MODES OF CONVERGENCE

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Modes of Convergence

If one has a sequence of complex numbers $(x_n)_{n\in\mathbb{N}}$, it is unambiguous what it means for that sequence to converge to a limit $x \in \mathbb{R}$. More generally, if we have a sequence $(v_n)_{n\in\mathbb{N}}$ of d-dimensional vectors in a real vector space $\mathbb{R}^n$, it is clear what it means for a sequence to converge to a limit. We usually consider convergence with respect to the Euclidean norm, but for the purposes of convergence, these norms are all equivalent.

If, however, one has a sequence of real-valued functions $(f_n)_{n\in\mathbb{N}}$ on a common domain Ω and a perceived limit f, there can now be many different ways how f_n may or may not converge to f. Since the function spaces we consider are infinite dimensional, the functions f_n have an infinite number of degrees of freedom, and this allows them to approach f in any number of inequivalent ways. We now introduce different convergence concepts for sequences of measurable functions and then compare them to each other.

Definition 1: Modes of Convergence. Let $(f_n)_{n\in\mathbb{N}}$ and $f : \Omega \to \overline{\mathbb{R}}$ be measurable functions. We say that (f_n) converges to f:

(1) μ-**almost everywhere** (μ-**a.e.**) if there is a measurable set N with $\mu(N) = 0$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in N^c$. We write $f_n \to f$ μ-a.e..
(2) **in measure** μ if for all $\epsilon > 0$, $\lim_{n\to\infty} \mu(\{x \in \Omega : |f(x) - f_n(x)| > \epsilon\}) = 0$. We write $f_n \xrightarrow{\mu} f$.
(3) **in** $L^1(\Omega,\mu)$ if $\lim_{n\to\infty} ||f_n - f||_{L^1(\Omega,\mu)} := \lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0$. We write $f_n \xrightarrow{L^1} f$.

The L^1 mode of convergence is a special case of the L^p mode of convergence. One particular advantage of L^1 convergence is that, in the case when the f_n are μ-summable, it implies convergence of the integrals $\int_\Omega f_n d\mu \to \int_\Omega f d\mu$. This follows directly by the triangle inequality, i.e., $|\int_\Omega f_n d\mu - \int_\Omega f d\mu| \le \int_\Omega |f_n - f| d\mu$.

Proposition 2: Simple Implications. Convergence in $L^1(\Omega,\mu)$ implies convergence in measure μ. Moreover, if $\mu(\Omega) < \infty$, then convergence μ-a.e. implies convergence in measure μ too.

Proof. By replacing f_n with $f_n - f$, we can assume that $f \equiv 0$ without loss of generality.

(1) Recall Chebyshev's inequality, which states that for every μ-summable $f : \Omega \to \overline{\mathbb{R}}$, we have $\mu(\{x \in \Omega : |f(x)| > \alpha) \le \frac{1}{\alpha} \int_\Omega |f| d\mu$ for all $\alpha > 0$. It follows that for all $\epsilon > 0$, $\mu(\{x \in \Omega : |f_n| > \epsilon\}) \le \frac{1}{\epsilon} \int_\Omega |f_n| d\mu = \frac{1}{\epsilon} ||f_n||_{L^1}$. Therefore, $L^1(\Omega,\mu)$-convergence implies convergence in measure.
(2) From Egorov's theorem, it follows that for every $\delta > 0$, there exists $F_\delta \subset \Omega$ measurable with $\mu(\Omega \backslash F_\delta) < \delta$ such that $(f_n)_n$ converges uniformly to f on F_δ. In other words, for any $\epsilon > 0$, there exists $N \ge 0$, any $n > N$, we have $\sup_{x\in F_\delta} |f_n(x) - f(x)| < \epsilon$. For $n \ge N$, $\{x \in \Omega : |f_n(x) - f(x)| > \epsilon\} \subset \Omega \backslash F_\delta$. Hence $\mu(\{x \in \Omega : |f_n(x) - f(x)| > \epsilon\}) \le \mu(\Omega \backslash F_\delta) < \delta$. Since $\delta > 0$ was arbitrary, we can conclude.
(3) Alternatively, the latter can be proven by applying the dominated convergence theorem (DCT) to the integral of $\mathbb{1}_{\{|f_n - f| > \epsilon\}}$, which is dominated by 1 on a finite measure space.

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Examples. All other implications between different convergence concepts are not true in general.

- **A.e. convergence does not imply in-measure convergence on spaces with infinite measure:** The sequence $f_n = \mathbb{1}_{[n,n+1]}$ shows that the finiteness assumption in Proposition 2 is necessary. The sequence converges to 0 pointwise (and thus μ-a.e.), but it does not converge in measure.
- **A.e. convergence does not imply L^1 convergence:** Let $\Omega = [0,1]$ and λ be the Lebesgue measure. The sequence $f_n := n\mathbb{1}_{(0,\frac{1}{n})}$, $n \in \mathbb{N}$, converges to 0 pointwise, hence also λ-a.e.. It also converges in measure because we are on a finite measure space. However, $\int f_n d\lambda = 1$, so (f_n) does not converge to 0 in $L^1([0,1],\lambda)$.
- **L^1 convergence does not imply a.e. convergence:** For $n \in \mathbb{N}$ and $k = 1,\dots,2^n$, define $f_{nk} := \mathbb{1}_{[(k-1)2^{-n},k2^{-n}]}$. Renumbering this double sequence to a single sequence $(g_m)_{m\in\mathbb{N}}$, we have $\int f_{nk} d\lambda = 2^{-n}$ and hence $g_m \to 0$ in L^1 as $m \to \infty$. The sequence also converges to 0 in measure. However, $\limsup_{m\to\infty} g_m = 1$ and $\liminf_{m\to\infty} g_m = 0$ show that (g_m) does not converge to 0 μ-a.e. Intuitively, this is a sequence of indicator functions of intervals of decreasing length, marching across the unit interval over and over again. This sequence is also known as the *typewriter sequence*.

