: for every $\varepsilon > 0$, find $\delta > 0$ such that

$$|u(x) - \varphi(y_0)| \le \varepsilon \qquad \forall x \in B_\delta(y_0) \cap B_R.$$

Using $\int_{\partial B_R} K(x,y)\, dS_y = 1$,

$$u(x) - \varphi(y_0) = \int_{\partial B_R} (\varphi(y) - \varphi(y_0)) K(x,y)\, dS_y.$$

Split the boundary into

$$\partial B_R \cap B_{\delta_1}(y_0) \quad \text{and} \quad \partial B_R \setminus B_{\delta_1}(y_0),$$

and denote the two integrals by I_1 and I_2.

Since $\varphi \in C(\partial B_R)$, choose δ_1 so that

$$|\varphi(y) - \varphi(y_0)| < \varepsilon/2 \qquad \forall y \in B_{\delta_1}(y_0) \cap \partial B_R.$$

Then

$$|I_1| \le \frac{\varepsilon}{2} \int_{\partial B_R} K(x,y)\, dS_y = \frac{\varepsilon}{2}.$$

For I_2, use

$$K(x,y) = \frac{R^2 - |x|^2}{n\omega_n R\, |x-y|^n}.$$

If $\delta < \delta_1$, $x \in B_\delta(y_0) \cap B_R$, and $y \in \partial B_R \setminus B_{\delta_1}(y_0)$, then

$$0 < K(x,y) \le \frac{(R+|x|)(R-|x|)}{n\omega_n R(\delta_1 - \delta)^n} \le \frac{2R}{n\omega_n R(\delta_1-\delta)^n}\, |y_0 - x| \le \frac{2\delta}{n\omega_n(\delta_1-\delta)^n}.$$

Set $M = \max_{\partial B_R} |\varphi|$, and choose $\delta \le 1$ so small that

$$0 < K(x,y) \le \frac{\varepsilon}{2M\omega_n R^{n-1}}$$

for all such x, y. Then

$$|I_2| \le \int_{\partial B_R \setminus B_{\delta_1}(y_0)} \frac{\varepsilon}{2M\omega_n R^{n-1}} \big(|\varphi(y)| + |\varphi(y_0)|\big)\, dS_y \le \frac{\varepsilon}{2}.$$

Hence

$$|u(x) - \varphi(y_0)| \le |I_1| + |I_2| \le \varepsilon,$$

so $\lim_{x\to y_0} u(x) = \varphi(y_0)$. □

Remark 2.2. The function

$$u(x) = \int_{\partial B_R} \varphi(y) K(x,y)\, dS_y$$

is a harmonic function in B_R.

The theorem above implies:

(1) $u \in C^\infty(B_R) \cap C(\overline{B_R})$.
(2) u solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_R, \\ u = \varphi & \text{on } \partial B_R. \end{cases}$$

2.6. **Harmonic functions.**

Theorem 2.3 (Mean value property)**.** *Let Ω be a domain in $\mathbb{R}^n$. If $u \in C^2(\Omega)$ is harmonic in Ω, then u satisfies the mean value property: for every $\overline{B_r(x_0)} \subset \Omega$,*

$$u(x_0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x_0)} u(y)\, dS_y = \frac{1}{\omega_n r^n} \int_{B_r(x_0)} u(y)\, dy.$$

Proof. After translation, assume $x_0 = 0$. Otherwise consider

$$v(x) = u(x + x_0).$$

For every $s \in (0, r]$, Green's third identity gives

$$0 = \int_{B_s} \Delta u\, dy = \int_{\partial B_s} \frac{\partial u}{\partial \nu}\, dS_y = \int_{\partial B_s} \sum_{i=1}^n \frac{y_i}{s} \partial_i u(y)\, dS_y.$$

Writing $w = y/s$, this becomes

$$\begin{aligned} s^{n-1} \int_{\partial B_1} \sum_{i=1}^n w_i \partial_i u(sw)\, dS_w &= s^{n-1} \int_{\partial B_1} \frac{\partial u}{\partial s}(sw)\, dS_w \\ &= s^{n-1} \frac{\partial}{\partial s} \int_{\partial B_1} u(sw)\, dS_w. \end{aligned}$$

Thus

$$\int_{\partial B_1} u(sw)\, dS_w$$

is constant in s. Letting $s \to 0$,

$$\int_{\partial B_1} u(sw)\, dS_w = n\omega_n u(0).$$

Therefore, with $y = sw$,

$$u(0) = \frac{1}{n\omega_n} \int_{\partial B_1} u(sw)\, dS_w = \frac{1}{n\omega_n s^{n-1}} \int_{\partial B_s} u(y)\, dS_y.$$

Choosing $s = r$,

$$u(0) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} u(y)\, dS_y.$$

Multiplying the sphere formula by $n\omega_n s^{n-1}$ and integrating in s from 0 to r,

$$\int_0^r n\omega_n s^{n-1} u(0)\, ds = \int_0^r \int_{\partial B_s} u(y)\, dS_y\, ds = \int_{B_r} u(y)\, dy.$$

Hence

$$\omega_n r^n u(0) = \int_{B_r} u(y)\,dy,$$

which proves

$$u(0) = \frac{1}{\omega_n r^n} \int_{B_r} u(y)\,dy.$$

□

Theorem 2.4 (Interior derivative estimate)**.** *If $u \in C^\infty(\Omega)$ is harmonic in Ω, then for $B_R(x_0) \Subset \Omega$, $r \in (0, R)$, and $k \in \mathbb{Z}_+$,*

$$\sup_{B_r(x_0)} \left|\nabla^k u\right| \le \frac{C(n,k)}{(R-r)^k} \max_{B_R(x_0)} |u|\,.$$

Here $C(n,k)$ denotes a constant depending only on n and k.

Proof. After translation, assume $x_0 = 0$. First take $k = 1$. For every $x \in B_r$,

$$u(x) = \frac{1}{\omega_n (R-r)^n} \int_{B_{R-r}(x)} u(y)\,dy = \frac{1}{\omega_n (R-r)^n} \int_{B_{R-r}} u(x+y)\,dy.$$

Thus

$$\partial_i u(x) = \frac{1}{\omega_n (R-r)^n} \int_{B_{R-r}} \partial_{y_i} u(x+y)\,dy = \frac{1}{\omega_n (R-r)^n} \int_{\partial B_{R-r}} u(x+y)\nu_i\,dS_y.$$

Therefore

$$|\partial_i u(x)| \le \frac{n}{R-r} \sup_{B_R} |u|\,,$$

and hence

$$\sup_{B_r} |\nabla u| \le \frac{C(n)}{R-r} \sup_{B_R} |u|\,.$$

For higher derivatives, argue by induction. Assume the estimate $(*_k)$ holds. Since $\Delta u = 0$,

$$\Delta(\nabla^k u) = \nabla^k(\Delta u) = 0,$$

so $\nabla^k u$ is harmonic. Set

$$\widehat{r} = \frac{R+r}{2}.$$

Applying the $k = 1$ case to $\nabla^k u$, replacing r, R by $r, \widehat{r}$,

$$\sup_{B_r} \left|\nabla^{k+1} u\right| \le \frac{C(n,1)}{\widehat{r} - r} \sup_{B_{\widehat{r}}} \left|\nabla^k u\right| = \frac{2C(n,1)}{R-r} \sup_{B_{\widehat{r}}} \left|\nabla^k u\right|.$$

By $(*_k)$, replacing r, R by $\widehat{r}, R$,

$$\sup_{B_{\widehat{r}}} \left|\nabla^k u\right| \le \frac{C(n,k)}{(R-\widehat{r})^k} \sup_{B_R} |u| = \frac{2^k C(n,k)}{(R-r)^k} \sup_{B_R} |u|\,.$$

Combining,

$$\sup_{B_r} \left|\nabla^{k+1} u\right| \le \frac{2^{k+1} C(n,1) C(n,k)}{(R-r)^{k+1}} \sup_{B_R} |u|\,.$$

This proves $(*_{k+1})$, with

$$C(n, k+1) = 2^{k+1} C(n,1) C(n,k).$$

□

3. Maximum Principle I

3.1. **Smoothness of harmonic functions.** Recall:

$$\begin{cases} \Delta u = 0 & \text{in } B_R, \\ u = \varphi & \text{on } \partial B_R \end{cases}$$

is solved by Green's function and the Poisson kernel. We also recalled two properties of harmonic functions:

$$u \in C^2 \Longrightarrow \text{mean value property},$$

and

$$u \in C^\infty, \quad \sup_{B_r(x_0)} \left|\nabla^k u\right| \leq \frac{C(n,k)}{(R-r)^k} \sup_{B_R(x_0)} |u| .$$

Theorem 3.1. *If $u \in C^2(\Omega)$ is harmonic, then $u \in C^\infty(\Omega)$.*

Proof. Step 1: construct a mollifier. Choose a one-variable function $\widehat{\eta} \in C^\infty([0,+\infty))$ such that

$$\widehat{\eta} \geq 0 \text{ in } [0,+\infty), \qquad \widehat{\eta} = 1 \text{ in } [0,1/4], \qquad \widehat{\eta} = 0 \text{ in } [3/4,+\infty).$$

Choose a constant $c > 0$ such that

$$n\omega_n \int_0^1 r^{n-1} c\,\widehat{\eta}(r)\,dr = 1.$$

Set $\eta(x) = c\widehat{\eta}(|x|)$, and

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

Then

$$\operatorname{supp}\eta \subset B_1, \qquad \operatorname{supp}\eta_\varepsilon \subset B_\varepsilon,$$

and

$$\int_{B_1} \eta(y)\,dy = \int_0^1 \int_{\partial B_r} \eta(y)\,dS_y\,dr = \int_0^1 n\omega_n r^{n-1}\eta(r)\,dr = 1,$$

so also

$$\int_{B_\varepsilon} \eta_\varepsilon(y)\,dy = 1.$$

Step 2: approximation. Let

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \varepsilon\}.$$

For $x \in \Omega_\varepsilon$, define

$$u_\varepsilon(x) = \int_\Omega u(y)\eta_\varepsilon(y-x)\,dy.$$

Then $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$. Since $\operatorname{supp}\eta_\varepsilon \subset B_\varepsilon$,

$$u_\varepsilon(x) = \int_{B_\varepsilon(x)} u(y)\eta_\varepsilon(y-x)\,dy.$$

Using

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right),$$

we compute

$$u_\varepsilon(x) = \frac{1}{\varepsilon^n}\int_{B_\varepsilon(x)} u(y)\eta\left(\frac{y-x}{\varepsilon}\right)\,dy.$$

With $w = (y-x)/\varepsilon$,

$$u_\varepsilon(x) = \int_{B_1} u(x+\varepsilon w)\eta(w)\, dw.$$

Because u is harmonic,

$$\Delta u_\varepsilon(x) = \int_{B_1} \Delta u(x+\varepsilon y)\eta(y)\, dy = 0.$$

Thus u_ε is a smooth harmonic function in Ω_ε.

Step 3: prove $u \in C^\infty(\Omega)$. It is enough to show that for every $B_R(x_0) \Subset \Omega$, $u \in C^\infty(B_R(x_0))$. Choose $d > 0$ such that

$$B_R(x_0) \subset B_{R+3d}(x_0) \subset \Omega.$$

Then for $\varepsilon \ll 1$,

$$B_{R+2d}(x_0) \Subset \Omega_\varepsilon.$$

Using $\int_{B_1} \eta(y)\, dy = 1$,

$$u(x) = \int_{B_1} u(x)\eta(y)\, dy, \qquad u_\varepsilon(x) = \int_{B_1} u(x+\varepsilon y)\eta(y)\, dy.$$

Hence

$$|u_\varepsilon(x) - u(x)| \le \int_{B_1} |u(x+\varepsilon y) - u(x)|\, \eta(y)\, dy.$$

By the mean value theorem,

$$|u_\varepsilon(x) - u(x)| \le \varepsilon\, \|\nabla u\|_{L^\infty(B_{R+2d}(x_0))} \int_{B_1} |y|\, \eta(y)\, dy \to 0.$$

Thus $u_\varepsilon \to u$ in $C^0(\overline{B_{R+d}(x_0)})$.

For $\varepsilon \ll 1$,

$$\sup_{B_{R+d}(x_0)} |u_\varepsilon| \le \sup_{B_{R+d}(x_0)} |u| + 1.$$

Since u_ε is smooth harmonic in Ω_ε, the interior estimates give, for every $k \in \mathbb{Z}_+$,

$$\|u_\varepsilon\|_{C^{k+1}(\overline{B_R(x_0)})} \le \frac{C(n,k)}{d^{k+1}} \sup_{B_{R+d}(x_0)} |u_\varepsilon| \le \frac{C(n,k)}{d^{k+1}} \left(\sup_{B_{R+d}(x_0)} |u| + 1 \right).$$

By the Arzela–Ascoli theorem, there exists $\varepsilon_i \to 0$ such that

$$u_{\varepsilon_i} \to v \quad \text{in } C^k(\overline{B_R(x_0)}).$$

Since also $u_{\varepsilon_i} \to u$ in C^0, we get

$$u = v \quad \text{in } \overline{B_R(x_0)}.$$

Therefore $u \in C^k(\overline{B_R(x_0)})$. Since k is arbitrary,

$$u \in C^\infty(B_R(x_0)).$$

$\square$

3.2. **Notations and definitions.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let a_{ij}, b_i, c be bounded continuous functions in Ω, with $a_{ij} = a_{ji}$. Define the operator

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i\partial_i + c.$$

Thus for $u \in C^2(\Omega)$,

$$Lu = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{i=1}^{n} b_i\partial_i u + cu.$$

Definition 3.1. (1) L is strictly elliptic if there exists $\lambda > 0$ such that

$$(a_{ij}(x)) \geq \lambda I_n,$$

i.e.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2 \qquad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n.$$

(2) L is uniformly elliptic if there exist $\lambda, \Lambda > 0$ such that

$$\lambda I_n \leq (a_{ij}(x)) \leq \Lambda I_n,$$

i.e.

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda |\xi|^2 .$$

Definition 3.2. Let $f \in C(\Omega)$. A linear second order PDE is

$$Lu = f. \tag{$*$}$$

If $u \in C^2(\Omega)$ and $Lu \geq f$, then u is called a subsolution of $(*)$. If $Lu \leq f$, then u is called a supersolution of $(*)$.

Remark 3.1. When $a_{ij} = \delta_{ij}$, $b_i = 0$, and $c = 0$, L becomes Laplace's operator:

$$L = \Delta.$$

If in addition $f = 0$, then $(*)$ is Laplace's equation. In this case, subsolution corresponds to subharmonic, and supersolution corresponds to superharmonic.

3.3. **Weak maximum principle.**

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, and*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_{i} b_i\partial_i + c.$$

Assume

$$a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega), \qquad (a_{ij}(x)) \geq \lambda I_n, \qquad c \leq 0.$$

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$Lu \geq 0 \quad \textit{in } \Omega,$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+, \qquad u^+ = \max(u, 0).$$

Proof. First consider the special case $Lu > 0$ in Ω. Since $u \in C(\overline{\Omega})$, there exists $x_0 \in \overline{\Omega}$ such that

$$u(x_0) = \max_{\overline{\Omega}} u.$$

If $x_0 \in \partial\Omega$, or $u(x_0) \le 0$, then the conclusion is immediate. Suppose $x_0 \notin \partial\Omega$ and $u(x_0) > 0$. Then x_0 is an interior maximum point, hence

$$\partial_i u(x_0) = 0, \qquad (\partial_{ij} u(x_0)) \le 0.$$

Then

$$0 < Lu(x_0) = \sum_{i,j} a_{ij}(x_0)\partial_{ij}u(x_0) + \sum_i b_i(x_0)\partial_i u(x_0) + c(x_0)u(x_0) \le 0.$$

Indeed, $(a_{ij}(x_0)) > 0$, so $(a_{ij}(x_0)) = P^T P$, and

$$\sum_{i,j} a_{ij}(x_0)\partial_{ij}u(x_0) = \mathrm{Tr}\big((a_{ij}(x_0))(\partial_{ij}u(x_0))\big) = \mathrm{Tr}\big(P(\partial_{ij}u(x_0))P^T\big) \le 0.$$

This is a contradiction.

For the general case $Lu \ge 0$, take $\varepsilon > 0$ and define

$$v(x) = u(x) + \varepsilon e^{Ax_1},$$

where $A > 0$ is chosen large. Then

$$Lv = Lu + \varepsilon L(e^{Ax_1}) = Lu + \varepsilon e^{Ax_1}(a_{11}A^2 + b_1 A + c).$$

Since $a_{11} \ge \lambda > 0$ and $b_i, c \in L^\infty(\Omega)$, choose $A \gg 1$ such that

$$a_{11}A^2 + b_1 A + c \ge \lambda A^2 - \|b_1\|_{L^\infty(\Omega)} A - \|c\|_{L^\infty(\Omega)} > 0.$$

Thus $Lv > 0$ in Ω. Applying the special case,

$$\max_{\overline{\Omega}} v \le \max_{\partial\Omega} v^+.$$

Since

$$v = u + \varepsilon e^{Ax_1},$$

we have

$$\max_{\overline{\Omega}} u \le \max_{\overline{\Omega}} v \le \max_{\partial\Omega} v^+ \le \max_{\partial\Omega} u^+ + \varepsilon \max_{\partial\Omega} e^{Ax_1}.$$

Because Ω is bounded, $\max_{\partial\Omega} e^{Ax_1} < \infty$. Letting $\varepsilon \to 0$, we get

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u^+.$$

□

Remark 3.2. If $c = 0$, then the stronger inequality holds:

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u.$$

Remark 3.3. The assumptions that Ω is bounded and $c \le 0$ are necessary.

Example 3.1 (Unbounded domain)**.** Let

$$\Omega = \{x \in \mathbb{R}^n : x_n > 0\}, \qquad u = x_n.$$

Then

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

but

$$\sup_{\Omega} u = +\infty > \sup_{\partial\Omega} u^+ = 0.$$

Example 3.2 (c not non-positive)**.** Let

$$\Omega = (0,\pi) \times \cdots \times (0,\pi) \subset \mathbb{R}^n, \qquad u = \prod_{i=1}^n \sin x_i.$$

Then

$$\Delta u + nu = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

but

$$\max_{\overline{\Omega}} u = 1 > \sup_{\partial\Omega} u^+ = 0.$$

Remark (Assumption ($*$)). Throughout the following corollaries, assume:

$$\Omega \text{ is a bounded domain in } \mathbb{R}^n,$$

$$L = \sum_{i,j=1}^n a_{ij}\partial_{ij} + \sum_{i=1}^n b_i\partial_i + c,$$

$$a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega), \qquad (a_{ij}) \geq \lambda I_n, \qquad c \leq 0.$$

Corollary 3.1. *Under the above assumption, if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$Lu \geq 0 \quad \textit{in } \Omega, \qquad u \leq 0 \quad \textit{on } \partial\Omega,$$

then

$$u \leq 0 \quad \textit{in } \Omega.$$

Proof. This is immediate from the weak maximum principle. □

Corollary 3.2 (Comparison principle)**.** *Under the above assumption, if $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$Lu \geq Lv \quad \textit{in } \Omega, \qquad u \leq v \quad \textit{on } \partial\Omega,$$

then

$$u \leq v \quad \textit{in } \Omega.$$

Proof. Define $w = u - v$. Then

$$Lw \geq 0 \quad \text{in } \Omega, \qquad w \leq 0 \quad \text{on } \partial\Omega.$$

The previous corollary gives $w \leq 0$, hence $u \leq v$. □

Corollary 3.3 (Uniqueness)**.** *Under the above assumption, for $f \in C(\Omega)$, $\varphi \in C(\partial\Omega)$, there exists at most one solution*

$$u \in C^2(\Omega) \cap C(\overline{\Omega})$$

of

$$\begin{cases} Lu = f & \textit{in } \Omega, \\ u = \varphi & \textit{on } \partial\Omega. \end{cases}$$

Proof. If u_1, u_2 are two solutions, then

$$Lu_1 = f = Lu_2 \quad \text{in } \Omega, \qquad u_1 = u_2 \quad \text{on } \partial\Omega.$$

The comparison principle gives $u_1 \leq u_2$ and $u_2 \leq u_1$, so $u_1 = u_2$. □

Corollary 3.4. *Under the above assumption, suppose $u, v, w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy*

$$Lu \geq f, \quad u = \varphi \text{ on } \partial\Omega,$$
$$Lv = f, \quad v = \varphi \text{ on } \partial\Omega,$$
$$Lw \leq f, \quad w = \varphi \text{ on } \partial\Omega.$$

Then

$$u \leq v \leq w \quad \text{in } \Omega.$$

Remark 3.4. This says:

$$\text{subsolution} \leq \text{solution} \leq \text{supersolution}.$$

3.4. **Hopf lemma.**

Theorem 3.3 (Hopf lemma)**.** *Let $B = B_R(y_0)$, and let $x_0 \in \partial B$. Let*

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_{ij} + \sum_{i=1}^{n} b_i\partial_i + c$$

with

$$(a_{ij}) \geq \lambda I_n, \qquad c \leq 0, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega).$$

If $u \in C^2(B) \cap C^1(\overline{B})$ satisfies

$$Lu \geq 0 \quad \text{in } B,$$
$$u(x) < u(x_0) \quad \forall x \in B,$$

and

$$u(x_0) \geq 0,$$

then

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

where ν is the unit outward normal to ∂B.