Example 2 shows that convergence in measure is a strictly weaker notion, as it is not implied by a.e. or L^1 convergence. Convergence μ-a.e. and convergence in $L^1(\Omega,\mu)$ do not seem to be related in general.

The dominated convergence theorem of Lebesgue states that μ-a.e. convergence together with the existence of a μ-summable bound for a sequence of measurable functions imply convergence in $L^1(\Omega,\mu)$. These conditions are only sufficient, but not necessary. Thus it is of interest to look for an even sharper result.

Example 3. Let $\Omega = [0,1]$ and consider the Lebesgue measure λ. We define the functions

$$f_n := \frac{n}{\log(n)}\mathbb{1}_{(0,\frac{1}{n}]} \quad \forall n \geq 1.$$

Then we have $f_n \to 0$ pointwise and hence also λ-a.e. Moreover

$$\int_{[0,1]} f_n d\lambda = \frac{1}{\log(n)} \to 0$$

so that $f_n \to 0$ in $L^1([0,1],\lambda)$ since $f_n \geq 0$. However, there exists no λ-summable function g with $g \geq f_n$ λ-a.e. for all n.

Indeed, such a g would have to satisfy $g \geq \frac{n}{\log(n)}\lambda$-a.e. on $(0,\frac{1}{n}]$ for all n. But then

$$\int_{[0,1]} g\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n}]} d\lambda \geq \frac{n}{\log(n)}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{(n+1)\log(n)}$$

and hence

$$\int_{[0,1]} g d\lambda = \sum_{n=1}^{\infty}\int_{[0,1]} g\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n}]} d\lambda \geq \sum_{n=2}^{\infty}\frac{1}{n\log(n)} = \infty.$$

Definition 3: Uniform Summability The family $(f_n)_{n\in\mathbb{N}}$ is called *uniformly μ-summable* if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ and $A \subset \Omega$ μ-measurable with $\mu(A) < \delta$ it holds

$$\int_A |f_n| d\mu < \epsilon.$$

This allows us to formulate a necessary and sufficient condition for L^1 convergence.

Vitali Convergence Theorem. The Lebesgue dominated convergence theorem states that μ-a.e. convergence together with the existence of a μ-summable bound imply convergence in $L^1(\Omega,\mu)$. These conditions are sufficient but not necessary.

Definition 3: Uniform Summability. The family $(f_n)_{n\in\mathbb{N}}$ is called uniformly μ-summable if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ and $A \subset \Omega$ μ-measurable with $\mu(A) < \delta$, it holds that $\int_A |f_n| d\mu < \epsilon$. This allows us to formulate a necessary and sufficient condition for L^1 convergence.

Theorem 4: Vitali Convergence Theorem. If $\mu(\Omega) < \infty$, the following conditions are equivalent:

(1) $f_n \to f$ in $L^1(\Omega, \mu)$.
(2) $f_n \xrightarrow{\mu} f$ and $(f_n)_{n\in\mathbb{N}}$ is uniformly μ-summable.

L^p **Convergence.** L^1-convergence is one particular case of a more general concept called L^p-convergence.

Definition 5: L^p Convergence. Let $p \in [1, \infty)$. For $f : \Omega \to \overline{\mathbb{R}}$, we define the $L^p(\Omega, \mu)$ norm by $||f||_{L^p(\Omega,\mu)} = (\int_\Omega |f|^p d\mu)^{\frac{1}{p}} \leq \infty$. For $p = \infty$, we define the $L^\infty(\Omega, \mu)$ norm by $||f||_{L^\infty(\Omega,\mu)} := \mu\text{-ess sup}_{x\in\Omega} |f(x)|$. A sequence of μ-measurable functions $(f_n)_{n\in\mathbb{N}}$ converges in $L^p(\Omega, \mu)$ to a measurable function f if $\lim_{n\to\infty} ||f_n - f||_{L^p(\Omega,\mu)} = 0$.

Proposition 6. If $\mu(\Omega) < \infty$, then for $1 \leq r < s \leq \infty$, we have $L^s(\Omega, \mu) \subset L^r(\Omega, \mu)$. In particular, convergence in $L^s(\Omega, \mu)$ implies convergence in $L^r(\Omega, \mu)$. To summarize, when $\mu(\Omega) < \infty$ and $1 \leq r \leq s \leq \infty$, we have the following implications: $L^s \Rightarrow L^r \Rightarrow L^1 \Rightarrow$ in measure μ.

$$\begin{tikzcd}
L^s \arrow[r, "r\leq s"] & L^r \arrow[r, "\Longrightarrow"] & L^1 \arrow[d, leftrightarrow, "\text{uniformly } \mu\text{-summable}"', "\Downarrow"] \\
 & & \text{in measure } \mu
\end{tikzcd}$$

$$\begin{tikzcd}
\mu\text{-a.e.} \arrow[r, "\Longrightarrow"] & \text{in measure } \mu
\end{tikzcd}$$