Remark 3.5. The non-strict inequality follows directly from the boundary maximum: for $t > 0$ small, $x_0 - t\nu \in B$, and

$$\frac{u(x_0) - u(x_0 - t\nu)}{t} \geq 0.$$

Letting $t \downarrow 0$ gives $\partial_\nu u(x_0) \geq 0$. The key point is strict positivity.

Proof. After translation, assume

$$B = B_R = B_R(0).$$

Define

$$V(x) = u(x) - u(x_0) + \varepsilon W(x), \qquad W(x) = e^{-A|x|^2} - e^{-AR^2},$$

where ε and A are constants to be determined. Write

$$D = B_R \setminus \overline{B_{R/2}}.$$

We claim that $A \gg 1$ and $\varepsilon \ll 1$ can be chosen so that

$$LV \geq 0 \quad \text{in } D, \qquad V \leq 0 \quad \text{on } \partial D.$$

Then the weak maximum principle implies

$$V \leq 0 \quad \text{in } D.$$

At x_0, since $|x_0| = R$,

$$V(x_0) = u(x_0) - u(x_0) + \varepsilon W(x_0) = 0.$$

Thus $V(x_0) = \max_{\overline{D}} V$, so

$$\frac{\partial V}{\partial \nu}(x_0) \geq 0.$$

Therefore

$$\frac{\partial u}{\partial \nu}(x_0) + \varepsilon \frac{\partial W}{\partial \nu}(x_0) \geq 0.$$

Since

$$\frac{\partial W}{\partial \nu}(x_0) = \nabla W(x_0) \cdot \frac{x_0}{R} = \sum_{i=1}^{n} (-2A) e^{-A|x_0|^2} x_i \frac{x_i}{R} = -2ARe^{-AR^2},$$

we get

$$\frac{\partial u}{\partial \nu}(x_0) \geq 2\varepsilon ARe^{-AR^2} > 0.$$

It remains to prove the claim.

First,

$$\partial_i W = -2Ae^{-A|x|^2} x_i,$$

and

$$\partial_{ij} W = 4A^2 e^{-A|x|^2} x_i x_j - 2Ae^{-A|x|^2} \delta_{ij}.$$

Hence

$$LW = e^{-A|x|^2} \left(4A^2 \sum_{i,j} a_{ij} x_i x_j - 2A \sum_i a_{ii} - 2A \sum_i b_i x_i + c \right) - c\, e^{-AR^2}.$$

Since $c \leq 0$,

$$LW \geq e^{-A|x|^2} \left(4A^2 \sum_{i,j} a_{ij} x_i x_j - 2A \sum_i a_{ii} - 2A \sum_i b_i x_i + c \right).$$

Using $(a_{ij}) \geq \lambda I_n$ and $R/2 \leq |x| \leq R$ in D,

$$LW \geq e^{-A|x|^2} \left(\lambda A^2 R^2 - 2A \sum_i a_{ii} - 2AR \sum_i |b_i| + c \right) \geq 0$$

by choosing $A \gg 1$. Therefore

$$LV = Lu - c\, u(x_0) + \varepsilon LW \geq 0 \qquad (u(x_0) \geq 0,\ c \leq 0)$$

in D.

Second, prove $V \leq 0$ on $\partial D = \partial B_{R/2} \cup \partial B_R$. Recall that $u(x) < u(x_0)$ in B_R. On $\partial B_{R/2}$, choose $\delta > 0$ such that

$$u(x) < u(x_0) - \delta.$$

Since

$$W(x) = e^{-A|x|^2} - e^{-AR^2} \leq e^{-A|x|^2} \leq 1,$$

we have

$$V(x) = u(x) - u(x_0) + \varepsilon W(x) \leq -\delta + \varepsilon \leq 0$$

for $\varepsilon \leq \delta$. On ∂B_R, $W(x) = 0$, so

$$V(x) = u(x) - u(x_0) \leq 0.$$

This proves the claim. □

Remark 3.6. If $c = 0$, then the assumption $u(x_0) \geq 0$ in Hopf's lemma can be removed.

4. Maximum Principle II

4.1. **Strong maximum principle.** Recall the weak maximum principle. Let Ω be a bounded domain in $\mathbb{R}^n$, and

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c,$$

where

$$a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega), \qquad (a_{ij}) \geq \lambda I_n, \qquad c \leq 0.$$

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ in Ω, then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+, \qquad u^+ = \max(u, 0).$$

If $c = 0$, then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u.$$

We also have comparison principles and uniqueness of solutions, and Hopf's lemma: if $B = B_R(y_0)$, $x_0 \in \partial B$, $Lu \geq 0$ in B,

$$u(x) < u(x_0) \quad \forall x \in B, \qquad u(x_0) \geq 0,$$

then

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

If $c = 0$, the assumption $u(x_0) \geq 0$ can be removed.

Theorem 4.1 (Strong maximum principle)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, and*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c,$$

with

$$(a_{ij}) \geq \lambda I_n, \qquad c \leq 0, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega).$$

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$Lu \geq 0 \quad \text{in } \Omega$$

and there exists $y_0 \in \Omega$ such that

$$u(y_0) = \max_{\overline{\Omega}} u^+,$$

then u is constant.

Proof. Let

$$M = \max_{\overline{\Omega}} u^+ \geq 0, \qquad D = \{x \in \Omega : u(x) = M\}.$$

Since $y_0 \in D$, $D \neq \varnothing$, and D is closed. We show that $D = \Omega$. Suppose not. Then $\Omega \setminus D \neq \varnothing$ is open. Choose

$$z_0 \in \Omega \setminus D \quad \text{such that} \quad \operatorname{dist}(z_0, D) < \operatorname{dist}(z_0, \partial\Omega).$$

Set $d_0 = \operatorname{dist}(z_0, D)$ and $B = B_{d_0}(z_0)$. Then

$$B \subset \Omega \setminus D,$$

and

$$\partial B \cap D \neq \varnothing.$$

Choose $x_0 \in \partial B \cap D$. Then

$$Lu \geq 0 \quad \text{in } B, \qquad u(x) < M = u(x_0) \quad \forall x \in B,$$

and $u(x_0) = M \geq 0$. By Hopf's lemma,

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

But x_0 is an interior maximum point of u, so

$$\nabla u(x_0) = 0, \qquad \frac{\partial u}{\partial \nu}(x_0) = 0,$$

a contradiction. Hence $D = \Omega$, so $u \equiv M$ in Ω. $\square$

Remark 4.1. When $c = 0$, the conclusion becomes: if there exists $y_0 \in \Omega$ such that

$$u(y_0) = \max_{\overline{\Omega}} u,$$

then u is constant.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c,$$

where $(a_{ij}) \geq \lambda I_n$ and $a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega)$. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$Lu \geq 0, \qquad u \leq 0 \quad \textit{in } \Omega,$$

then either

$$u < 0 \quad \textit{in } \Omega,$$

or

$$u \equiv 0 \quad \textit{in } \Omega.$$

There is no restriction on c, but we need $u \leq 0$ in Ω.

Proof. Let

$$c^+ = \max(c, 0), \qquad c^- = \max(-c, 0),$$

so $c = c^+ - c^-$. From $Lu \geq 0$ and $u \leq 0$,

$$\sum_{i,j} a_{ij}\partial_{ij}u + \sum_i b_i\partial_i u - c^- u \geq -c^+ u \geq 0.$$

Define

$$\widetilde{L} = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i - c^-.$$

Then

$$\widetilde{L}u \geq 0, \qquad u \leq 0 \quad \text{in } \Omega,$$

and $\widetilde{L}$ has non-positive zero order coefficient. Hence

$$\max_{\overline{\Omega}} u^+ = 0.$$

If $u(y) < 0$ for all $y \in \Omega$, we are done. If there exists $y_0 \in \Omega$ such that $u(y_0) = 0$, then the strong maximum principle applied to $\widetilde{L}$ gives $u \equiv 0$. $\square$

4.2. **A priori estimates.** The motivation is the continuity method. To solve a PDE, one often constructs a sequence of functions u_j. If $\{u_j\}$ has a convergent subsequence, then the solution is the limit. A priori estimates of the form

$$\|u_j\|_{C^k} \leq C$$

with C independent of j, together with Arzela–Ascoli, provide the convergent subsequence.

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let L be an elliptic operator satisfying*

$$(a_{ij}) \geq \lambda I_n, \qquad c \leq 0, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega).$$

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$Lu \geq f \quad \text{in } \Omega$$

for $f \in L^\infty(\Omega) \cap C(\Omega)$, then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ + C \max_{\Omega} f^-,$$

where

$$C = C\big(n, \lambda, \operatorname{diam}(\Omega), \|b_i\|_{L^\infty(\Omega)}\big).$$

Proof. Let $d = \operatorname{diam}(\Omega)$. After translation, assume

$$\Omega \subset \{0 < x_1 < d\}.$$

Write

$$M = \max_{\partial\Omega} u^+ \geq 0, \qquad F = \max_{\Omega} f^- \geq 0.$$

Then

$$Lu \geq f \geq -F \quad \text{in } \Omega, \qquad u \leq M \quad \text{on } \partial\Omega.$$

Define

$$v(x) = M + d^2 F \left(e^A - e^{Ax_1/d}\right),$$

where $A > 0$ is to be chosen. Compute

$$Lv = -(a_{11}A^2 + db_1A)Fe^{Ax_1/d} + cM + cd^2F\left(e^A - e^{Ax_1/d}\right).$$

Since $c \leq 0$,

$$Lv \leq -(a_{11}A^2 + db_1A)Fe^{Ax_1/d} \leq -(\lambda A^2 + d\,|b_1|\,A)Fe^{Ax_1/d}.$$

Choose $A = A(\lambda, d, \|b_1\|_{L^\infty(\Omega)})$ such that

$$\lambda A^2 - d\,\|b_1\|_{L^\infty(\Omega)}\,A \geq 1.$$

Then

$$Lv \leq -F \quad \text{in } \Omega.$$

Also

$$v(x) = M + d^2F\left(e^A - e^{Ax_1/d}\right) \geq M \quad \text{on } \overline{\Omega},$$

since $0 < x_1 < d$. Thus

$$Lv \leq -F \leq f \leq Lu \quad \text{in } \Omega, \qquad v \geq M \geq u \quad \text{on } \partial\Omega.$$

By comparison,

$$u \leq v \quad \text{in } \Omega.$$

Therefore

$$\max_{\overline{\Omega}} u \leq M + d^2Fe^A = \max_{\partial\Omega} u^+ + C\max_{\Omega} f^-,$$

with $C = d^2 e^A$. $\square$

Corollary 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and L an elliptic operator with*

$$(a_{ij}) \geq \lambda I_n, \qquad c \leq 0, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega).$$

If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^\infty(\Omega) \cap C(\Omega)$ and $\varphi \in C(\partial\Omega)$, then

$$\max_{\overline{\Omega}} |u| \leq \max_{\partial\Omega} |\varphi| + C \max_{\Omega} |f| .$$

Proof. Apply the previous theorem to u and to $-u$. $\square$

Proposition 4.4. *Let $B = B_R(y_0) \subset \mathbb{R}^n$, and let L be an elliptic operator with*

$$(a_{ij}) \geq \lambda I_n, \qquad c \leq 0, \qquad a_{ij}, b_i, c \in L^\infty(B) \cap C(B).$$

Let $\beta \in (0,1)$, $f \in C(B)$, and

$$\sup_{x \in B} d_x^{2-\beta} |f(x)| < +\infty, \qquad d_x = \operatorname{dist}(x, \partial B).$$

If $u \in C^2(B) \cap C(\overline{B})$ satisfies

$$\begin{cases} Lu = f & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

then

$$\sup_{x \in B} d_x^{-\beta} |u(x)| \leq C \sup_{x \in B} d_x^{2-\beta} |f(x)| ,$$

where

$$C = C(n, \beta, \lambda, R, \|b_i\|_{L^\infty(B)}).$$

Proof. After translation assume $B = B_R(0) = B_R$. Set

$$F = \sup_{x \in B_R} d_x^{2-\beta} |f(x)| , \qquad d_x = R - |x| .$$

The goal is

$$\sup_{x \in B_R} d_x^{-\beta} |u(x)| \leq CF,$$

equivalently, for every $x_0 \in B_R$,

$$|u(x_0)| \leq C F d_{x_0}^{\beta}.$$

Fix $x_0 \in B_R$. By rotation assume

$$x_0 = (r_0, 0, \ldots, 0), \qquad r_0 \geq 0,$$

so

$$d_{x_0} = R - r_0.$$

Define

$$W_1(x) = R^{-\beta}(R^2 - |x|^2)^\beta, \qquad W_2(x) = R^\beta \left(e^A - e^{Ax_1/R} \right),$$

and

$$W = \gamma_1 W_1 + \gamma_2 W_2.$$

The claim is that A, γ_1, γ_2 can be chosen so that

$$\begin{cases} LW \le -(R-|x|)^{\beta-2} & \text{in } B_R, \\ W \ge 0 & \text{on } \partial B_R. \end{cases}$$

Since

$$F = \sup_{x \in B_R} d_x^{2-\beta} |f(x)|,$$

we have

$$|f(x)| = d_x^{2-\beta} |f(x)| \, d_x^{\beta-2} \le F d_x^{\beta-2} = F(R-|x|)^{\beta-2}.$$

Thus

$$L(\pm u) = \pm f \ge -F(R-|x|)^{\beta-2} \ge L(FW) \quad \text{in } B_R,$$

and

$$\pm u = 0 \le FW \quad \text{on } \partial B_R.$$

By the comparison principle,

$$|u| \le FW \quad \text{in } B_R.$$

At $x_0 = (r_0, 0, \dots, 0)$,

$$|u(x_0)| \le FW(x_0) = F\gamma_1 R^{-\beta}(R^2 - r_0^2)^\beta + F\gamma_2 R^\beta \left(e^A - e^{Ar_0/R}\right).$$

For the first term,

$$I_1 = F\gamma_1 R^{-\beta}(R+r_0)^\beta (R-r_0)^\beta \le F\gamma_1 R^{-\beta}(2R)^\beta d_{x_0}^\beta \le CF d_{x_0}^\beta.$$

For the second term, use

$$e^{Ax} - 1 \le Ae^A x, \qquad x \in [0,1].$$

Then

$$I_2 = F\gamma_2 R^\beta e^{Ar_0/R}\left(e^{A(1-r_0/R)} - 1\right) \le F\gamma_2 R^\beta e^A A e^A \frac{R-r_0}{R} \le CF d_{x_0}^\beta.$$

Therefore

$$|u(x_0)| \le CF d_{x_0}^\beta,$$

which is the desired estimate.

It remains to prove the claim. Recall

$$W_1(x) = R^{-\beta}(R^2 - |x|^2)^\beta, \qquad W_2(x) = R^\beta \left(e^A - e^{Ax_1/R}\right).$$

Choose $W = \gamma_1 W_1 + \gamma_2 W_2$ after choosing A, γ_1, γ_2 suitably.

Step 1: estimate W_1. We compute

$$\partial_i W_1(x) = -2\beta R^{-\beta}(R^2 - |x|^2)^{\beta-1} x_i,$$

$$\partial_{ij} W_1(x) = 4\beta(\beta-1) R^{-\beta}(R^2 - |x|^2)^{\beta-2} x_i x_j - 2\beta R^{-\beta}(R^2 - |x|^2)^{\beta-1}\delta_{ij}.$$

Hence

$$LW_1(x) = \beta R^{-\beta}(R^2 - |x|^2)^{\beta-2}\Bigg[4(\beta-1)\sum_{i,j} a_{ij}x_i x_j - 2(R^2 - |x|^2)\sum_i a_{ii}$$
$$-2(R^2 - |x|^2)\sum_i b_i x_i + cR^{-\beta}(R^2 - |x|^2)^2\Bigg].$$

Let

$$M_b = \max_i \|b_i\|_{L^\infty(B_R)}.$$

Choose $\theta \in (0,1)$, close to 1, such that

$$2nM_b R(1-\theta^2) \le 3(1-\beta)\lambda\theta^2.$$

If $\theta R \le |x| < R$, then

$$\begin{aligned} LW_1(x) &\le \beta R^{-\beta}(R^2-|x|^2)^{\beta-2}\left[-(1-\beta)\lambda\,|x|^2 - 3(1-\beta)\lambda\theta^2R^2 + 2nM_bR(1-\theta^2)R^2\right] \\ &\le -\beta R^{-\beta}(R-|x|)^{\beta-2}(R+|x|)^{\beta-2}(1-\beta)\lambda\,|x|^2 \le -C_1^{-1}(R-|x|)^{\beta-2}. \end{aligned}$$

If $|x| < \theta R$, then

$$LW_1(x) \le \beta R^{-\beta}(R^2-|x|^2)^{\beta-2}\,2nM_bR(R^2-|x|^2) \le C_2(R-|x|)^{\beta-2}.$$

Thus

$$LW_1(x) \le \begin{cases} -C_1^{-1}(R-|x|)^{\beta-2}, & \theta R \le |x| < R, \\ C_2(R-|x|)^{\beta-2}, & |x| < \theta R. \end{cases}$$

Step 2: estimate W_2. Since

$$W_2(x) = R^\beta\left(e^A - e^{Ax_1/R}\right),$$

we have, by choosing $A \gg 1$,

$$LW_2(x) = -(a_{11}A^2+b_1RA)R^{\beta-2}e^{Ax_1/R} + cR^\beta\left(e^A - e^{Ax_1/R}\right) \le -R^{\beta-2}e^{Ax_1/R} \le -e^{-A}R^{\beta-2}.$$

If $\theta R \le |x| < R$, this gives only

$$LW_2(x) \le 0.$$

If $|x| < \theta R$, then

$$LW_2(x) \le -e^{-A}(1-\theta)^{2-\beta}(R-\theta R)^{\beta-2} \le -C_3^{-1}(R-|x|)^{\beta-2}.$$

Thus

$$LW_2(x) \le \begin{cases} 0, & \theta R \le |x| < R, \\ -C_3^{-1}(R-|x|)^{\beta-2}, & |x| < \theta R. \end{cases}$$

Choose

$$\gamma_1 = C_1, \qquad \gamma_2 = (C_1C_2+1)C_3.$$

Then

$$W = \gamma_1 W_1 + \gamma_2 W_2$$

satisfies

$$\begin{cases} LW \le -(R-|x|)^{\beta-2} & \text{in } B_R, \\ W \ge 0 & \text{on } \partial B_R. \end{cases}$$

This proves the claim, and hence the proposition. □

5. Poisson's Equation and Newtonian Potentials I

Recall. We have proved the strong maximum principle. We also applied the maximum principle to establish some a priori estimates. The key point was to construct a suitable auxiliary function; for example

$$W = \gamma_1 W_1 + \gamma_2 W_2, \qquad W_1 = R^{-\beta}(R^2-|x|^2)^\beta, \qquad W_2 = R^\beta\left(e^A - e^{Ax_1/R}\right).$$

1. Poisson's Equation. Let Ω be a domain in $\mathbb{R}^n$. The equation

$$\Delta u = f \qquad \text{in } \Omega$$

is called Poisson's equation.

Remark. If $u \in C^2(\Omega)$, then $\Delta u \in C(\Omega)$. Hence it is natural to ask: given $f \in C(\Omega)$, is there a C^2-solution of Poisson's equation?

The answer is no: there exists $f \in C(\Omega)$ such that $\Delta u = f$ does not admit any C^2-solution. Thus $C(\Omega)$ is not a good function space for Poisson's equation.

2. Holder Spaces. Let Ω be a bounded domain in $\mathbb{R}^n$, let $\alpha \in (0,1)$, and let f be a function defined in Ω.

Definition.

(1) For every $\Omega' \Subset \Omega$, if there exists a constant $C_{\Omega'}$, possibly depending on Ω', such that

$$|f(x) - f(y)| \leq C_{\Omega'} |x-y|^\alpha \qquad \forall x, y \in \Omega',$$

then f is called locally Holder continuous with respect to α in Ω. The set of all such functions is denoted by $C^\alpha(\Omega)$.

(2) If there exists a constant C such that

$$|f(x) - f(y)| \leq C |x-y|^\alpha \qquad \forall x, y \in \Omega,$$

then f is called uniformly Holder continuous with respect to α in Ω. The set of all such functions is denoted by $C^\alpha(\overline{\Omega})$.

Remark. If $f \in C^\alpha(\Omega)$, then $f \in C(\Omega)$. If $f \in C^\alpha(\overline{\Omega})$, then f can be extended continuously to $\overline{\Omega}$.

Definition. For $f \in C^\alpha(\overline{\Omega})$, define its Holder seminorm by

$$[f]_{C^\alpha(\overline{\Omega})} = [f]_{C^\alpha(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

For $k \in \mathbb{Z}_{\geq 0}$, define

$$C^{k,\alpha}(\Omega) = \left\{ u \in C^k(\Omega) \mid \nabla^k u \in C^\alpha(\Omega) \right\},$$

$$C^{k,\alpha}(\overline{\Omega}) = \left\{ u \in C^k(\overline{\Omega}) \mid \nabla^k u \in C^\alpha(\overline{\Omega}) \right\}.$$

For $u \in C^{k,\alpha}(\overline{\Omega})$, define

$$\|u\|_{C^k(\Omega)} = \sum_{i=0}^k \sup_\Omega \left|\nabla^i u\right|, \qquad \|u\|_{C^{k,\alpha}(\Omega)} = \|u\|_{C^k(\Omega)} + [\nabla^k u]_{C^\alpha(\Omega)}.$$

Remark.

$$\left(C^k(\overline{\Omega}), \|\cdot\|_{C^k(\Omega)}\right), \qquad \left(C^{k,\alpha}(\overline{\Omega}), \|\cdot\|_{C^{k,\alpha}(\Omega)}\right)$$

are Banach spaces.

3. Newtonian Potential. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $f \in L^\infty(\Omega)$.

Definition. The Newtonian potential of f in Ω is defined by

$$W_{\Omega,f}(x) = \int_\Omega \Gamma(x-y)f(y)\,dy, \qquad x \in \mathbb{R}^n,$$

where Γ is the fundamental solution

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & n = 2, \\ \frac{1}{n(2-n)\omega_n}|x|^{2-n}, & n \geq 3. \end{cases}$$

Remark. Recall that $\Gamma(x-\cdot)$ is L^1-integrable near x by the fundamental-solution estimates above. Hence $W = W_{\Omega,f}$ is well-defined in $\mathbb{R}^n$.

Recall Green's identity:

$$u(x) = \int_\Omega \Gamma(x-y)\Delta u(y)\,dy - \int_{\partial\Omega}\left[\Gamma(x-y)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial\Gamma}{\partial\nu_y}(x-y)\right]dS_y.$$

The Newtonian potential W is the first term of the right hand side of Green's identity.

Lemma. *Let $\Omega = B_1$, and let $f \in L^\infty(B_1)$. Set*

$$W(x) = \int_{B_1}\Gamma(x-y)f(y)\,dy.$$

Then

(1) $W \in C^1(B_1)$, and

$$\partial_i W(x) = \int_{B_1}\partial_{x_i}\Gamma(x-y)f(y)\,dy.$$

(2)

$$\|W\|_{C^1(B_1)} \leq C(n)\|f\|_{L^\infty(B_1)}.$$

Proof. Set

$$V_i(x) = \int_{B_1}\partial_{x_i}\Gamma(x-y)f(y)\,dy, \qquad x \in B_1.$$

Since

$$|\partial_{x_i}\Gamma(x-y)| \leq C(n)|y-x|^{1-n},$$

we have $\partial_{x_i}\Gamma(x-y) \in L^1(B_1)$. Indeed, if $R > 0$ and $B_R(x) \Subset B_1$, then

$$\int_{B_1}|\partial_i\Gamma(x-y)|\,dy = \int_{B_R(x)}|\partial_i\Gamma(x-y)|\,dy + \int_{B_1\setminus B_R(x)}|\partial_i\Gamma(x-y)|\,dy,$$

and the only possibly singular part is

$$C(n)\int_{B_R(x)}|y-x|^{1-n}\,dy = C(n)\int_0^R\int_{\partial B_r(x)}|y-x|^{1-n}\,dS_y\,dr = C(n)\int_0^R n\omega_n r^{n-1}r^{1-n}\,dr < \infty.$$

Thus V_i is well-defined.

Step 1: construct a smooth approximation W_ε of W. Choose $\eta \in C^\infty(\mathbb{R})$ such that

$$0 \leq \eta \leq 1, \qquad \eta = 0 \text{ in } (-\infty, 1], \qquad \eta = 1 \text{ in } [2,\infty),$$
$$0 \leq \eta' \leq 100 \quad \text{in } \mathbb{R}.$$

For every $\varepsilon > 0$, define

$$W_\varepsilon(x) = \int_{B_1} \Gamma_\varepsilon(x-y) f(y)\, dy, \qquad \Gamma_\varepsilon(x-y) = \Gamma(x-y)\eta_\varepsilon, \qquad \eta_\varepsilon = \eta\left(\frac{|x-y|}{\varepsilon}\right).$$

Then $W_\varepsilon \in C^\infty(\overline{B}_1)$, and

$$\partial_i W_\varepsilon(x) = \int_{B_1} \partial_{x_i}\left(\Gamma\, \eta_\varepsilon\right) f(y)\, dy.$$

Step 2: as $\varepsilon \to 0$,

$$W_\varepsilon \to W \quad \text{in } C^0(\overline{B}_1), \qquad \partial_i W_\varepsilon \to V_i \quad \text{in } C^0(\overline{B}_1).$$

Then conclusion (1) follows from Step 2.

It remains to prove the convergence. We only write the proof for $n \geq 3$; the case $n = 2$ is similar. First,

$$\begin{aligned}
|W_\varepsilon(x) - W(x)| &\leq \int_{B_1} |\Gamma(1-\eta_\varepsilon)|\, |f(y)|\ dy \\
&\leq \|f\|_{L^\infty(B_1)} \int_{B_{2\varepsilon}(x)} |\Gamma|\ dy \\
&\leq C(n)\, \|f\|_{L^\infty(B_1)} \int_0^{2\varepsilon} r^{2-n} r^{n-1}\, dr \leq C(n)\, \|f\|_{L^\infty(B_1)}\, \varepsilon^2 \to 0.
\end{aligned}$$

Also,

$$\begin{aligned}
|\partial_i W_\varepsilon(x) - V_i(x)| &\leq \|f\|_{L^\infty(B_1)} \int_{B_{2\varepsilon}(x)} |\partial_{x_i}\left(\Gamma(1-\eta_\varepsilon)\right)|\ dy \\
&\leq \|f\|_{L^\infty(B_1)} \int_{B_{2\varepsilon}(x)} \left(|\partial_{x_i}\Gamma| + \frac{100}{\varepsilon}\,|\Gamma|\right)\, dy \\
&\leq C(n)\, \|f\|_{L^\infty(B_1)}\, \varepsilon \to 0.
\end{aligned}$$

Thus Step 2 holds.

For (2), for every $x \in B_1$,

$$|W(x)| \leq \|f\|_{L^\infty(B_1)} \int_{B_1} |\Gamma|\ dy \leq C(n)\, \|f\|_{L^\infty(B_1)}\,,$$

and

$$|\partial_i W(x)| \leq \|f\|_{L^\infty(B_1)} \int_{B_1} |\nabla\Gamma|\ dy \leq C(n)\, \|f\|_{L^\infty(B_1)}\,.$$

Therefore

$$\|W\|_{C^1(B_1)} \leq C(n)\, \|f\|_{L^\infty(B_1)}\,.$$

□

Lemma. *Let $\Omega = B_1$, and let $f \in C^\alpha(\overline{B}_1)$ for some $\alpha \in (0,1)$. Then*

(1) $W \in C^2(B_1)$, and

$$\begin{aligned}
\partial_{ij} W(x) = &\int_{B_1} \partial_{x_i x_j}\Gamma(x-y)\big(f(y) - f(x)\big)\, dy \\
&- f(x)\int_{\partial B_1} \partial_{x_j}\Gamma(x-y)\nu_i(y)\, dS_y. \qquad (*)
\end{aligned}$$

Moreover, $\Delta W = f$.

(2)

$$\|W\|_{C^2(B_{1/2})} \le C(\alpha,n)\,\|f\|_{C^\alpha(B_1)}\,.$$

Proof. For (1), set $V_{ij}(x)$ to be the right hand side of $(*)$. Since $f \in C^\alpha(\overline{B}_1)$,

$$|f(y)-f(x)| \le [f]_{C^\alpha(B_1)}\,|x-y|^\alpha\,.$$

Hence

$$\left|\partial_{x_i x_j}\Gamma(x-y)(f(y)-f(x))\right| \le C(n)[f]_{C^\alpha(B_1)}\,|x-y|^{\alpha-n} \in L^1(B_1),$$

because

$$\int_{B_R(x)} |x-y|^{\alpha-n}\;dy = \int_0^R C(n) r^{n-1} r^{\alpha-n}\,dr < \infty.$$

Thus V_{ij} is well-defined in B_1. Notice that $|x-y|^{-n} \notin L^1(B_1)$, so the assumption $f \in C^\alpha(\overline{B}_1)$ is important.

Step 1: smooth approximation $V_{j\varepsilon}$ of $\partial_j W$. For every $\varepsilon > 0$, define

$$V_{j\varepsilon}(x) = \int_{B_1} \partial_{x_j}\Gamma\,\eta_\varepsilon f(y)\,dy,$$

where η_ε is the same function as in the previous lemma. Then

$$\begin{aligned}
\partial_i V_{j\varepsilon}(x) &= \int_{B_1} \partial_{x_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right) f(y)\,dy \\
&= \int_{B_1} \partial_{x_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right)\left(f(y)-f(x)\right) dy \\
&\qquad + f(x)\int_{B_1} \partial_{x_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right)\,dy \\
&= \int_{B_1} \partial_{x_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right)\left(f(y)-f(x)\right) dy \\
&\qquad - f(x)\int_{B_1} \partial_{y_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right)\,dy.
\end{aligned}$$

Since $\partial_{x_j}\Gamma\,\eta_\varepsilon$ depends only on $x-y$,

$$\partial_{x_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right) = -\partial_{y_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right).$$

Therefore

$$\begin{aligned}
\partial_i V_{j\varepsilon}(x) &= \int_{B_1} \partial_{x_i}\left(\partial_{x_j}\Gamma\,\eta_\varepsilon\right)\left(f(y)-f(x)\right) dy \\
&\qquad - \int_{\partial B_1} \partial_{x_j}\Gamma\,\eta_\varepsilon\,\nu_i\,dS_y.
\end{aligned}$$

Step 2: for every $r \in (0,1)$, as $\varepsilon \to 0$,

$$V_{j\varepsilon} \to \partial_j W, \qquad \partial_i V_{j\varepsilon} \to V_{ij} \quad \text{in } C^0(\overline{B}_r).$$

Indeed,

$$|V_{j\varepsilon}(x) - \partial_j W(x)| \le \|f\|_{L^\infty(B_1)} \int_{B_{2\varepsilon}(x)} \left|\partial_{x_j}\Gamma(1-\eta_\varepsilon)\right|\,dy \le C(n)\,\|f\|_{L^\infty(B_1)}\,\varepsilon \to 0.$$

If $\varepsilon < \frac{1}{2}(1-r)$, then $\eta_\varepsilon = 1$ on ∂B_1 for $x \in B_r$. Thus

$$\begin{aligned}
|\partial_i V_{j\varepsilon}(x) - V_{ij}(x)| &\le \int_{B_{2\varepsilon}(x)} \left|\partial_{x_i}\left(\partial_{x_j}\Gamma(1-\eta_\varepsilon)\right)\right| |f(y)-f(x)|\, dy \\
&\le [f]_{C^\alpha(B_1)} \int_{B_{2\varepsilon}(x)} \left(\left|\partial_{x_i x_j}\Gamma\right| + \frac{100}{\varepsilon}\left|\partial_{x_j}\Gamma\right|\right) |y-x|^\alpha \, dy \\
&\le C(n,\alpha)[f]_{C^\alpha(B_1)}\varepsilon^\alpha \to 0.
\end{aligned}$$

Hence Step 2 gives $W \in C^2(B_1)$ and $\partial_{ij}W = V_{ij}$.

Step 3: $\Delta W = f$. Summing the formula for V_{ii}, we obtain

$$\begin{aligned}
\Delta W(x) &= \int_{B_1} \Delta_x \Gamma(x-y)\big(f(y)-f(x)\big)\, dy - f(x)\int_{\partial B_1} \partial_{x_i}\Gamma(x-y)\nu_i(y)\, dS_y \\
&= 0 + f(x)\int_{\partial B_1} \partial_{\nu_y}\Gamma(x-y)\, dS_y = f(x),
\end{aligned}$$

where the last identity is the normalization of the fundamental solution.

For (2), let $x \in B_{1/2}$. From $(*)$,

$$\begin{aligned}
|\partial_{ij}W(x)| \le & \int_{B_1} \left|\partial_{x_i x_j}\Gamma(x-y)\right| |f(y)-f(x)|\, dy \\
&+ |f(x)| \int_{\partial B_1} \left|\partial_{x_j}\Gamma(x-y)\right|\, dS_y \\
&\le C(n)[f]_{C^\alpha(B_1)} \int_{B_1} |y-x|^{\alpha-n}\, dy + C(n)\left\|f\right\|_{L^\infty(B_1)} \int_{\partial B_1} |y-x|^{1-n}\, dS_y \\
&\le C(\alpha,n)[f]_{C^\alpha(B_1)} + C(n)\left\|f\right\|_{L^\infty(B_1)} \le C(\alpha,n)\left\|f\right\|_{C^\alpha(B_1)}.
\end{aligned}$$

Combining this with

$$\left\|W\right\|_{C^1(B_1)} \le C(n)\left\|f\right\|_{L^\infty(B_1)},$$

from the previous lemma gives

$$\left\|W\right\|_{C^2(B_{1/2})} \le C(\alpha,n)\left\|f\right\|_{C^\alpha(B_1)}.$$

□

6. Poisson's Equation and Newtonian Potentials II

Recall. We recall the main objects from the preceding lecture:

$$\Delta u = f, \qquad W = W_{\Omega,f} = \int_\Omega \Gamma(x-y)f(y)\, dy.$$

When $\Omega = B_1$ and $f \in C^\alpha(\overline{B}_1)$, we proved

$$W \in C^1(\overline{B}_1), \qquad \partial_i W(x) = \int_{B_1} \partial_{x_i}\Gamma(x-y)f(y)\, dy,$$

and

$$\left\|W\right\|_{C^1(B_1)} \le C(n)\left\|f\right\|_{L^\infty(B_1)}.$$

We also proved

$$W \in C^2(B_{1/2}),$$

$$\partial_{ij}W(x) = \int_{B_1} \partial_{x_i x_j}\Gamma(x-y)\big(f(y)-f(x)\big)\, dy - f(x)\int_{\partial B_1} \partial_{x_j}\Gamma(x-y)\nu_i\, dS_y,$$

and

$$\|W\|_{C^2(B_{1/2})} \le C(\alpha, n) \|f\|_{C^\alpha(B_1)}.$$

How should one understand the second derivative formula? Formally one would write

$$\partial_{ij} W(x) \text{ “} = \text{”} \int_{B_1} \partial_{x_i x_j} \Gamma(x-y) f(y)\, dy.$$

But $\nabla^2\Gamma \sim |x-y|^{-n}$, and for $x \in B_{1/2}$,

$$\int_{B_1} \left|\nabla^2\Gamma(x-y)\right| \, dy \ge \frac{1}{C(n)} \int_{B_{1/2}(x)} |x-y|^{-n} \, dy = \frac{1}{C(n)} \int_0^{1/2} n\omega_n r^{n-1} r^{-n} \, dr = +\infty.$$

Thus the expression must be rewritten as

$$\begin{aligned} \partial_{ij} W(x) \text{ “} = \text{”} & \int_{B_1} \partial_{x_i x_j} \Gamma(x-y) \big(f(y) - f(x)\big)\, dy \\ & + f(x) \int_{B_1} \partial_{x_i x_j} \Gamma(x-y)\, dy. \end{aligned}$$

The second term is interpreted by integration by parts:

$$f(x) \int_{B_1} \partial_{x_i x_j} \Gamma(x-y)\, dy = -f(x) \int_{B_1} \partial_{y_i} \partial_{x_j} \Gamma(x-y)\, dy = -f(x) \int_{\partial B_1} \partial_{x_j} \Gamma(x-y) \nu_i \, dS_y.$$

1. $C^{2,\alpha}$-Regularity and Estimate of W.

Lemma. *Let $\Omega = B_1$, and let $f \in C^\alpha(\overline{B}_1)$ for some $\alpha \in (0,1)$. Then*

$$W \in C^{2,\alpha}(\overline{B}_{1/2}), \qquad \|W\|_{C^{2,\alpha}(B_{1/2})} \le C(\alpha, n) \|f\|_{C^\alpha(B_1)}.$$

Proof. Let $x', x'' \in B_{1/2}$, and set

$$r = |x' - x''|.$$

It is enough to prove

$$|\partial_{ij} W(x') - \partial_{ij} W(x'')| \le C(\alpha, n) \|f\|_{C^\alpha(B_1)} \, r^\alpha.$$

Together with the C^2-estimate for Newtonian potentials above, this gives the desired estimate.

Case 1: $r \ge \frac{1}{4}$. Then

$$\begin{aligned} |\partial_{ij} W(x') - \partial_{ij} W(x'')| &\le 2 \|W\|_{C^2(B_{1/2})} \le 2 \|W\|_{C^2(B_{1/2})} \, 4^\alpha r^\alpha \\ &\le C(\alpha, n) \|f\|_{C^\alpha(B_1)} \, r^\alpha. \end{aligned}$$

Case 2: $r < \frac{1}{4}$. Let

$$x^* = \frac{1}{2}(x' + x'')$$

be the midpoint. Using the second-derivative formula for the Newtonian potential at x' and x'', write

$$\partial_{ij} W(x') - \partial_{ij} W(x'') = \sum_{k=1}^{6} I_k,$$

where

$$I_1 = f(x'') \int_{\partial B_1} \big(\partial_{x_i} \Gamma(x''-y) - \partial_{x_i} \Gamma(x'-y)\big) \nu_j \, dS_y,$$

$$I_2 = \big(f(x') - f(x'')\big) \int_{\partial B_1} \partial_{x_i}\Gamma(x' - y)\nu_j \, dS_y,$$

$$I_3 = \int_{B_r(x^*)} \partial_{x_i x_j}\Gamma(x' - y)\big(f(y) - f(x')\big)\, dy,$$

$$I_4 = \int_{B_r(x^*)} \partial_{x_i x_j}\Gamma(x'' - y)\big(f(x'') - f(y)\big)\, dy,$$

$$I_5 = \big(f(x'') - f(x')\big) \int_{B_1 \setminus B_r(x^*)} \partial_{x_i x_j}\Gamma(x' - y)\, dy,$$

and

$$I_6 = \int_{B_1 \setminus B_r(x^*)} \big(\partial_{x_i x_j}\Gamma(x' - y) - \partial_{x_i x_j}\Gamma(x'' - y)\big)\big(f(y) - f(x'')\big)\, dy.$$

The new goal is

$$\sum_{k=1}^{6} |I_k| \le C(\alpha, n) \, \|f\|_{C^\alpha(B_1)} \, r^\alpha.$$

For I_1, the mean value theorem gives a point $\overline{x} \in \overline{x'x''}$ such that

$$|\partial_{x_i}\Gamma(x'' - y) - \partial_{x_i}\Gamma(x' - y)| \le |\nabla \partial_{x_i}\Gamma(\overline{x} - y)| \, |x' - x''| \, .$$

Hence

$$\begin{aligned}
|I_1| &\le \|f\|_{L^\infty(B_1)} \, r \int_{\partial B_1} \left|\nabla^2\Gamma(\overline{x} - y)\right| \, dS_y \\
&\le \|f\|_{L^\infty(B_1)} \, r^{1-\alpha} r^\alpha \int_{\partial B_1} C(n) \, |\overline{x} - y|^{-n} \, dS_y \\
&\le C(\alpha, n) \, \|f\|_{C^\alpha(B_1)} \, r^\alpha,
\end{aligned}$$

because $x', x'' \in B_{1/2}$, $\overline{x} \in B_{1/2}$, and $r < \frac{1}{4}$.

For I_2,

$$|I_2| \le \|f\|_{C^\alpha(B_1)} \, r^\alpha \int_{\partial B_1} |\partial_{x_i}\Gamma(x' - y)| \, dS_y \le C(n) \, \|f\|_{C^\alpha(B_1)} \, r^\alpha.$$

For I_3, since $B_r(x^*) \subset B_{3r/2}(x') \subset B_1$,

$$\begin{aligned}
|I_3| &\le \int_{B_{3r/2}(x')} \left|\partial_{x_i x_j}\Gamma(x' - y)\right| |f(y) - f(x')| \, dy \\
&\le C(n) \, \|f\|_{C^\alpha(B_1)} \int_{B_{3r/2}(x')} |y - x'|^{-n+\alpha} \, dy \\
&\le C(\alpha, n) \, \|f\|_{C^\alpha(B_1)} \, r^\alpha.
\end{aligned}$$

The same argument gives

$$|I_4| \le C(\alpha, n) \, \|f\|_{C^\alpha(B_1)} \, r^\alpha.$$

For I_5, we first estimate

$$\begin{aligned}\left|\int_{B_1\setminus B_r(x^*)} \partial_{x_ix_j}\Gamma(x'-y)\,dy\right| &= \left|-\int_{B_1\setminus B_r(x^*)} \partial_{y_i}\partial_{x_j}\Gamma(x'-y)\,dy\right|\\ &= \left|-\int_{\partial(B_1\setminus B_r(x^*))} \partial_{x_j}\Gamma(x'-y)\nu_i\,dS_y\right|\\ &\le \int_{\partial B_r(x^*)} \left|\partial_{x_j}\Gamma(x'-y)\right|\,dS_y + \int_{\partial B_1}\left|\partial_{x_j}\Gamma(x'-y)\right|\,dS_y.\end{aligned}$$

For $y\in\partial B_r(x^*)$, one has

$$\frac{r}{2}\le |x'-y|\le \frac{3}{2}r,$$

and for $x'\in B_{1/2}$, $y\in\partial B_1$, one has $|x'-y|\ge\frac{1}{2}$.

Therefore

$$\int_{\partial B_r(x^*)} C(n)r^{1-n}\,dS_y + \int_{\partial B_1} C(n)\,dS_y \le C(n),$$

and consequently

$$|I_5|\le |f(x')-f(x'')|\,C(n)\le C(n)\,\|f\|_{C^\alpha(B_1)}\,r^\alpha.$$

It remains to estimate I_6. By the mean value theorem, there exists $\widehat{x}\in\overline{x'x''}$ such that

$$\begin{aligned}&\left|\partial_{x_ix_j}\Gamma(x'-y)-\partial_{x_ix_j}\Gamma(x''-y)\right|\\ &\quad\le \left|\nabla\partial_{x_ix_j}\Gamma(\widehat{x}-y)\right|\,|x'-x''|\le C(n)\frac{r}{|y-\widehat{x}|^{n+1}}.\end{aligned}$$

For $y\in B_1\setminus B_r(x^*)$,

$$|y-x''|\le |y-x^*|+|x^*-x''| = |y-x^*|+\frac{r}{2}\le\frac{3}{2}\,|y-x^*|\,,$$

and

$$|y-\widehat{x}|\ge |y-x^*|-|x^*-\widehat{x}|\ge |y-x^*|-r\ge\frac{1}{2}\,|y-x^*|\,.$$

Thus

$$\begin{aligned}|I_6| &\le C(n)r\int_{B_1\setminus B_r(x^*)}\frac{|f(y)-f(x'')|}{|y-\widehat{x}|^{n+1}}\,dy\\ &\le C(n)\,\|f\|_{C^\alpha(B_1)}\,r\int_{B_1\setminus B_r(x^*)}\frac{|y-x''|^\alpha}{|y-\widehat{x}|^{n+1}}\,dy\\ &\le C(\alpha,n)\,\|f\|_{C^\alpha(B_1)}\,r\int_{B_2(x^*)\setminus B_r(x^*)}|y-x^*|^{\alpha-n-1}\,dy\\ &\le C(\alpha,n)\,\|f\|_{C^\alpha(B_1)}\,r\int_r^2 s^{\alpha-2}\,ds\le C(\alpha,n)\,\|f\|_{C^\alpha(B_1)}\,r^\alpha.\end{aligned}$$

The lemma follows. □

Theorem. *Let $\alpha\in(0,1)$, and let $f\in C^\alpha(\overline{B}_1)$. If*

$$u\in L^\infty(B_1)\cap C^2(B_1),\qquad \Delta u=f\quad \text{in } B_1,$$

then

$$u\in C^{2,\alpha}(\overline{B}_{1/2}),$$

and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\alpha, n) \left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)} \right).$$

Proof. Let W be the Newtonian potential of f in B_1. Then

$$\Delta W = f \quad \text{in } B_1, \qquad W \in C^1(\overline{B}_1) \cap C^{2,\alpha}(\overline{B}_{1/2}) \cap C^2(B_1),$$

$$\|W\|_{C^1(B_1)} \leq C(n) \|f\|_{L^\infty(B_1)}, \qquad \|W\|_{C^{2,\alpha}(B_{1/2})} \leq C(\alpha, n) \|f\|_{C^\alpha(B_1)}.$$

Define

$$V = u - W.$$

Then

$$\Delta V = \Delta u - \Delta W = f - f = 0 \quad \text{in } B_1.$$

Thus V is harmonic, and hence $V \in C^\infty(B_1)$ by the smoothness of harmonic functions. By the derivative estimates for harmonic functions,

$$\|V\|_{C^{2,\alpha}(B_{1/2})} \leq C(\alpha, n) \|V\|_{C^3(B_{1/2})} \leq C(\alpha, n) \|V\|_{L^\infty(B_1)}.$$

Since $u = W + V$,

$$u \in C^{2,\alpha}(\overline{B}_{1/2}),$$

and

$$\begin{aligned} \|u\|_{C^{2,\alpha}(B_{1/2})} &\leq \|W\|_{C^{2,\alpha}(B_{1/2})} + \|V\|_{C^{2,\alpha}(B_{1/2})} \\ &\leq C(\alpha, n) \|f\|_{C^\alpha(B_1)} + C(\alpha, n) \|V\|_{L^\infty(B_1)} \\ &\leq C(\alpha, n) \|f\|_{C^\alpha(B_1)} + C(\alpha, n) \left(\|u\|_{L^\infty(B_1)} + \|W\|_{L^\infty(B_1)} \right) \\ &\leq C(\alpha, n) \left(\|f\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)} \right). \end{aligned}$$

□

Corollary. *Let $\alpha \in (0, 1)$, and let $f \in C^\alpha(\overline{B_R(x_0)})$. If*

$$u \in L^\infty(B_R(x_0)) \cap C^2(B_R(x_0)), \qquad \Delta u = f \quad \text{in } B_R(x_0),$$

then $u \in C^{2,\alpha}(\overline{B_{R/2}(x_0)})$, and

$$\begin{aligned} &\|u\|_{L^\infty(B_{R/2}(x_0))} + R \|\nabla u\|_{L^\infty(B_{R/2}(x_0))} + R^2 \left\|\nabla^2 u\right\|_{L^\infty(B_{R/2}(x_0))} \\ &\quad + R^{2+\alpha} [\nabla^2 u]_{C^\alpha(B_{R/2}(x_0))} \\ &\leq C(\alpha, n) \left(\|u\|_{L^\infty(B_R(x_0))} + R^2 \|f\|_{L^\infty(B_R(x_0))} + R^{2+\alpha} [f]_{C^\alpha(B_R(x_0))} \right). \end{aligned}$$

Proof. After translation, assume $x_0 = 0$, so $B_R(x_0) = B_R$. Define

$$u_R(x) = u(Rx), \qquad f_R(x) = f(Rx), \qquad x \in B_1.$$

Then

$$\Delta u_R = R^2 f_R \quad \text{in } B_1.$$

Applying the previous theorem gives

$$(*) \qquad \|u_R\|_{C^{2,\alpha}(B_{1/2})} \leq C(\alpha, n) \left(\|u_R\|_{L^\infty(B_1)} + \left\|R^2 f_R\right\|_{C^\alpha(B_1)} \right).$$

For $(*)$, compute

$$\begin{aligned} \text{LHS} = \|u\|_{L^\infty(B_{R/2})} &+ R \|\nabla u\|_{L^\infty(B_{R/2})} + R^2 \left\|\nabla^2 u\right\|_{L^\infty(B_{R/2})} \\ &+ R^{2+\alpha} [\nabla^2 u]_{C^\alpha(B_{R/2})}, \end{aligned}$$

and

$$\text{RHS} = C(\alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2 \|f\|_{L^\infty(B_R)} + R^{2+\alpha}[f]_{C^\alpha(B_R)}\right).$$

This is exactly the desired estimate. □

Corollary. *Let Ω be a bounded domain in $\mathbb{R}^n$, let $f \in C^\alpha(\Omega)$, and let $\alpha \in (0,1)$. If*

$$u \in L^\infty(\Omega) \cap C^2(\Omega), \qquad \Delta u = f \quad \text{in } \Omega,$$

then

$$u \in C^{2,\alpha}(\Omega).$$

Proof. For $\Omega' \Subset \Omega$, set

$$d = \operatorname{dist}(\Omega', \partial\Omega) > 0.$$

It is enough to prove

$$\sup_{\substack{x', x'' \in \Omega' \\ x' \neq x''}} \frac{|\partial_{ij} u(x') - \partial_{ij} u(x'')|}{|x' - x''|^\alpha} < +\infty.$$

Case 1: $|x' - x''| \geq d/2$. Then

$$\frac{|\partial_{ij} u(x') - \partial_{ij} u(x'')|}{|x' - x''|^\alpha} \leq \frac{2\left\|\nabla^2 u\right\|_{L^\infty(\Omega')}}{(d/2)^\alpha} < \infty.$$

Case 2: $|x' - x''| < d/2$. Then

$$x'' \in B_{d/2}(x') \subset B_d(x') \Subset \Omega.$$

By the previous corollary,

$$\begin{aligned}
\frac{|\partial_{ij} u(x') - \partial_{ij} u(x'')|}{|x' - x''|^\alpha} &\leq [\nabla^2 u]_{C^\alpha(B_{d/2}(x'))} \\
&\leq \frac{C(\alpha, n)}{d^{2+\alpha}}\left(\|u\|_{L^\infty(B_d(x'))} + d^2 \|f\|_{L^\infty(B_d(x'))} + d^{2+\alpha}[f]_{C^\alpha(B_d(x'))}\right) \\
&\leq \frac{C(\alpha, n)}{d^{2+\alpha}}\left(\|u\|_{L^\infty(\Omega)} + d^2 \|f\|_{L^\infty(\Omega)} + d^{2+\alpha}[f]_{C^\alpha(\Omega)}\right) < \infty.
\end{aligned}$$

□

2. Dirichlet Problem.

Theorem. *Let $\alpha \in (0,1)$, let*

$$f \in C^\alpha(\overline{B_R(x_0)}), \qquad \varphi \in C(\partial B_R(x_0)).$$

Then there exists a unique solution

$$u \in C(\overline{B_R(x_0)}) \cap C^{2,\alpha}(B_R(x_0))$$

of the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } B_R(x_0), \\ u = \varphi & \text{on } \partial B_R(x_0). \end{cases}$$

Proof. After translation, assume $x_0 = 0$.

First consider the special case $R = 1$. Let W be the Newtonian potential of f in B_1. Then

$$\Delta W = f \quad \text{in } B_1, \qquad W \in C^1(\overline{B}_1) \cap C^{2,\alpha}(B_1).$$

By the solvability of the harmonic Dirichlet problem in balls, there exists a solution

$$V \in C(\overline{B}_1) \cap C^2(B_1)$$

of the harmonic Dirichlet problem

$$\begin{cases} \Delta V = 0 & \text{in } B_1, \\ V = \varphi - W & \text{on } \partial B_1. \end{cases}$$

Moreover $V \in C^\infty(B_1)$ by the smoothness of harmonic functions. Set

$$u = V + W.$$

Then

$$\begin{cases} \Delta u = \Delta V + \Delta W = f & \text{in } B_1, \\ u = \varphi - W + W = \varphi & \text{on } \partial B_1, \end{cases}$$

and

$$u \in C(\overline{B}_1) \cap C^{2,\alpha}(B_1).$$

Uniqueness follows from the maximum principle.

For the general case, use scaling. Define

$$u_R(x) = u(Rx), \qquad f_R(x) = f(Rx), \qquad \varphi_R(x) = \varphi(Rx), \qquad x \in B_1.$$

Then the problem

$$\begin{cases} \Delta u = f & \text{in } B_R, \\ u = \varphi & \text{on } \partial B_R \end{cases} \tag{*1}$$

is equivalent to

$$\begin{cases} \Delta u_R = R^2 f_R & \text{in } B_1, \\ u_R = \varphi_R & \text{on } \partial B_1. \end{cases} \tag{*2}$$

The special case gives a solution u_R of $(*2)$. Then

$$u(x) = u_R\left(\frac{x}{R}\right)$$

solves $(*1)$. Uniqueness again follows from the maximum principle. □

7. Schauder Theory I: Weighted Norms and Interpolation

Recall. We have proved the $C^{2,\alpha}$-regularity and estimate for the Newtonian potential, and the following estimate for Poisson's equation in balls. If $\alpha \in (0,1)$, $f \in C^\alpha(\overline{B_R(x_0)})$, and

$$u \in L^\infty(B_R(x_0)) \cap C^2(B_R(x_0)), \qquad \Delta u = f \quad \text{in } B_R(x_0),$$

then $u \in C^{2,\alpha}(\overline{B_{R/2}(x_0)})$, and

$$\begin{aligned} &R\left\|\nabla u\right\|_{L^\infty(B_{R/2}(x_0))} + R^2\left\|\nabla^2 u\right\|_{L^\infty(B_{R/2}(x_0))} + R^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{R/2}(x_0))} \\ &\leq C(\alpha, n)\left(\|u\|_{L^\infty(B_R(x_0))} + R^2\|f\|_{L^\infty(B_R(x_0))} + R^{2+\alpha}[f]_{C^\alpha(B_R(x_0))}\right). \end{aligned}$$

We also proved the Dirichlet problem for Poisson's equation in balls.

Definition. For $k \in \mathbb{Z}_{\geq 0}$, $\alpha \in (0,1)$, define the weighted norm

$$\|u\|^*_{C^{k,\alpha}(B_R(x_0))} = \sum_{i=0}^{k} R^i \left\|\nabla^i u\right\|_{L^\infty(B_R(x_0))} + R^{k+\alpha}[\nabla^k u]_{C^\alpha(B_R(x_0))}.$$

Also set

$$\|u\|^*_{C^k(B_R(x_0))} = \|u\|^*_{C^{k,0}(B_R(x_0))}, \qquad \|u\|^*_{C^\alpha(B_R(x_0))} = \|u\|^*_{C^{0,\alpha}(B_R(x_0))}.$$

With this notation, the preceding estimate can be rewritten as follows: if $\alpha \in (0,1)$, $f \in C^\alpha(\overline{B_R(x_0)})$, and

$$u \in L^\infty(B_R(x_0)) \cap C^2(B_R(x_0)), \qquad \Delta u = f \quad \text{in } B_R(x_0),$$

then

$$u \in C^{2,\alpha}(\overline{B_{R/2}(x_0)})$$

and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2}(x_0))} \leq C(\alpha,n)\left(\|u\|_{L^\infty(B_R(x_0))} + R^2\|f\|^*_{C^\alpha(B_R(x_0))}\right).$$

Our goal is to generalize the above theorem to

$$Lu = f, \qquad L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c.$$

1. Interpolation Inequalities.

Lemma. *Let $\alpha \in (0,1)$, $\tau \in (0,1)$, and*

$$u \in C^{2,\alpha}(\overline{B}_R).$$

Then

$$\tau^\alpha R^\alpha [u]_{C^\alpha(B_R)} \leq C(\alpha,n)\left(\tau R\|\nabla u\|_{L^\infty(B_R)} + \|u\|_{L^\infty(B_R)}\right), \tag{1}$$

$$\tau R\|\nabla u\|_{L^\infty(B_R)} \leq C(\alpha,n)\left((\tau R)^{1+\alpha}[\nabla u]_{C^\alpha(B_R)} + \|u\|_{L^\infty(B_R)}\right), \tag{2}$$

and

$$\tau R\|\nabla u\|_{L^\infty(B_R)} \leq C(\alpha,n)\left((\tau R)^2\left\|\nabla^2 u\right\|_{L^\infty(B_R)} + \|u\|_{L^\infty(B_R)}\right). \tag{3}$$

Proof. By scaling, assume $R = 1$.

For (1), take $x', x'' \in B_1$. If $|x' - x''| \leq \tau$, then by the mean value theorem,

$$\frac{|u(x') - u(x'')|}{|x' - x''|^\alpha} \leq \|\nabla u\|_{L^\infty(B_1)} |x' - x''|^{1-\alpha} \leq \|\nabla u\|_{L^\infty(B_1)} \tau^{1-\alpha}.$$

If $|x' - x''| > \tau$, then

$$\frac{|u(x') - u(x'')|}{|x' - x''|^\alpha} \leq \frac{2\|u\|_{L^\infty(B_1)}}{\tau^\alpha}.$$

Therefore

$$[u]_{C^\alpha(B_1)} \leq \tau^{1-\alpha}\|\nabla u\|_{L^\infty(B_1)} + 2\tau^{-\alpha}\|u\|_{L^\infty(B_1)},$$

which is (1).

For (2), fix $x \in B_1$. Set

$$\overline{e} = -\frac{x}{|x|},$$

and if $x = 0$, choose $\overline{e} = (1, 0, \ldots, 0)$. Let

$$S = \{\overline{a} \text{ is a unit vector} \mid \angle(\overline{a}, \overline{e}) \leq \pi/4\}.$$

For $\overline{a} \in S$, define

$$x_\tau = x + \tau \overline{a}.$$

Then $x_\tau \in B_1$ and $|x_\tau - x| = \tau$. By the mean value theorem, there exists $\overline{x} \in \overline{x x_\tau}$ such that

$$|\partial_{\overline{a}} u(\overline{x})| = \frac{|u(x_\tau) - u(x)|}{|x_\tau - x|} \le \frac{2\,\|u\|_{L^\infty(B_1)}}{\tau}.$$

Moreover $|\overline{x} - x| \le |x_\tau - x| = \tau$. Hence

$$\begin{aligned}|\partial_{\overline{a}} u(x)| &\le |\partial_{\overline{a}} u(\overline{x})| + |\partial_{\overline{a}} u(\overline{x}) - \partial_{\overline{a}} u(x)| \\ &\le 2\tau^{-1}\,\|u\|_{L^\infty(B_1)} + \tau^\alpha [\nabla u]_{C^\alpha(B_1)}.\end{aligned}$$

If $\nabla u(x) \neq 0$, we choose $\overline{a} \in S$ such that

$$\angle(\overline{a}, \nabla u(x)) \le \frac{\pi}{4} \quad \text{or} \quad \angle(\overline{a}, -\nabla u(x)) \le \frac{\pi}{4}.$$

Then

$$|\overline{a} \cdot \nabla u(x)| \ge \frac{1}{\sqrt{2}}\,|\nabla u(x)|.$$

Thus

$$(*) \qquad |\nabla u(x)| \le \sqrt{2}\,|\partial_{\overline{a}} u(x)| \le 2\sqrt{2}\,\tau^{-1}\,\|u\|_{L^\infty(B_1)} + \sqrt{2}\,\tau^\alpha [\nabla u]_{C^\alpha(B_1)}.$$

If $\nabla u(x) = 0$, the desired estimate is immediate because the right hand side is nonnegative. This proves (2).

For (3), repeat the same construction with $x_\tau = x + \tau\overline{a}$. The mean value theorem again gives

$$|\partial_{\overline{a}} u(\overline{x})| \le \frac{2\,\|u\|_{L^\infty(B_1)}}{\tau}$$

for some $\overline{x} \in \overline{x x_\tau}$, while

$$|\partial_{\overline{a}} u(x) - \partial_{\overline{a}} u(\overline{x})| \le \tau \left\|\nabla^2 u\right\|_{L^\infty(B_1)}.$$

Choosing $\overline{a}$ as above yields

$$|\nabla u(x)| \le 2\sqrt{2}\,\tau^{-1}\,\|u\|_{L^\infty(B_1)} + \sqrt{2}\,\tau \left\|\nabla^2 u\right\|_{L^\infty(B_1)}.$$

Multiplying by τ proves (3). $\square$

Corollary. *Let $\alpha \in (0,1)$, $\tau \in (0,1)$, and $u \in C^{2,\alpha}(\overline{B}_R)$. Then*

$$\sum_{i=0}^{2} (\tau R)^i \left\|\nabla^i u\right\|_{L^\infty(B_R)} + \sum_{i=0}^{1} (\tau R)^{i+\alpha} [\nabla^i u]_{C^\alpha(B_R)} \le C(\alpha, n) \left((\tau R)^{2+\alpha} [\nabla^2 u]_{C^\alpha(B_R)} + \|u\|_{L^\infty(B_R)} \right).$$

Proof. By scaling, assume $R = 1$. For example, we prove

$$(*) \qquad \tau^2 \left\|\nabla^2 u\right\|_{L^\infty(B_1)} \le C(\alpha, n) \left(\tau^{2+\alpha} [\nabla^2 u]_{C^\alpha(B_1)} + \|u\|_{L^\infty(B_1)} \right).$$

Apply Lemma (2) to ∇u:

$$\left\|\nabla^2 u\right\|_{L^\infty(B_1)} \le C(\alpha, n) \left(\tau^\alpha [\nabla^2 u]_{C^\alpha(B_1)} + \tau^{-1}\,\|\nabla u\|_{L^\infty(B_1)} \right).$$

Apply Lemma (3) to u:

$$\|\nabla u\|_{L^\infty(B_1)} \le C(\alpha, n) \left(\widehat{\tau} \left\|\nabla^2 u\right\|_{L^\infty(B_1)} + \widehat{\tau}^{-1}\,\|u\|_{L^\infty(B_1)} \right),$$

where $\widehat{\tau}$ is to be chosen. Hence
$$\left\|\nabla^2 u\right\|_{L^\infty(B_1)} \le C(\alpha,n)\left(\tau^\alpha[\nabla^2 u]_{C^\alpha(B_1)} + \tau^{-1}\widehat{\tau}\left\|\nabla^2 u\right\|_{L^\infty(B_1)} + (\tau\widehat{\tau})^{-1}\|u\|_{L^\infty(B_1)}\right).$$
Choose
$$\widehat{\tau} = \frac{\tau}{2C(\alpha,n)} \in (0,1).$$
Then the middle term is absorbed into the left hand side, giving $(*)$. The remaining estimates are obtained in the same way. ☐

Corollary. *Let $\alpha \in (0,1)$, and let $\varepsilon \in (0,1)$. Then*
$$(**) \qquad \|u\|^*_{C^2(B_R)} \le \varepsilon R^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_R)} + C(\varepsilon,\alpha,n)\|u\|_{L^\infty(B_R)}.$$

Proof. By the previous corollary,
$$\sum_{i=0}^{2}(\tau R)^i\left\|\nabla^i u\right\|_{L^\infty(B_R)} \le C(\alpha,n)\left((\tau R)^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_R)} + \|u\|_{L^\infty(B_R)}\right).$$
For $\tau \in (0, C^{-1}(\alpha,n))$,
$$\|u\|^*_{C^2(B_R)} = \sum_{i=0}^{2} R^i\left\|\nabla^i u\right\|_{L^\infty(B_R)} \le \tau^{-2}\sum_{i=0}^{2}(\tau R)^i\left\|\nabla^i u\right\|_{L^\infty(B_R)}.$$
Therefore
$$\|u\|^*_{C^2(B_R)} \le C(\alpha,n)\tau^\alpha R^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_R)} + C(\alpha,n)\tau^{-2}\|u\|_{L^\infty(B_R)}.$$
Choose τ such that $C(\alpha,n)\tau^\alpha = \varepsilon$, and $(**)$ follows. ☐

2. Extremely Special Case.

Proposition. *Let $A = (a_{ij})$ be a constant symmetric matrix satisfying*
$$\lambda I_n \le A \le \Lambda I_n, \qquad \lambda, \Lambda > 0.$$
If
$$u \in L^\infty(B_R) \cap C^2(B_R), \qquad \sum_{i,j} a_{ij}\partial_{ij}u = f \quad \text{in } B_R,$$
equivalently
$$\operatorname{tr}(A\nabla^2 u) = f \quad \text{in } B_R,$$
where $f \in C^\alpha(\overline{B}_R)$, $\alpha \in (0,1)$, then
$$u \in C^{2,\alpha}(\overline{B}_{R/2}),$$
and
$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

Proof. By scaling, assume $R = 1$. There exists an orthogonal matrix P such that
$$PAP^T = \operatorname{diag}(\lambda_1,\dots,\lambda_n), \qquad \lambda \le \lambda_i \le \Lambda.$$
Define
$$D = \operatorname{diag}(\lambda_1^{-1/2},\dots,\lambda_n^{-1/2}), \qquad Q = DP.$$
Then
$$QAQ^T = I_n, \qquad Q^{-1}Q^{-T} = A.$$
Consider the map $x \mapsto Qx$, and denote $Q(B_1)$ by an ellipsoid. Define
$$u_Q(x) = u(Q^{-1}x), \qquad f_Q(x) = f(Q^{-1}x), \qquad x \in Q(B_1).$$

Then

$$\begin{aligned}\Delta u_Q(x) &= \operatorname{tr}(\nabla^2 u_Q(x)) = \operatorname{tr}\left(Q^{-T}\nabla^2 u(Q^{-1}x)Q^{-1}\right)\\ &= \operatorname{tr}\left(Q^{-1}Q^{-T}\nabla^2 u(Q^{-1}x)\right) = \operatorname{tr}\left(A\nabla^2 u(Q^{-1}x)\right)\\ &= \sum_{i,j} a_{ij}\partial_{ij}u(Q^{-1}x) = f(Q^{-1}x) = f_Q(x).\end{aligned}$$

Moreover

$$|Qx|^2 = x^T Q^T Qx = x^T A^{-1}x,$$

so

$$\Lambda^{-1/2}|x| \le |Qx| \le \lambda^{-1/2}|x|.$$

For $Q(B_{1/2}) \Subset Q(B_1)$,

$$d\big(Q(B_{1/2}), \partial Q(B_1)\big) \ge \Lambda^{-1/2}\operatorname{dist}(B_{1/2}, \partial B_1) = \frac{\Lambda^{-1/2}}{2}.$$

Set

$$\gamma = \frac{\Lambda^{-1/2}}{4}.$$

There exist $N = N(\lambda, \Lambda, n)$ and

$$y_1, \dots, y_N \in Q(B_{1/2})$$

such that

$$Q(B_{1/2}) \subset \bigcup_{i=1}^{N} B_{\gamma/4}(y_i).$$

Notice that $B_\gamma(y_i) \subset Q(B_1)$. Apply the previous Poisson estimate to $B_\gamma(y_i)$. Then

$$u_Q \in C^{2,\alpha}(\overline{B_{\gamma/2}(y_i)})$$

and

$$(*) \qquad \|u_Q\|^*_{C^{2,\alpha}(B_{\gamma/2}(y_i))} \le C(\alpha, n)\left(\|u_Q\|_{L^\infty(B_\gamma(y_i))} + \gamma^2\|f_Q\|^*_{C^\alpha(B_\gamma(y_i))}\right).$$

We claim

$$[\nabla^2 u_Q]_{C^\alpha(Q(B_{1/2}))} \le C(\alpha)\left(\sum_{i=1}^{N}[\nabla^2 u_Q]_{C^\alpha(B_{\gamma/2}(y_i))} + \gamma^{-2-\alpha}\sum_{i=1}^{N}\|u_Q\|_{L^\infty(B_{\gamma/2}(y_i))}\right).$$

Combining $(*)$, the claim, and the interpolation inequality gives

$$u_Q \in C^{2,\alpha}(\overline{Q(B_{1/2})}),$$

and

$$\|u_Q\|_{C^{2,\alpha}(Q(B_{1/2}))} \le C(\alpha, n, \gamma)\left(\|u_Q\|_{L^\infty(Q(B_1))} + \|f_Q\|_{C^\alpha(Q(B_1))}\right).$$

Returning to the definitions of u_Q and f_Q, we get

$$u \in C^{2,\alpha}(\overline{B}_{1/2})$$

and

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right).$$

The corresponding weighted estimate for general R follows by scaling.

It remains to prove the claim. Let

$$x', x'' \in Q(B_{1/2}) \subset \bigcup_{i=1}^{N} B_{\gamma/4}(y_i).$$

Assume $x' \in B_{\gamma/4}(y_k)$.

Case 1: if

$$|x' - x''| \leq \frac{\gamma}{100},$$

then $x'' \in B_{\gamma/2}(y_k)$. Hence

$$\frac{\left|\nabla^2 u_Q(x') - \nabla^2 u_Q(x'')\right|}{|x' - x''|^{\alpha}} \leq [\nabla^2 u_Q]_{C^{\alpha}(B_{\gamma/2}(y_k))} \leq \sum_{i=1}^{N} [\nabla^2 u_Q]_{C^{\alpha}(B_{\gamma/2}(y_i))}.$$

Case 2: if

$$|x' - x''| > \frac{\gamma}{100},$$

then $x'' \in B_{\gamma/4}(y_\ell)$ for some ℓ, and

$$\begin{aligned}\frac{\left|\nabla^2 u_Q(x') - \nabla^2 u_Q(x'')\right|}{|x' - x''|^{\alpha}} &\leq \frac{\left\|\nabla^2 u_Q\right\|_{L^\infty(B_{\gamma/2}(y_k))} + \left\|\nabla^2 u_Q\right\|_{L^\infty(B_{\gamma/2}(y_\ell))}}{(\gamma/100)^{\alpha}} \\ &\leq C(\alpha)\gamma^{-\alpha} \sum_{i=1}^{N} \left\|\nabla^2 u_Q\right\|_{L^\infty(B_{\gamma/2}(y_i))}.\end{aligned}$$

By the interpolation inequality, the last term is bounded by

$$C(\alpha)\gamma^{-\alpha} \sum_{i=1}^{N} \left(\left(\frac{\gamma}{2}\right)^{\alpha} [\nabla^2 u_Q]_{C^{\alpha}(B_{\gamma/2}(y_i))} + \left(\frac{\gamma}{2}\right)^{-2} \|u_Q\|_{L^\infty(B_{\gamma/2}(y_i))} \right),$$

which proves the claim. □

3. Special Case: Small Oscillation.

Proposition. *Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^{\alpha}(\overline{B}_R), \quad \alpha \in (0,1),$$

and suppose

$$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n, \qquad \lambda, \Lambda > 0.$$

Assume the weighted coefficient bounds

$$\|a_{ij}\|^*_{C^{\alpha}(B_R)}, \quad R\|b_i\|^*_{C^{\alpha}(B_R)}, \quad R^2\|c\|^*_{C^{\alpha}(B_R)} \leq \Lambda.$$

Suppose

$$u \in L^\infty(B_R) \cap C^2(B_R), \qquad Lu = f \quad \text{in } B_R,$$

for some $f \in C^{\alpha}(\overline{B}_R)$. Then there exists

$$\varepsilon = \varepsilon(\lambda, \Lambda, \alpha, n) > 0$$

such that if

$$\sup_{x', x'' \in B_R} |a_{ij}(x') - a_{ij}(x'')| \leq \varepsilon \qquad \text{for all } i, j,$$

then

$$u \in C^{2,\alpha}(\overline{B}_{R/2}),$$

and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

8. Schauder Theory II: Interior Estimates

Recall. We recall the interpolation inequalities and the extremely special case of the Schauder estimate. If $A = (a_{ij})$ is a constant symmetric matrix,

$$\lambda I_n \le A \le \Lambda I_n, \qquad \lambda, \Lambda > 0,$$

and

$$u \in L^\infty(B_R) \cap C^2(B_R), \qquad \sum_{i,j} a_{ij}\partial_{ij}u = f \quad \text{in } B_R,$$

with $f \in C^\alpha(\overline{B}_R)$, then

$$u \in C^{2,\alpha}(\overline{B}_{R/2})$$

and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

The goal is to generalize this result to $Lu = f$.

3. Special Case: Small Oscillation. We continue the small oscillation proposition from the preceding section. Let

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{B}_R), \quad \alpha \in (0,1),$$

and assume

$$\lambda I_n \le (a_{ij}) \le \Lambda I_n, \qquad \|a_{ij}\|^*_{C^\alpha(B_R)},\ R\|b_i\|^*_{C^\alpha(B_R)},\ R^2\|c\|^*_{C^\alpha(B_R)} \le \Lambda.$$

Suppose

$$u \in L^\infty(B_R) \cap C^2(B_R), \qquad Lu = f \quad \text{in } B_R,$$

for some $f \in C^\alpha(\overline{B}_R)$. Then there exists $\varepsilon = \varepsilon(\lambda, \Lambda, \alpha, n) > 0$ such that, if

$$\sup_{x', x'' \in B_R} |a_{ij}(x') - a_{ij}(x'')| \le \varepsilon,$$

then

$$u \in C^{2,\alpha}(\overline{B}_{R/2})$$

and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_R)} + R^2\|f\|^*_{C^\alpha(B_R)}\right).$$

Proof. By scaling, assume $R = 1$. In general the scaling is

$$a^R_{ij}(x) = a_{ij}(Rx), \qquad b^R_i(x) = Rb_i(Rx), \qquad c^R(x) = R^2c(Rx),$$

$$u_R(x) = u(Rx), \qquad f_R(x) = R^2f(Rx),$$

and

$$L_R = \sum_{i,j} a^R_{ij}\partial_{ij} + \sum_i b^R_i\partial_i + c^R.$$

Then $Lu = f$ in B_R is equivalent to

$$L_Ru_R = f_R \quad \text{in } B_1.$$

It is enough to prove

$$\text{(Goal)} \qquad [\nabla^2u]_{C^\alpha(B_{1/2})} \le C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right).$$

Together with the interpolation inequality, this gives the proposition.

From $Lu = f$, we write

$$(*) \qquad \sum_{i,j} a_{ij}(0)\partial_{ij}u = \widehat{f},$$

where

$$\widehat{f} = f - cu - \sum_i b_i \partial_i u - \sum_{i,j} \big(a_{ij} - a_{ij}(0)\big)\partial_{ij}u.$$

Apply the extremely special case to $(*)$ in $B_r \subset B_1$. We get

$$r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \leq C(\lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_r)} + r^2 \left\|\widehat{f}\right\|^*_{C^\alpha(B_r)}\right).$$

We estimate $\widehat{f}$. For $V, W \in C^\alpha(B_r)$,

$$[VW]_{C^\alpha(B_r)} \leq [V]_{C^\alpha(B_r)} \|W\|_{L^\infty(B_r)} + \|V\|_{L^\infty(B_r)} [W]_{C^\alpha(B_r)}.$$

Hence

$$\begin{aligned}
[\widehat{f}]_{C^\alpha(B_r)} \leq {} & [f]_{C^\alpha(B_r)} + [c]_{C^\alpha(B_r)} \|u\|_{L^\infty(B_r)} + \|c\|_{L^\infty(B_r)} [u]_{C^\alpha(B_r)} \\
& + \sum_i [b_i]_{C^\alpha(B_r)} \|\nabla u\|_{L^\infty(B_r)} + \sum_i \|b_i\|_{L^\infty(B_r)} [\nabla u]_{C^\alpha(B_r)} \\
& + \sum_{i,j} [a_{ij} - a_{ij}(0)]_{C^\alpha(B_r)} \left\|\nabla^2 u\right\|_{L^\infty(B_r)} \\
& + \sum_{i,j} \|a_{ij} - a_{ij}(0)\|_{L^\infty(B_r)} [\nabla^2 u]_{C^\alpha(B_r)}.
\end{aligned}$$

By the assumptions,

$$\|a_{ij}\|_{C^\alpha(B_r)},\ \|b_i\|_{C^\alpha(B_r)},\ \|c\|_{C^\alpha(B_r)} \leq \Lambda, \qquad \|a_{ij} - a_{ij}(0)\|_{L^\infty(B_r)} \leq \varepsilon.$$

Therefore

$$r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \leq C\left(r^2 \|f\|^*_{C^\alpha(B_1)} + \|u\|^*_{C^2(B_r)} + \varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r)}\right).$$

Using the interpolation inequality above,

$$\|u\|^*_{C^2(B_r)} \leq \varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r)} + C(\varepsilon, \alpha, n)\|u\|_{L^\infty(B_r)}.$$

Thus

$$(1) \qquad r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \leq C\varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r)} + C\|f\|_{C^\alpha(B_1)} + C_\varepsilon \|u\|_{L^\infty(B_1)}.$$

By the same argument, this holds for any ball $B_r(x_0) \subset B_1$:

(1')

$$r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \leq C\varepsilon r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_r(x_0))} + C\|f\|_{C^\alpha(B_1)} + C_\varepsilon \|u\|_{L^\infty(B_1)}.$$

Define

$$K = \sup\left\{r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \mid B_r(x_0) \subset B_1\right\}.$$

For every $B_r(x_0) \subset B_1$, there exist $N = N(n)$ and

$$y_1, \ldots, y_N \in B_{r/2}(x_0)$$

such that

$$B_{r/2}(x_0) \subset \bigcup_{i=1}^N B_{r/16}(y_i).$$

Apply (1′) to $B_{r/4}(y_i) \subset B_1$. Then

$$\left(\frac{r}{4}\right)^{2+\alpha} [\nabla^2 u]_{C^\alpha(B_{r/8}(y_i))} \le C\varepsilon \left(\frac{r}{4}\right)^{2+\alpha} [\nabla^2 u]_{C^\alpha(B_{r/4}(y_i))} + C \|f\|_{C^\alpha(B_1)} + C_\varepsilon \|u\|_{L^\infty(B_1)}.$$

The same covering argument as in the extremely special case gives

$$\begin{aligned} r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \le C\varepsilon \sum_{i=1}^{N} \left(\frac{r}{4}\right)^{2+\alpha} [\nabla^2 u]_{C^\alpha(B_{r/4}(y_i))} \\ + C \|f\|_{C^\alpha(B_1)} + C_\varepsilon \|u\|_{L^\infty(B_1)}. \end{aligned}$$

By the definition of K and $B_{r/4}(y_i) \subset B_1$,

$$r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2}(x_0))} \le C\varepsilon K + C \|f\|_{C^\alpha(B_1)} + C_\varepsilon \|u\|_{L^\infty(B_1)}.$$

Taking the supremum over all $B_r(x_0) \subset B_1$,

$$K \le C\varepsilon K + C \|f\|_{C^\alpha(B_1)} + C_\varepsilon \|u\|_{L^\infty(B_1)}.$$

Choose

$$\varepsilon = \frac{1}{2C}.$$

Then

$$K \le C \|f\|_{C^\alpha(B_1)} + C \|u\|_{L^\infty(B_1)}.$$

By the definition of K, for every $r \in (0,1)$ with $B_r \subset B_1$,

$$r^{2+\alpha}[\nabla^2 u]_{C^\alpha(B_{r/2})} \le C \|f\|_{C^\alpha(B_1)} + C \|u\|_{L^\infty(B_1)}.$$

Letting $r \to 1$ gives the goal. □

4. General Case.

Theorem. *Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{B}_R), \quad \alpha \in (0,1),$$

and suppose

$$\lambda I_n \le (a_{ij}) \le \Lambda I_n, \qquad \lambda, \Lambda > 0,$$
$$\|a_{ij}\|^*_{C^\alpha(B_R)}, \ R\|b_i\|^*_{C^\alpha(B_R)}, \ R^2 \|c\|^*_{C^\alpha(B_R)} \le \Lambda.$$

If

$$u \in L^\infty(B_R) \cap C^2(B_R), \qquad Lu = f \quad \text{in } B_R,$$

for some $f \in C^\alpha(\overline{B}_R)$*, then*

$$u \in C^{2,\alpha}(\overline{B}_{R/2})$$

and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \le C(\lambda, \Lambda, \alpha, n) \left(\|u\|_{L^\infty(B_R)} + R^2 \|f\|^*_{C^\alpha(B_R)} \right).$$

Proof. By scaling, assume $R = 1$. For any $B_{r_0}(x_0) \subset B_1$,

$$\sup_{x,x' \in B_{r_0}(x_0)} |a_{ij}(x) - a_{ij}(x')| \le [a_{ij}]_{C^\alpha(B_1)} |x - x'|^\alpha \le \Lambda (2r_0)^\alpha.$$

Choose $r_0 \in (0,1)$ such that

$$\Lambda(2r_0)^\alpha = \varepsilon,$$

where ε is the constant in the small oscillation case. Then

$$r_0 = r_0(\lambda, \Lambda, \alpha, n).$$

Apply the special case to $B_{r_0}(x_0)$. This yields the local estimate on balls of radius $r_0/2$. It is enough to prove

$$[\nabla^2 u]_{C^\alpha(B_{1/2})} \le C(\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right),$$

because the interpolation inequality then gives the theorem.

Choose $N = N(n, r_0)$ and

$$y_1,\dots,y_N \in B_{1/2}$$

such that

$$B_{1/2} \subset \bigcup_{i=1}^N B_{r_0/8}(y_i).$$

By a covering argument as before,

$$\begin{aligned}[\nabla^2 u]_{C^\alpha(B_{1/2})} &\le C\sum_{i=1}^N \left([\nabla^2 u]_{C^\alpha(B_{r_0/2}(y_i))} + r_0^{-2-\alpha}\|u\|_{L^\infty(B_{r_0/2}(y_i))}\right)\\ &\le C\|u\|_{L^\infty(B_1)} + C\|f\|_{C^\alpha(B_1)}.\end{aligned}$$

□

1. $C^{2,\alpha}$-Regularity and Estimate in Domains.

Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{\Omega}), \quad \alpha \in (0,1),$$

and assume

$$\lambda I_n \le (a_{ij}) \le \Lambda I_n, \qquad \lambda,\Lambda > 0,$$
$$\|a_{ij}\|_{C^\alpha(\Omega)},\ \|b_i\|_{C^\alpha(\Omega)},\ \|c\|_{C^\alpha(\Omega)} \le \Lambda.$$

If

$$u \in L^\infty(\Omega)\cap C^2(\Omega), \qquad Lu = f \quad \text{in } \Omega$$

for some $f \in C^\alpha(\Omega)$, then

$$u \in C^{2,\alpha}(\Omega),$$

and for every $\Omega' \Subset \Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega')} \le C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)}\right),$$

where

$$C = C(\Omega', \operatorname{dist}(\Omega',\partial\Omega), \lambda, \Lambda, \alpha, n).$$

Proof. Set

$$d = \operatorname{dist}(\Omega',\partial\Omega).$$

There exist $N = N(\Omega', d, n)$ and points

$$y_1,\dots,y_N \in \Omega'$$

such that

$$\Omega' \subset \bigcup_{i=1}^N B_{d/4}(y_i).$$

Apply the Schauder estimate in balls to $B_d(y_i) \Subset \Omega$. Then

$$\|u\|_{C^{2,\alpha}(B_{d/2}(y_i))} \le C(d,\lambda,\Lambda,\alpha,n)\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)}\right).$$

Using the same covering argument as before, and since Ω' is arbitrary, we obtain $u \in C^{2,\alpha}(\Omega)$ and the estimate. $\square$

2. Weighted Holder Norms. Let Ω be a bounded domain in $\mathbb{R}^n$. For $x, y \in \Omega$, define

$$d_x = \operatorname{dist}(x, \partial\Omega), \qquad d_{x,y} = \min(d_x, d_y).$$

For $k \in \mathbb{Z}_{\geq 0}$, $\alpha \in (0,1)$, and $\sigma \in \mathbb{R}$, define

$$[u]_{C^k_\sigma(\Omega)} = \sup_{x\in\Omega} d_x^{k+\sigma} \left|\nabla^k u(x)\right|, \qquad \|u\|_{C^k_\sigma(\Omega)} = \sum_{i=0}^{k} [u]_{C^i_\sigma(\Omega)}.$$

Also

$$\|u\|_{C_\sigma(\Omega)} = \|u\|_{C^0_\sigma(\Omega)}.$$

Define

$$[u]_{C^{k,\alpha}_\sigma(\Omega)} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} d_{x,y}^{k+\alpha+\sigma} \frac{\left|\nabla^k u(x) - \nabla^k u(y)\right|}{|x-y|^\alpha},$$

and

$$\|u\|_{C^{k,\alpha}_\sigma(\Omega)} = \|u\|_{C^k_\sigma(\Omega)} + [u]_{C^{k,\alpha}_\sigma(\Omega)}.$$

Set

$$\|u\|_{C^\alpha_\sigma(\Omega)} = \|u\|_{C^{0,\alpha}_\sigma(\Omega)}.$$

Finally,

$$C^k_\sigma(\Omega) = \left\{u \in C^k(\Omega) \mid \|u\|_{C^k_\sigma(\Omega)} < \infty\right\},$$
$$C^{k,\alpha}_\sigma(\Omega) = \left\{u \in C^{k,\alpha}(\Omega) \mid \|u\|_{C^{k,\alpha}_\sigma(\Omega)} < \infty\right\}.$$

Remark. (1) σ may be negative.

(2)
$$\left(C^k_\sigma(\Omega), \|\cdot\|_{C^k_\sigma(\Omega)}\right), \qquad \left(C^{k,\alpha}_\sigma(\Omega), \|\cdot\|_{C^{k,\alpha}_\sigma(\Omega)}\right)$$
are Banach spaces.

(3) If $\sigma \geq 0$, then
$$\|WV\|_{C^\alpha_\sigma(\Omega)} \leq \|W\|_{C^\alpha_0(\Omega)} \|V\|_{C^\alpha_\sigma(\Omega)}.$$

3. Schauder Estimate Revisited.

Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $\alpha, \beta \in (0,1)$. Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{\Omega}),$$

and assume

$$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n, \qquad \lambda, \Lambda > 0,$$
$$\|a_{ij}\|_{C^\alpha(\Omega)}, \ \|b_i\|_{C^\alpha(\Omega)}, \ \|c\|_{C^\alpha(\Omega)} \leq \Lambda.$$

If $u \in C^2(\Omega)$ satisfies

$$Lu = f \quad in\ \Omega$$

for some $f \in C^\alpha(\Omega)$, with

$$\|u\|_{C_{-\beta}(\Omega)}, \qquad \|f\|_{C^\alpha_{2-\beta}(\Omega)} < \infty,$$

then

$$\|u\|_{C^{2,\alpha}_{-\beta}(\Omega)} \leq C(\lambda, \Lambda, \alpha, \beta, n)\left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)}\right).$$

Proof. For $x \in \Omega$, set

$$d_x = \operatorname{dist}(x, \partial\Omega), \qquad d = \frac{d_x}{2}.$$

Apply the Schauder estimate in balls to $B_d(x) \Subset \Omega$. We get

$$d\,|\nabla u(x)| + d^2 \left|\nabla^2 u(x)\right| + d^{2+\alpha} \sup_{\substack{y \in B_d(x)\\ y \neq x}} \frac{\left|\nabla^2 u(x) - \nabla^2 u(y)\right|}{|x-y|^\alpha}$$

$$\leq C \left(\sup_{y \in B_d(x)} |u(y)| + d^2 \sup_{y \in B_d(x)} |f(y)| + d^{2+\alpha} \sup_{\substack{y,z \in B_d(x)\\ y \neq z}} \frac{|f(y) - f(z)|}{|y-z|^\alpha} \right).$$

For $y, z \in B_d(x)$,

$$d \leq d_y, d_z \leq 3d.$$

Multiplying both sides by $d_x^{-\beta}$ gives

$$d_x^{1-\beta}\,|\nabla u(x)| + d_x^{2-\beta} \left|\nabla^2 u(x)\right| + d_x^{2+\alpha-\beta} \sup_{\substack{y \in B_d(x)\\ y \neq x}} \frac{\left|\nabla^2 u(x) - \nabla^2 u(y)\right|}{|x-y|^\alpha}$$

$$\leq C \left(\sup_{y \in B_d(x)} d_y^{-\beta}\,|u(y)| + \sup_{y \in B_d(x)} d_y^{2-\beta}\,|f(y)| + \sup_{\substack{y,z \in B_d(x)\\ y \neq z}} d_{y,z}^{2+\alpha-\beta} \frac{|f(y) - f(z)|}{|y-z|^\alpha} \right)$$

$$\leq C \left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)} \right). \qquad (1)$$

Thus

$$\|u\|_{C^2_{-\beta}(\Omega)} \leq C \left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)} \right). \tag{2}$$

It remains to control

$$[u]_{C^{2,\alpha}_{-\beta}(\Omega)}.$$

Let $x, y \in \Omega$.

Case 1: if $|x-y| < d_x/2$, then $y \in B_d(x)$, and by (1),

$$d_{x,y}^{2+\alpha-\beta} \frac{\left|\nabla^2 u(x) - \nabla^2 u(y)\right|}{|x-y|^\alpha} \leq C \left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)} \right).$$

Case 2: if $|x-y| \geq d_x/2$, then

$$\begin{aligned}
d_{x,y}^{2+\alpha-\beta} \frac{\left|\nabla^2 u(x) - \nabla^2 u(y)\right|}{|x-y|^\alpha} &\leq d_{x,y}^{2+\alpha-\beta} \frac{\left|\nabla^2 u(x)\right| + \left|\nabla^2 u(y)\right|}{(d_x/2)^\alpha} \\
&\leq 4^\alpha d_{x,y}^{2-\beta} \left(\left|\nabla^2 u(x)\right| + \left|\nabla^2 u(y)\right| \right) \\
&\leq 4^\alpha \left(d_x^{2-\beta} \left|\nabla^2 u(x)\right| + d_y^{2-\beta} \left|\nabla^2 u(y)\right| \right) \\
&\leq 2 \cdot 4^\alpha \|u\|_{C^2_{-\beta}(\Omega)}.
\end{aligned}$$

Using (2), this is bounded by

$$C \left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)} \right).$$

Therefore

$$[u]_{C^{2,\alpha}_{-\beta}(\Omega)} \leq C \left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)} \right). \tag{3}$$

Combining (2) and (3) proves the theorem. ☐

Corollary. *Let $\alpha, \beta \in (0,1)$, let $B = B_R(x_0)$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c,$$

where

$$a_{ij}, b_i, c \in C^\alpha(\overline{B}), \qquad \lambda I_n \le (a_{ij}) \le \Lambda I_n, \qquad c \le 0,$$

and

$$\|a_{ij}\|_{C^\alpha(B)}, \ \|b_i\|_{C^\alpha(B)}, \ \|c\|_{C^\alpha(B)} \le \Lambda.$$

If

$$u \in C(\overline{B}) \cap C^2(B)$$

satisfies

$$\begin{cases} Lu = f & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

for some $f \in C^\alpha(B)$ with

$$\|f\|_{C^\alpha_{2-\beta}(B)} < \infty,$$

then

$$\|u\|_{C^{2,\alpha}_{-\beta}(B)} \le C \|f\|_{C^\alpha_{2-\beta}(B)},$$

where

$$C = C(\lambda, \Lambda, \alpha, \beta, R, n).$$

Proof. The previous theorem gives

(1) $$\|u\|_{C^{2,\alpha}_{-\beta}(B)} \le C \left(\|u\|_{C_{-\beta}(B)} + \|f\|_{C^\alpha_{2-\beta}(B)} \right).$$

By the boundary estimate obtained from the maximum principle,

$$\sup_B d_x^{-\beta} |u(x)| \le C \sup_{x\in B} d_x^{2-\beta} |f(x)|.$$

Hence

(2) $$\|u\|_{C_{-\beta}(B)} \le C \|f\|_{C_{2-\beta}(B)}.$$

Combining (1) and (2) proves the corollary. ☐

9. Schauder Estimates and Regularity

The preceding arguments only prove the Schauder estimate. They do not show any $C^{2,\alpha}$-regularity. Furthermore, there we used a $C^{2,\alpha}$-assumption in places where one should replace the C^2 assumption by a $C^{2,\alpha}$-assumption. Actually, the C^2-assumption is enough. This will be proved below.

At this point, what has been proved is the following.

Proposition (Special case: small oscillation)**.** *Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{B}_R), \quad \alpha \in (0,1),$$

and suppose

$$\lambda I_n \le (a_{ij}) \le \Lambda I_n, \qquad \|a_{ij}\|^*_{C^\alpha(\overline{B}_R)} + R\|b_i\|^*_{C^\alpha(\overline{B}_R)} + R^2 \|c\|^*_{C^\alpha(\overline{B}_R)} \le \Lambda.$$

If $u \in C^{2,\alpha}(\overline{B}_R)$ satisfies $Lu = f$ in B_R for some $f \in C^\alpha(\overline{B}_R)$, then there exists $\varepsilon = \varepsilon(\lambda, \Lambda, \alpha, n) > 0$ such that, if

$$\sup_{x', x'' \in B_R} |a_{ij}(x') - a_{ij}(x'')| \leq \varepsilon,$$

then

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \leq C(\lambda, \Lambda, \alpha, n) \left(\|u\|_{L^\infty(B_R)} + R^2 \|f\|^*_{C^\alpha(B_R)} \right).$$

Remark. The regularity assumption $u \in C^{2,\alpha}(\overline{B}_R)$ guarantees that $f \in C^\alpha(\overline{B}_R)$ and the constant K used in the proof is finite. But this assumption can be replaced by a weaker assumption as follows.

Proposition. *Let L be as above. Suppose*

$$u \in L^\infty(B_R) \cap C^2(B_R), \qquad Lu = f \quad \text{in } B_R,$$

for some $f \in C^\alpha(\overline{B}_R)$. Then there exists $\varepsilon = \varepsilon(\lambda, \Lambda, \alpha, n) > 0$ such that, if

$$\sup_{x', x'' \in B_R} |a_{ij}(x') - a_{ij}(x'')| \leq \varepsilon,$$

then $u \in C^{2,\alpha}(B_{R/2})$ and

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \leq C(\lambda, \Lambda, \alpha, n) \left(\|u\|_{L^\infty(B_R)} + R^2 \|f\|^*_{C^\alpha(B_R)} \right).$$

Proof. For every $r \in (0, R)$, we have $u \in C^{2,\alpha}(B_r)$. The previous proposition gives

$$\|u\|^*_{C^{2,\alpha}(B_{r/2})} \leq C(\lambda, \Lambda, \alpha, n) \left(\|u\|_{L^\infty(B_r)} + r^2 \|f\|^*_{C^\alpha(B_r)} \right).$$

Letting $r \to R$ proves the proposition. $\square$

Theorem (General case)**.** *Let*

$$L = \sum_{i,j} a_{ij} \partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{B}_R), \quad \alpha \in (0,1),$$

and assume

$$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n, \qquad \|a_{ij}\|^*_{C^\alpha(B_R)} + R \|b_i\|^*_{C^\alpha(B_R)} + R^2 \|c\|^*_{C^\alpha(B_R)} \leq \Lambda.$$

If

$$u \in L^\infty(B_R) \cap C^2(B_R), \qquad Lu = f \quad \text{in } B_R,$$

for some $f \in C^\alpha(\overline{B}_R)$, then

$$\|u\|^*_{C^{2,\alpha}(B_{R/2})} \leq C(\lambda, \Lambda, \alpha, n) \left(\|u\|_{L^\infty(B_R)} + R^2 \|f\|^*_{C^\alpha(B_R)} \right).$$

Theorem (Schauder estimate in domains)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, and let*

$$L = \sum_{i,j} a_{ij} \partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{\Omega}), \quad \alpha \in (0,1),$$

with

$$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n, \qquad \|a_{ij}\|_{C^\alpha(\Omega)}, \ \|b_i\|_{C^\alpha(\Omega)}, \ \|c\|_{C^\alpha(\Omega)} \leq \Lambda.$$

If $u \in L^\infty(\Omega) \cap C^2(\Omega)$ satisfies $Lu = f$ in Ω for some $f \in C^\alpha(\overline{\Omega})$, then for every $\Omega' \Subset \Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)} \right),$$

where

$$C = C(\Omega', \operatorname{dist}(\Omega', \partial\Omega), \lambda, \Lambda, \alpha, n).$$

Theorem (Schauder estimate revisited)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, let $\alpha, \beta \in (0,1)$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{\Omega}),$$

with

$$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n, \qquad \|a_{ij}\|_{C^\alpha(\Omega)}, \ \|b_i\|_{C^\alpha(\Omega)}, \ \|c\|_{C^\alpha(\Omega)} \leq \Lambda.$$

If $u \in C^{2,\alpha}(\Omega)$ satisfies $Lu = f$ in Ω for some $f \in C^\alpha(\Omega)$, and

$$\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)} < \infty,$$

then

$$\|u\|_{C^{2,\alpha}_{-\beta}(\Omega)} \leq C(\lambda, \Lambda, \alpha, \beta, n) \left(\|u\|_{C_{-\beta}(\Omega)} + \|f\|_{C^\alpha_{2-\beta}(\Omega)} \right).$$

In particular,

$$u \in C^{2,\alpha}_{-\beta}(\Omega).$$

Corollary. *Let $B = B_R(x_0)$, $\alpha, \beta \in (0,1)$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{B}_R).$$

Assume

$$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n, \qquad \|a_{ij}\|_{C^\alpha(B_R)}, \ \|b_i\|_{C^\alpha(B_R)}, \ \|c\|_{C^\alpha(B_R)} \leq \Lambda, \qquad c \leq 0.$$

If $u \in C(\overline{B}_R) \cap C^{2,\alpha}(B_R)$ satisfies

$$\begin{cases} Lu = f & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

for some $f \in C^\alpha(B_R)$ with

$$\|f\|_{C^\alpha_{2-\beta}(B_R)} < \infty,$$

then

$$u \in C^{2,\alpha}_{-\beta}(B_R)$$

and

$$\|u\|_{C^{2,\alpha}_{-\beta}(B_R)} \leq C(R, \lambda, \Lambda, \alpha, n) \|f\|_{C^\alpha_{2-\beta}(B_R)}.$$

10. Dirichlet Problem in Balls

Zero Boundary Value.

Theorem. *Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{B}_R), \quad \alpha, \beta \in (0,1),$$

and assume

$$\lambda I_n \leq (a_{ij}) \leq \Lambda I_n, \qquad \|a_{ij}\|_{C^\alpha(B_R)}, \ \|b_i\|_{C^\alpha(B_R)}, \ \|c\|_{C^\alpha(B_R)} \leq \Lambda, \qquad c \leq 0.$$

For every $f \in C^\alpha_{2-\beta}(B_R)$, there exists a unique solution

$$u \in C^{2,\alpha}_{-\beta}(B_R)$$

of the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases}$$

Remark. Here

$$C^{\alpha}_{2-\beta}(B_R) \subset C^{\alpha}(B_R), \qquad C^{2,\alpha}_{-\beta}(B_R) \subset C(\overline{B}_R) \cap C^{2,\alpha}(B_R).$$

Proof. By scaling, assume $R = 1$. We use the continuity method. Define

$$L_t = tL + (1-t)\Delta, \qquad t \in [0,1].$$

Then

$$L_0 = \Delta, \qquad L_1 = L.$$

Write

$$L_t = \sum_{i,j} a^t_{ij}\partial_{ij} + \sum_i b^t_i \partial_i + c^t,$$

where

$$a^t_{ij} = ta_{ij} + (1-t)\delta_{ij}, \qquad b^t_i = tb_i, \qquad c^t = tc.$$

The assumptions imply

$$\min(\lambda, 1) I_n \le (a^t_{ij}) \le \max(\Lambda, 1) I_n,$$

and

$$\left\|a^t_{ij}\right\|_{C^{\alpha}(B_1)}, \ \left\|b^t_i\right\|_{C^{\alpha}(B_1)}, \ \left\|c^t\right\|_{C^{\alpha}(B_1)} \le \Lambda + 1.$$

For $v \in C^{2,\alpha}_{-\beta}(B_1)$, compute

$$\|L_t v\|_{C^{\alpha}_{2-\beta}(B_1)} \le \sum_{i,j} \left\|a^t_{ij}\partial_{ij} v\right\|_{C^{\alpha}_{2-\beta}(B_1)} + \sum_i \left\|b^t_i \partial_i v\right\|_{C^{\alpha}_{2-\beta}(B_1)} + \left\|c^t v\right\|_{C^{\alpha}_{2-\beta}(B_1)}.$$

Recall that, for $\sigma \ge 0$,

$$\|wv\|_{C^{\alpha}_{\sigma}(B_1)} \le \|w\|_{C^{\alpha}_{0}(B_1)} \|v\|_{C^{\alpha}_{\sigma}(B_1)}.$$

Also, for $x, y \in B_1$, we have $d_x, d_y, d_{x,y} \le 1$. Hence

$$\|w\|_{C^{\alpha}_{0}(B_1)} \le \|w\|_{C^{\alpha}(B_1)}, \qquad \|w\|_{C^{\alpha}_{2-\beta}(B_1)} \le \|w\|_{C^{\alpha}_{1-\beta}(B_1)} \le \|w\|_{C^{\alpha}_{-\beta}(B_1)}.$$

Therefore

$$\begin{aligned} \|L_t v\|_{C^{\alpha}_{2-\beta}(B_1)} &\le \sum_{i,j} \left\|a^t_{ij}\right\|_{C^{\alpha}(B_1)} \|\partial_{ij} v\|_{C^{\alpha}_{2-\beta}(B_1)} \\ &\quad + \sum_i \left\|b^t_i\right\|_{C^{\alpha}(B_1)} \|\partial_i v\|_{C^{\alpha}_{1-\beta}(B_1)} + \left\|c^t\right\|_{C^{\alpha}(B_1)} \|v\|_{C^{\alpha}_{-\beta}(B_1)} \\ &\le C(\Lambda, n) \|v\|_{C^{2,\alpha}_{-\beta}(B_1)}. \end{aligned}$$

Thus

$$L_t : C^{2,\alpha}_{-\beta}(B_1) \longrightarrow C^{\alpha}_{2-\beta}(B_1)$$

is a bounded linear operator.

Consider the Dirichlet problem

$$(*_t) \qquad \begin{cases} L_t u = h & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

Define

$$I = \left\{ t \in [0,1] \;\middle|\; (*_t) \text{ is solvable in } C^{2,\alpha}_{-\beta}(B_1) \text{ for any } h \in C^{\alpha}_{2-\beta}(B_1) \right\}.$$

The goal is to prove $1 \in I$. Uniqueness follows from the maximum principle.

Step 1. We prove $0 \in I$, i.e. $L_0 = \Delta$. Let

$$h \in C^{\alpha}_{2-\beta}(B_1).$$

The function h may not belong to $C^\alpha(\overline{B}_1)$, so we cannot apply the unweighted Poisson-equation solvability result directly. To avoid this, take $\eta_\varepsilon \in C^\infty(\overline{B}_1)$ such that

$$0 \le \eta_\varepsilon \le 1 \quad \text{in } \overline{B}_1, \qquad \eta_\varepsilon \equiv 1 \quad \text{in } B_{1-2\varepsilon}, \qquad \eta_\varepsilon \equiv 0 \quad \text{in } B_1 \setminus B_{1-\varepsilon}.$$

Set

$$h_\varepsilon = \eta_\varepsilon h.$$

Then $h_\varepsilon \in C^\alpha(\overline{B}_1)$. By the solvability result for the Poisson equation in a ball, there exists

$$u_\varepsilon \in C(\overline{B}_1) \cap C^{2,\alpha}(B_1)$$

such that

$$\begin{cases} \Delta u_\varepsilon = h_\varepsilon & \text{in } B_1, \\ u_\varepsilon = 0 & \text{on } \partial B_1. \end{cases}$$

The boundary estimate obtained from the maximum principle gives

$$\begin{aligned} \sup_{x \in B_1} d_x^{-\beta} \left| u_\varepsilon(x) \right| &\le C \sup_{x \in B_1} d_x^{2-\beta} \left| h_\varepsilon(x) \right| \\ &\le C \left\| h \right\|_{C_{2-\beta}(B_1)}. \end{aligned}$$

Hence

$$\left| u_\varepsilon(x) \right| \le C \left\| h \right\|_{C_{2-\beta}(B_1)} d_x^\beta, \qquad x \in B_1, \tag{1}$$

and, since $\beta \in (0,1)$,

$$\left\| u_\varepsilon \right\|_{L^\infty(B_1)} \le C \left\| h \right\|_{C_{2-\beta}(B_1)}. \tag{2}$$

For $r \in (0,1)$, set $\hat{r} = (r+1)/2$. If $\varepsilon \le (1-\hat{r})/100$, then $h_\varepsilon = h$ in $B_{\hat{r}}$. Since

$$h \in C^\alpha_{2-\beta}(B_1),$$

we have $h \in C^\alpha(\overline{B}_{\hat{r}})$. For

$$u_\varepsilon \in C(\overline{B}_1) \cap C^{2,\alpha}(B_1),$$

the Schauder estimate gives

$$\begin{aligned} \left\| u_\varepsilon \right\|_{C^{2,\alpha}(B_r)} &\le C(r, \lambda, \Lambda, \alpha, \beta, n) \left(\left\| u_\varepsilon \right\|_{L^\infty(B_1)} + \left\| h \right\|_{C^\alpha(B_{\hat{r}})} \right) \\ &\le C(r, \lambda, \Lambda, \alpha, \beta, n) \left(\left\| h \right\|_{C_{2-\beta}(B_1)} + \left\| h \right\|_{C^\alpha(B_{\hat{r}})} \right), \end{aligned}$$

which is independent of ε. By the Arzela–Ascoli theorem and a diagonal argument, there exist $\varepsilon_i \to 0$ and

$$u \in C^2_{\mathrm{loc}}(B_1)$$

such that

$$u_{\varepsilon_i} \to u \quad \text{in } C^2_{\mathrm{loc}}(B_1).$$

Passing to the limit in

$$\Delta u_{\varepsilon_i} = h_{\varepsilon_i} \quad \text{in } B_1$$

gives

$$\Delta u = h \quad \text{in } B_1.$$

Moreover, by (1),

$$\left| u(x) \right| \le C \left\| h \right\|_{C_{2-\beta}(B_1)} d_x^\beta,$$

so u can be extended continuously to $\overline{B}_1$ with

$$u = 0 \quad \text{on } \partial B_1.$$

Thus

$$u \in C(\overline{B}_1) \cap C^2(B_1)$$

solves

$$\begin{cases} \Delta u = h & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

Since $h \in C^{\alpha}_{2-\beta}(B_1)$, the weighted Schauder corollary above yields

$$u \in C^{2,\alpha}_{-\beta}(B_1).$$

Step 2. We prove that there exists $\delta > 0$ such that, if $t \in I$, $s \in [0,1]$, and $|s-t| < \delta$, then $s \in I$.

Together with Step 1, this proves the goal $1 \in I$: divide $[0,1]$ into finitely many subintervals I_i with $|I_i| < \delta$. Then

$$0 \in I \quad \Longrightarrow \quad 1 \in I.$$

For $t \in I$, $(*_t)$ is solvable in $C^{2,\alpha}_{-\beta}(B_1)$ for any $h \in C^{\alpha}_{2-\beta}(B_1)$. By the maximum principle, the solution is unique. Denote the unique solution of $(*_t)$ by

$$L_t^{-1} h \in C^{2,\alpha}_{-\beta}(B_1),$$

i.e.

$$\begin{cases} L_t(L_t^{-1} h) = h & \text{in } B_1, \\ L_t^{-1} h = 0 & \text{on } \partial B_1. \end{cases}$$

The weighted Schauder estimate above gives

$$\left\| L_t^{-1} h \right\|_{C^{2,\alpha}_{-\beta}(B_1)} \leq C \left\| h \right\|_{C^{\alpha}_{2-\beta}(B_1)}, \tag{3}$$

where C is independent of t. Hence

$$L_t^{-1} : C^{\alpha}_{2-\beta}(B_1) \longrightarrow C^{2,\alpha}_{-\beta}(B_1)$$

is a bounded linear operator.

Define

$$T : C^{2,\alpha}_{-\beta}(B_1) \longrightarrow C^{2,\alpha}_{-\beta}(B_1)$$

by

$$Tv = L_t^{-1}\left(h + (t-s)(Lv - \Delta v)\right).$$

The motivation is

$$L_s V = h \Longleftrightarrow L_t V = h + L_t V - L_s V = h + (t-s)(LV - \Delta V),$$

where

$$L_t = tL + (1-t)\Delta, \qquad L_s = sL + (1-s)\Delta.$$

Thus

$$\begin{cases} L_s V = h & \text{in } B_1, \\ V = 0 & \text{on } \partial B_1 \end{cases} \Longleftrightarrow \begin{cases} L_t V = h + (t-s)(LV - \Delta V) & \text{in } B_1, \\ V = 0 & \text{on } \partial B_1 \end{cases}$$

and hence

$$V = L_t^{-1}\left(h + (t-s)(LV - \Delta V)\right) \Longleftrightarrow V = TV.$$

Therefore V solves $(*_s)$ if and only if V is a fixed point of T.

For $v, w \in C^{2,\alpha}_{-\beta}(B_1)$, compute

$$\begin{aligned}\|Tv - Tw\|_{C^{2,\alpha}_{-\beta}(B_1)} &= \left\|(t-s)L_t^{-1}\left((L-\Delta)(v-w)\right)\right\|_{C^{2,\alpha}_{-\beta}(B_1)}\\ &\le C\,|t-s|\,\|(L-\Delta)(v-w)\|_{C^{\alpha}_{2-\beta}(B_1)}\\ &\le C\,|t-s|\,\|v-w\|_{C^{2,\alpha}_{-\beta}(B_1)}\,.\end{aligned}$$

Choose

$$\delta = \frac{1}{2C}.$$

Then, when $|t-s| < \delta$, T is a contraction. The contraction mapping theorem gives a $V \in C^{2,\alpha}_{-\beta}(B_1)$ such that $TV = V$. Equivalently, V solves $(*_s)$, so $s \in I$. This finishes the proof. $\square$

Continuous Boundary Value.

Theorem. *Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\overline{B}_R), \quad \alpha \in (0,1),$$

and assume

$$\lambda I_n \le (a_{ij}) \le \Lambda I_n, \qquad \|a_{ij}\|_{C^\alpha(B_R)}, \ \|b_i\|_{C^\alpha(B_R)}, \ \|c\|_{C^\alpha(B_R)} \le \Lambda, \qquad c \le 0.$$

For every $f \in C^\alpha(\overline{B}_R)$ and every $\varphi \in C(\partial B_R)$, there exists a unique solution

$$u \in C(\overline{B}_R) \cap C^{2,\alpha}(B_R)$$

of

$$\begin{cases} Lu = f & \text{in } B_R,\\ u = \varphi & \text{on } \partial B_R.\end{cases}$$

Proof. By scaling, assume $R = 1$. Let $\{\varphi_k\}$ be a sequence of functions in $C^2(\overline{B}_1)$ such that

$$\|\varphi_k - \varphi\|_{C(\partial B_1)} \to 0.$$

Then

$$f - L\varphi_k \in C^\alpha_{2-\beta}(B_1) \qquad \text{for } \beta \in (0,1).$$

The previous theorem gives

$$v_k \in C^{2,\alpha}_{-\beta}(B_1) \subset C(\overline{B}_1) \cap C^{2,\alpha}(B_1)$$

solving

$$\begin{cases} Lv_k = f - L\varphi_k & \text{in } B_1,\\ v_k = 0 & \text{on } \partial B_1.\end{cases}$$

Set

$$u_k = v_k + \varphi_k.$$

Then $u_k \in C(\overline{B}_1) \cap C^{2,\alpha}(B_1)$ satisfies

$$\begin{cases} Lu_k = f & \text{in } B_1,\\ u_k = \varphi_k & \text{on } \partial B_1.\end{cases}$$

For any k, l, we have

$$\begin{cases} L(u_k - u_l) = 0 & \text{in } B_1, \\ u_k - u_l = \varphi_k - \varphi_l & \text{on } \partial B_1. \end{cases}$$

The maximum principle gives

$$\|u_k - u_l\|_{C(\overline{B}_1)} \le \|\varphi_k - \varphi_l\|_{C(\partial B_1)} \to 0.$$

Hence there exists $u \in C(\overline{B}_1)$ such that

$$u_k \to u \quad \text{in } C(\overline{B}_1),$$

and $u = \varphi$ on ∂B_1.

For $r \in (0,1)$, the Schauder estimate gives

$$\begin{aligned} \|u_k\|_{C^{2,\alpha}(B_r)} &\le C(r, \lambda, \Lambda, \alpha, n)\left(\|u_k\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)}\right) \\ &\le C(r, \lambda, \Lambda, \alpha, n)\left(\|u\|_{L^\infty(B_1)} + 1 + \|f\|_{C^\alpha(B_1)}\right), \end{aligned}$$

which is independent of k. By the Arzela–Ascoli theorem and a diagonal argument, there exists a subsequence u_{k_i} such that

$$u_{k_i} \to u \quad \text{in } C^2_{\text{loc}}(B_1).$$

Therefore

$$u \in C^{2,\alpha}(B_1), \qquad Lu = f \quad \text{in } B_1.$$

Uniqueness follows from the maximum principle. □

$C^{2,\alpha}$-Regularity.

Corollary. *Let Ω be a domain, not necessarily bounded, in $\mathbb{R}^n$. Let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\Omega), \quad \alpha \in (0,1),$$

and assume

$$\lambda I_n \le (a_{ij}) \le \Lambda I_n.$$

If $u \in C^2(\Omega)$ satisfies

$$Lu = f \quad \textit{in } \Omega$$

for some $f \in C^\alpha(\Omega)$, then

$$u \in C^{2,\alpha}(\Omega).$$

Proof. Define

$$\widetilde{L} = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i, \qquad \widetilde{f} = f - cu.$$

For every ball $B = B_R(x_0) \Subset \Omega$, we have

$$a_{ij}, b_i, c, \widetilde{f} \in C^\alpha(\overline{B}), \qquad u \in C(\partial B).$$

The previous theorem gives a unique

$$v \in C(\overline{B}) \cap C^{2,\alpha}(B)$$

solving

the Dirichlet problem

$$\begin{cases} \widetilde{L}v = \widetilde{f} & \text{in } B, \\ v = u & \text{on } \partial B. \end{cases}$$

Since $Lu = f$, equivalently $\widetilde{L}u = f - cu = \widetilde{f}$ in B, and $u = u$ on ∂B. Thus u is also a solution of this problem. By uniqueness,

$$u = v \in C(\overline{B}) \cap C^{2,\alpha}(B).$$

Thus $u \in C^{2,\alpha}(B)$. Since $B \Subset \Omega$ is arbitrary, we get

$$u \in C^{2,\alpha}(\Omega).$$

□

Remark. Using this corollary, in the previous Schauder estimate the $C^{2,\alpha}$-assumption can be replaced by a C^2-assumption.

11. Dirichlet Problem in Domains

The method is Perron's method.

Subsolution and Supersolution.

Definition. Let Ω be a bounded domain in $\mathbb{R}^n$. Let

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c, \qquad a_{ij}, b_i, c \in C(\Omega),$$

with $(a_{ij}) > 0$, $c \leq 0$, and let $f \in C(\Omega)$.

A function $u \in C(\Omega)$ is called a subsolution of $Lu = f$ in Ω if the following holds: for every ball

$$B = B_R(x_0) \Subset \Omega$$

and every $w \in C(\overline{B}) \cap C^2(B)$ solving

$$Lw = f \quad \text{in } B,$$

if

$$u \leq w \quad \text{on } \partial B,$$

then

$$u \leq w \quad \text{in } B. \tag{*}$$

Replacing the signs "$\leq$" by "$\geq$" in $(*)$ gives the definition of a supersolution.

Remark. If $u \in C^2(\Omega)$ and $Lu \geq f$, then u is a subsolution as above.

Lemma. *Let Ω be a bounded domain in $\mathbb{R}^n$, $\alpha \in (0,1)$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\Omega),$$

with $(a_{ij}) > 0$, $c \leq 0$, and $f \in C^\alpha(\Omega)$. Suppose

$$u \in C(\overline{\Omega})$$

is a subsolution in Ω, and

$$v \in C(\overline{\Omega})$$

is a supersolution in Ω. If

$$u \leq v \quad \text{on } \partial\Omega,$$

then

$$u \leq v \quad \text{in } \Omega.$$

Proof. Set
$$M = \max_{\overline{\Omega}}(u - v).$$
If $M \le 0$, then $u \le v$ in Ω.

Assume $M > 0$. Define
$$D = \{x \in \Omega \mid u(x) - v(x) = M\}.$$
We claim that D is open and closed. The claim implies either $D = \emptyset$ or $D = \Omega$. If $D = \emptyset$, then
$$M = \max_{\partial\Omega}(u - v) \le 0,$$
a contradiction. If $D = \Omega$, then $u - v = M$ in Ω, and hence
$$u - v = M \quad \text{on } \partial\Omega,$$
again contradicting $M > 0$ and $u \le v$ on $\partial\Omega$.

It remains to prove the claim. Closedness follows from the continuity of $u - v$. To prove openness, take $x_0 \in D$ and choose
$$0 < r < d := \operatorname{dist}(x_0, \partial\Omega), \qquad B = B_r(x_0) \Subset \Omega.$$
Let
$$\overline{u}, \overline{v} \in C(\overline{B}) \cap C^{2,\alpha}(B)$$
solve
$$\begin{cases} L\overline{u} = f & \text{in } B, \\ \overline{u} = u & \text{on } \partial B, \end{cases} \qquad \begin{cases} L\overline{v} = f & \text{in } B, \\ \overline{v} = v & \text{on } \partial B. \end{cases}$$
By the definitions of subsolution and supersolution,
$$u \le \overline{u} \quad \text{in } B, \qquad v \ge \overline{v} \quad \text{in } B.$$
Thus
$$\overline{u} - \overline{v} \ge u - v \quad \text{in } B. \tag{1}$$
Moreover,
$$\begin{cases} L(\overline{u} - \overline{v}) = 0 & \text{in } B, \\ \overline{u} - \overline{v} = u - v & \text{on } \partial B. \end{cases}$$
By the maximum principle,
$$\max_{\overline{B}}(\overline{u} - \overline{v}) \le \max_{\partial B}(u - v)^+ \le M.$$
Since $x_0 \in D$, by (1),
$$M = (u - v)(x_0) \le (\overline{u} - \overline{v})(x_0) \le M.$$
Hence
$$(\overline{u} - \overline{v})(x_0) = M.$$
The strong maximum principle gives
$$\overline{u} - \overline{v} \equiv M \quad \text{in } B,$$
and therefore
$$\overline{u} - \overline{v} \equiv M \quad \text{on } \partial B.$$
Since $\overline{u} - \overline{v} = u - v$ on $\partial B = \partial B_r(x_0)$, we obtain
$$u - v = M \quad \text{on } \partial B_r(x_0).$$

The radius $r \in (0, d)$ was arbitrary, so

$$u - v = M \quad \text{in } B_d(x_0).$$

Thus

$$B_d(x_0) \subset D,$$

and D is open. This proves the claim and hence the lemma. $\square$

Lemma (Lifting lemma)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, $\alpha \in (0,1)$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i + c, \qquad a_{ij}, b_i, c \in C^\alpha(\Omega),$$

with $(a_{ij}) > 0$, $c \le 0$, and $f \in C^\alpha(\Omega)$. Let

$$u \in C(\Omega)$$

be a subsolution in Ω, and let $B = B_R(x_0) \Subset \Omega$. Define

$$\widehat{u} = \begin{cases} u & \text{in } \Omega \setminus B, \\ v & \text{in } B, \end{cases}$$

where v solves

$$\begin{cases} Lv = f & \text{in } B, \\ v = u & \text{on } \partial B. \end{cases}$$

Then $\widehat{u}$ is a subsolution in Ω, and

$$\widehat{u} \ge u \quad \text{in } \Omega.$$

Remark. $\widehat{u}$ is called the lifting of u in B.

Proof. On ∂B we have $u = v$. Since v solves $Lv = f$ in B, the definition of subsolution applied to the ball B gives $u \le v$ in B. Outside B, $\widehat{u} = u$. Hence $u \le \widehat{u}$ in Ω.

Let $B' \Subset \Omega$ be a ball, and let

$$w \in C(\overline{B'}) \cap C^2(B')$$

satisfy

$$\begin{cases} Lw = f & \text{in } B', \\ w \ge \widehat{u} & \text{on } \partial B'. \end{cases}$$

The goal is to show

$$\widehat{u} \le w \quad \text{in } B'.$$

If $B \cap B' = \emptyset$, this follows directly from the definition of subsolution. Assume

$$B \cap B' \ne \emptyset.$$

Step 1. We prove

$$\widehat{u} \le w \quad \text{in } B' \setminus B.$$

On $\partial B'$ we have $u \le \widehat{u} \le w$. Since u is a subsolution, this gives

$$u \le w \quad \text{in } B'.$$

Hence, in $B' \setminus B$,

$$\widehat{u} = u \le w.$$

Step 2. We prove

$$\widehat{u} \le w \quad \text{in } B' \cap B.$$

By Step 1,
$$\widehat{u} \leq w \quad \text{in } B' \setminus B,$$
and therefore
$$v = \widehat{u} \leq w \quad \text{on } \overline{B'} \cap \partial B.$$
Also, from
$$\widehat{u} \leq w \quad \text{on } \partial B',$$
we get
$$v = \widehat{u} \leq w \quad \text{on } \partial B' \cap \overline{B}.$$
In $B' \cap B$,
$$Lv = f = Lw.$$
Thus
$$\begin{cases} Lv = Lw & \text{in } B' \cap B, \\ v \leq w & \text{on } \partial(B' \cap B). \end{cases}$$
The comparison principle gives
$$v \leq w \quad \text{in } B' \cap B.$$
Since $\widehat{u} = v$ in B, this proves
$$\widehat{u} \leq w \quad \text{in } B' \cap B.$$
Combining Steps 1 and 2 proves the goal, and hence $\widehat{u}$ is a subsolution in Ω. $\square$

Recall. Recall the following points.

- The Dirichlet problem in balls was solved by the continuity method.
- The Dirichlet problem in domains will be solved by Perron's method.
- We defined subsolutions and supersolutions. If u is a subsolution and v is a supersolution with $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.
- If u is a subsolution and $B \Subset \Omega$, then the lifting
$$\widehat{u} = \begin{cases} u & \text{in } \Omega \setminus B, \\ v & \text{in } B, \end{cases} \qquad \begin{cases} Lv = f & \text{in } B, \\ v = u & \text{on } \partial B, \end{cases}$$
is still a subsolution and $\widehat{u} \geq u$.

Perron's Method.

Proposition. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let*
$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega), \quad \alpha \in (0,1).$$
Assume
$$(a_{ij}) \geq \lambda I_n \quad (\lambda > 0), \qquad c \leq 0,$$
and let
$$f \in L^\infty(\Omega) \cap C^\alpha(\Omega), \qquad \varphi \in C(\partial\Omega).$$
Define
$$\mathcal{S}_\varphi = \left\{ v \;\middle|\; v \in C(\overline{\Omega}) \textit{ is a subsolution in } \Omega,\ v \leq \varphi \textit{ on } \partial\Omega \right\},$$
and
$$u_\varphi(x) = \sup\{v(x) \mid v \in \mathcal{S}_\varphi\}, \qquad x \in \Omega.$$
Then
$$u_\varphi \in C^{2,\alpha}(\Omega), \qquad Lu_\varphi = f \quad \textit{in } \Omega.$$

Proof. Step 1. We show that u_φ is well-defined, i.e.

$$\mathcal{S}_\varphi \neq \emptyset, \qquad |u_\varphi(x)| < +\infty \quad \text{for all } x \in \Omega.$$

After a translation, assume

$$\Omega \subset \{0 < x_1 < d\}.$$

Set

$$F = \|f\|_{L^\infty(\Omega)}, \qquad \Phi = \|\varphi\|_{L^\infty(\partial\Omega)}.$$

Define

$$V_+(x) = \Phi + \left(e^{Ad} - e^{Ax_1}\right) F, \qquad V_-(x) = -V_+(x),$$

where A is a constant to be determined. Compute

$$LV_+ = -\left(a_{11}A^2 + b_1 A\right) F e^{Ax_1} + cV_+.$$

Since $a_{11} \geq \lambda$ and $b_1 \in L^\infty(\Omega)$, choose $A \gg 1$ so that

$$a_{11}A^2 + b_1 A \geq 1.$$

Then

$$LV_+ \leq -Fe^{Ax_1} + cV_+.$$

Because $V_+ \geq 0$, $c \leq 0$, and $e^{Ax_1} \geq 1$,

$$LV_+ \leq -F \leq f \quad \text{in } \Omega.$$

Moreover,

$$V_+ \geq \Phi \geq \varphi \quad \text{on } \partial\Omega.$$

Thus V_+ is a supersolution.

Recall $V_- = -V_+$. Then

$$\begin{cases} LV_- \geq F \geq f & \text{in } \Omega, \\ V_- \leq -\Phi \leq \varphi & \text{on } \partial\Omega. \end{cases}$$

Thus V_- is a subsolution. In particular $V_- \in \mathcal{S}_\varphi$, so

$$\mathcal{S}_\varphi \neq \emptyset.$$

For every $v \in \mathcal{S}_\varphi$, we have

$$v \leq \varphi \leq V_+ \quad \text{on } \partial\Omega.$$

By comparison,

$$v \leq V_+ \quad \text{in } \Omega.$$

Therefore

$$V_- \leq u_\varphi \leq V_+ \quad \text{in } \Omega,$$

and u_φ is well-defined.

Step 2. If $v_1, v_2 \in \mathcal{S}_\varphi$, then

$$V = \max(v_1, v_2) \in \mathcal{S}_\varphi.$$

Indeed, since $v_i \leq \varphi$ on $\partial\Omega$, we get

$$V \leq \varphi \quad \text{on } \partial\Omega. \tag{1}$$

To prove that V is a subsolution, take any ball $B \Subset \Omega$ and any $w \in C(\overline{B}) \cap C^2(B)$ such that

$$\begin{cases} Lw = f & \text{in } B, \\ w \geq V & \text{on } \partial B. \end{cases}$$

Then
$$w \geq v_i \quad \text{on } \partial B, \qquad i = 1, 2.$$
Since each v_i is a subsolution,
$$w \geq v_i \quad \text{in } B, \qquad i = 1, 2.$$
Hence $w \geq V$ in B, so V is a subsolution. Together with (1), this proves $V \in \mathcal{S}_\varphi$.

Step 3. We prove
$$u_\varphi \in C^{2,\alpha}(\Omega), \qquad Lu_\varphi = f.$$
It is enough to show that for every ball $B_r(x_0) \Subset \Omega$,
$$u_\varphi \in C^{2,\alpha}(B_r(x_0)), \qquad Lu_\varphi = f \quad \text{in } B_r(x_0).$$

By definition,
$$u_\varphi(x_0) = \sup_{v \in \mathcal{S}_\varphi} v(x_0).$$
Choose $v_i \in \mathcal{S}_\varphi$ such that
$$\lim_{i \to \infty} v_i(x_0) = u_\varphi(x_0).$$
We may assume
$$V_- \leq v_i \leq u_\varphi \quad \text{in } \Omega;$$
otherwise replace v_i by $\max(v_i, V_-)$, using Step 2. Let w_i be the lifting of v_i in $B_r(x_0)$. Then $w_i \in \mathcal{S}_\varphi$, and
$$V_- \leq v_i \leq w_i \leq u_\varphi \leq V_+ \quad \text{in } \Omega,$$
with
$$Lw_i = f \quad \text{in } B_r(x_0), \qquad \lim_{i \to \infty} w_i(x_0) = u_\varphi(x_0).$$
Therefore
$$\|w_i\|_{L^\infty(B_r(x_0))} \leq C,$$
where C is independent of i. By the Schauder estimate, for every $r' \in (0, r)$,
$$\|w_i\|_{C^{2,\alpha}(B_{r'}(x_0))} \leq C_{r'},$$
where $C_{r'}$ is independent of i.

By the Arzela–Ascoli theorem and a diagonal argument, there is a subsequence, still denoted by w_i, such that
$$w_i \to w \quad \text{in } C^2_{\mathrm{loc}}(B_r(x_0))$$
for some
$$w \in C^{2,\alpha}(B_r(x_0)).$$
It follows that
$$w \leq u_\varphi, \qquad Lw = f \quad \text{in } B_r(x_0), \qquad w(x_0) = u_\varphi(x_0).$$
We claim that
$$w = u_\varphi \quad \text{in } B_r(x_0).$$
The claim implies Step 3.

Let $\overline{x} \in B_r(x_0)$. Since
$$u_\varphi(\overline{x}) = \sup_{v \in \mathcal{S}_\varphi} v(\overline{x}),$$
there exist $\overline{v}_i \in \mathcal{S}_\varphi$ such that
$$\lim_{i \to \infty} \overline{v}_i(\overline{x}) = u_\varphi(\overline{x}).$$

We may assume
$$w_i \le \overline{v}_i \le u_\varphi \quad \text{in } \Omega;$$
otherwise replace $\overline{v}_i$ by $\max(\overline{v}_i, w_i)$, using Step 2. Let $\overline{w}_i$ be the lifting of $\overline{v}_i$ in $B_r(x_0)$. Then
$$V_- \le v_i \le w_i \le \overline{v}_i \le \overline{w}_i \le u_\varphi \le V_+ \quad \text{in } \Omega,$$
and
$$\lim_{i\to\infty} \overline{w}_i(\overline{x}) = u_\varphi(\overline{x}).$$
The same Schauder estimate and compactness argument gives a subsequence such that
$$\overline{w}_i \to \overline{w} \quad \text{in } C^2_{\mathrm{loc}}(B_r(x_0))$$
for some $\overline{w} \in C^{2,\alpha}(B_r(x_0))$. Hence
$$w \le \overline{w} \le u_\varphi, \qquad Lw = L\overline{w} = f \quad \text{in } B_r(x_0),$$
and
$$w(x_0) = \overline{w}(x_0) = u_\varphi(x_0), \qquad \overline{w}(\overline{x}) = u_\varphi(\overline{x}).$$
Thus
$$\begin{cases} L(w - \overline{w}) = 0 & \text{in } B_r(x_0), \\ w - \overline{w} \le 0. \end{cases}$$
Since
$$(w - \overline{w})(x_0) = 0,$$
the strong maximum principle gives
$$w - \overline{w} \equiv 0 \quad \text{in } B_r(x_0).$$
Therefore
$$w(\overline{x}) = \overline{w}(\overline{x}) = u_\varphi(\overline{x}).$$
Since $\overline{x} \in B_r(x_0)$ is arbitrary, this proves
$$w = u_\varphi \quad \text{in } B_r(x_0).$$
The proposition follows. □

Proposition (Boundary behavior under a barrier)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$, and let*
$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega), \quad \alpha \in (0,1).$$
Assume
$$(a_{ij}) \ge \lambda I_n \quad (\lambda > 0), \qquad c \le 0,$$
and let
$$f \in L^\infty(\Omega) \cap C^\alpha(\Omega), \qquad \varphi \in C(\partial\Omega).$$
For $x_0 \in \partial\Omega$, suppose there exists
$$W_{x_0} \in C(\overline{\Omega}) \cap C^2(\Omega)$$
such that
$$LW_{x_0} \le -1 \quad \textit{in } \Omega, \qquad W_{x_0}(x_0) = 0, \qquad W_{x_0} > 0 \quad \textit{on } \partial\Omega \setminus \{x_0\}.$$
Then
$$\lim_{x\to x_0} u_\varphi(x) = \varphi(x_0).$$

Remark. The function W_{x_0} is called a barrier function at x_0.

Proof. Write $W = W_{x_0}$ and
$$\Phi = \|\varphi\|_{L^\infty(\partial\Omega)} .$$
Since $\varphi \in C(\partial\Omega)$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|\varphi - \varphi(x_0)| < \varepsilon \quad \text{on } B_\delta(x_0) \cap \partial\Omega.$$
Because $W > 0$ on $\partial\Omega \setminus \{x_0\}$,
$$\min_{\partial\Omega \setminus B_\delta(x_0)} W > 0.$$
Therefore there exists $K_\varepsilon > 0$ such that
$$|\varphi - \varphi(x_0)| \le 2\Phi \le K_\varepsilon W \quad \text{on } \partial\Omega \setminus B_\delta(x_0).$$
Consequently,
$$|\varphi - \varphi(x_0)| \le \varepsilon + K_\varepsilon W \quad \text{on } \partial\Omega. \tag{1}$$

Set
$$A_\varepsilon = \max\left(K_\varepsilon, \|f - c\varphi(x_0)\|_{L^\infty(\Omega)}\right).$$
Then
$$\begin{aligned} L\left(\varphi(x_0) - \varepsilon - A_\varepsilon W\right) &= c\varphi(x_0) - c\varepsilon - A_\varepsilon LW \\ &\ge c\varphi(x_0) + A_\varepsilon \\ &\ge f \quad \text{in } \Omega. \end{aligned}$$
Thus
$$\varphi(x_0) - \varepsilon - A_\varepsilon W$$
is a subsolution. By (1),
$$\varphi(x_0) - \varepsilon - A_\varepsilon W \le \varphi \quad \text{on } \partial\Omega.$$
Hence
$$\varphi(x_0) - \varepsilon - A_\varepsilon W \in \mathcal{S}_\varphi.$$
Recalling that
$$u_\varphi = \sup_{v \in \mathcal{S}_\varphi} v,$$
we get
$$\varphi(x_0) - \varepsilon - A_\varepsilon W \le u_\varphi \quad \text{in } \Omega. \tag{2}$$

Similarly,
$$\begin{cases} L\left(\varphi(x_0) + \varepsilon + A_\varepsilon W\right) \le f & \text{in } \Omega, \\ \varphi(x_0) + \varepsilon + A_\varepsilon W \ge \varphi & \text{on } \partial\Omega. \end{cases}$$
Thus
$$\varphi(x_0) + \varepsilon + A_\varepsilon W$$
is a supersolution with boundary value above φ. By comparison,
$$u_\varphi \le \varphi(x_0) + \varepsilon + A_\varepsilon W \quad \text{in } \Omega. \tag{3}$$
Combining (2) and (3),
$$|u_\varphi - \varphi(x_0)| \le \varepsilon + A_\varepsilon W \quad \text{in } \Omega.$$
Therefore
$$\limsup_{x \to x_0} |u_\varphi(x) - \varphi(x_0)| \le \varepsilon + A_\varepsilon \limsup_{x \to x_0} W(x) = \varepsilon.$$

Letting $\varepsilon \to 0$ gives

$$\lim_{x \to x_0} |u_\varphi(x) - \varphi(x_0)| = 0.$$

□

Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega), \quad \alpha \in (0,1).$$

Assume

$$(a_{ij}) \geq \lambda I_n \quad (\lambda > 0), \qquad c \leq 0,$$

and let

$$f \in L^\infty(\Omega) \cap C^\alpha(\Omega), \qquad \varphi \in C(\partial\Omega).$$

If for every $x_0 \in \partial\Omega$ there exists a barrier function at x_0, then there exists a unique solution

$$u \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$$

of the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Remark. Reviewing Perron's method, the interior part uses the maximum principle and solvability of the Dirichlet problem in balls. The boundary part uses barrier functions, which encode local geometry of the domain.

The natural question is how to guarantee the existence of barrier functions.

Exterior Sphere Condition.

Definition. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $x_0 \in \partial\Omega$. If there exists a ball $B_R(y_0)$ such that

$$\Omega \cap B_R(y_0) = \emptyset, \qquad \overline{\Omega} \cap \overline{B}_R(y_0) = \{x_0\},$$

then we say that Ω satisfies the exterior sphere condition at x_0.

If Ω satisfies the exterior sphere condition at every boundary point, then we say that Ω satisfies the exterior sphere condition.

Lemma. *Let Ω be a bounded domain in $\mathbb{R}^n$, and let*

$$L = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C(\Omega).$$

Assume

$$(a_{ij}) \geq \lambda I_n \quad (\lambda > 0), \qquad c \leq 0.$$

If Ω satisfies the exterior sphere condition at $x_0 \in \partial\Omega$, then there exists a barrier function at x_0.

Proof. By the exterior sphere condition, there exists $B_R(y_0)$ such that

$$\Omega \cap B_R(y_0) = \emptyset, \qquad \overline{\Omega} \cap \overline{B}_R(y_0) = \{x_0\}.$$

Write

$$d(x) = |x - y_0|, \qquad y_0 = (y_1^0, \dots, y_n^0).$$

Then

$$R \leq d(x) \leq R + \operatorname{diam}(\Omega) \quad \text{in } \overline{\Omega}.$$

Define

$$W(x) = K\left(R^{-A} - d(x)^{-A}\right) \quad \text{in } \overline{\Omega},$$

where K and A are constants to be determined. Then

$$W(x_0) = 0, \qquad W > 0 \quad \text{on } \partial\Omega \setminus \{x_0\}.$$

Set $z_i = x_i - y_i^0$. A direct computation gives

$$\begin{aligned} L\left(R^{-A} - d^{-A}\right) = Ad^{-A-4} &\left[-(A+2)\sum_{i,j} a_{ij} z_i z_j + d^2 \sum_i a_{ii} + d^2 \sum_i b_i z_i\right] \\ &+ c\left(R^{-A} - d^{-A}\right). \end{aligned}$$

Since $(a_{ij}) \geq \lambda I_n$,

$$\sum_{i,j} a_{ij} z_i z_j \geq \lambda \left|x - y_0\right|^2 = \lambda d^2.$$

Also $c \leq 0$ and $R^{-A} - d^{-A} \geq 0$. Hence

$$\begin{aligned} L\left(R^{-A} - d^{-A}\right) &\leq Ad^{-A-4}\left[-(A+2)\lambda d^2 + d^2 \sum_i a_{ii} + d^2 \sum_i b_i z_i\right] \\ &= Ad^{-A-2}\left[-(A+2)\lambda + \sum_i a_{ii} + \sum_i b_i z_i\right]. \end{aligned}$$

Choose $A \gg 1$ so that the bracket is ≤ -1 in Ω. Then

$$L\left(R^{-A} - d^{-A}\right) \leq -Ad^{-A-2}.$$

Since $R \leq d \leq R + \operatorname{diam}(\Omega)$,

$$L\left(R^{-A} - d^{-A}\right) \leq -A\left(R + \operatorname{diam}(\Omega)\right)^{-A-2}.$$

Choose

$$K = A^{-1}\left(R + \operatorname{diam}(\Omega)\right)^{A+2}.$$

Then

$$LW = KL\left(R^{-A} - d^{-A}\right) \leq -1.$$

Thus W is a barrier function at x_0. $\square$

Example. Let Ω be a bounded domain in $\mathbb{R}^n$.

(1) If Ω is convex, then Ω satisfies the exterior sphere condition.
(2) If $\partial\Omega$ is C^2, then Ω satisfies the exterior sphere condition.

Proof. If Ω is convex and $x_0 \in \partial\Omega$, choose a supporting hyperplane H at x_0, and let ν be the unit normal pointing to the exterior side. For any $R > 0$, set $y_0 = x_0 + R\nu$. Then the ball $B_R(y_0)$ lies in the open half-space outside H, while Ω lies in the opposite closed half-space. Hence

$$\Omega \cap B_R(y_0) = \emptyset, \qquad \overline{\Omega} \cap \overline{B}_R(y_0) = \{x_0\},$$

so the exterior sphere condition holds at x_0.

For the second statement, let $x_0 \in \partial\Omega$. Since $\partial\Omega$ is C^2, near x_0 the boundary can be expressed as a graph of a C^2 function. Choose coordinates so that $x_0 = 0$ and

$$\partial\Omega: \quad x_n = F(x_1, \ldots, x_{n-1}), \qquad F(0) = 0, \qquad \nabla F(0) = 0.$$

Write
$$x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$
By Taylor expansion,
$$F(x') = \frac{1}{2} \sum_{i,j=1}^{n-1} \partial_{ij} F(0) x_i x_j + o(|x'|^2) \geq -C |x'|^2$$
near 0. Set
$$S_r(x') = \sqrt{r^2 - |x'|^2} - r.$$
Then
$$S_r(x') = \frac{1}{2} \sum_{i,j} \partial_{ij} S_r(0) x_i x_j + o(|x'|^2) = -\frac{|x'|^2}{2r} + o(|x'|^2).$$
Choosing $r > 0$ sufficiently small gives
$$S_r(x') \leq -2C |x'|^2.$$
Thus
$$F(x') \geq S_r(x')$$
near 0. Taking
$$y_0 = (0, \dots, 0, -r),$$
the ball $B_r(y_0)$ is the desired exterior ball. □

Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that either Ω is convex or $\partial\Omega$ is C^2. Let*
$$L = \sum_{i,j} a_{ij} \partial_{ij} + \sum_i b_i \partial_i + c, \qquad a_{ij}, b_i, c \in L^\infty(\Omega) \cap C^\alpha(\Omega), \quad \alpha \in (0,1),$$
and assume
$$(a_{ij}) \geq \lambda I_n \quad (\lambda > 0), \qquad c \leq 0.$$
If
$$f \in L^\infty(\Omega) \cap C^\alpha(\Omega), \qquad \varphi \in C(\partial\Omega),$$
then there exists a unique solution
$$u \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$$
of the Dirichlet problem
$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

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